MA203: Numerical Methods
Numerical Analysis of Microscopic Traffic Flow Models

22110279-Vannsh Jani
22110295-Yash Bachwana
22110275-Tripti Vishwakarma
22110276-Vatsal Trivedi
22110221-Bhanu Repalle

April 22, 2024

1 Problem Statement

Traffic congestion is a persistent and escalating problem in urban areas worldwide. As cities grow, traffic volume increases, leading to inefficient transportation systems, longer commute times, increased fuel consumption, environmental pollution, and reduced overall quality of life. Safety and capacity of the roadway system are both dependent on the the car-following characteristics of the driver. Car-following models are an integral part of traffic simulation tools that attempt to model the driver behavior in the real world. Car-following models is a type of microscopic traffic simulator as it is based on the behavior of the driver and not of all the vehicles on the road.

Microscopic traffic flow models like the car-following model captures the intricate interactions between individual vehicles, considering factors such as acceleration, deceleration, and safe following distances. This level of detail is crucial for understanding the complex nature of traffic flow in urban environments. Understanding microscopic traffic flow is crucial for assessing the environmental impact of traffic congestion, including emissions and fuel consumption.

Microscopic traffic flow models enable the optimization of traffic signal timings at intersections. By simulating the behavior of individual vehicles approaching intersections, traffic signals can be synchronized to minimize stops and delays, thereby improving traffic flow efficiency.

Car-following models are based on the assumption that each driver reacts in a specific fashion to stimulus, which leads to an actuation of acceleration. Stimulus mentioned here could be change in environmental conditions or a change in the headway distance. Using a system of differential equations, the car-following model can be generalized or extended to a platoon of vehicles.

1.1 OVM Microscopic Model

We describe the movement of car i using the following equation [4]:

$$\frac{d^2x_i(t)}{dt^2} = a(V(\Delta x_i(t)) - \frac{dx_i(t)}{dt})$$

Where $x_i(t)$ is the position of the car i at time t, $\Delta x_i(t)$ is the distance between car i and the car in front of it at time t, $V(\Delta x_i(t))$ is the optimal velocity of the car, The velocity of the car i is given by $\frac{dx_i(t)}{dt}$.

2 Objective

- Computing different traffic flow models efficiently. Different numerical methods vary in their computational efficiency, impacting the speed and resources required for simulations. Balancing accuracy and efficiency is essential, especially for complex traffic flow scenarios.
- Conduct an analysis of two microscopic models, OVM and an updated version of classical "follow the leader model" in terms of their predictive accuracy and suitability for different traffic scenarios.
- Explore and implement numerical techniques such as runge-kutta methods to solve the ordinary differential equations in the traffic flow models.
- Create simulations of the OVM and our models to gain insights into their dynamic behavior.
- Develop visualizations and plot graphs to effectively communicate the simulation results and model results to a wider audience.

3 Physical Situation

A single lane contains a line of vehicles, with each vehicle positioned behind the one in front. When the signal turns green at t = 0, all the vehicles begin to move. There is a time gap between when each vehicle starts moving compared to the vehicle in front of it. The first vehicle initiates its motion independently, while the others are required to follow the motion of the vehicle in front to prevent accidents. The lead vehicle can reach a maximum speed of v. Eventually, after a certain period following the green signal, the maximum speed of all the vehicles behind it also becomes v.

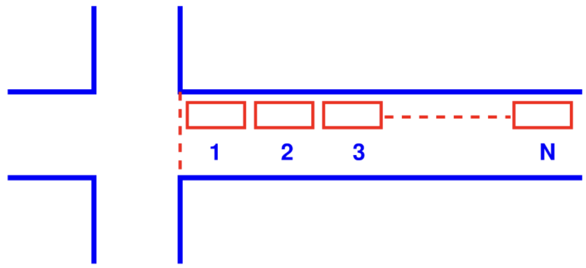

Figure 1: Traffic Jam at a signal [2]

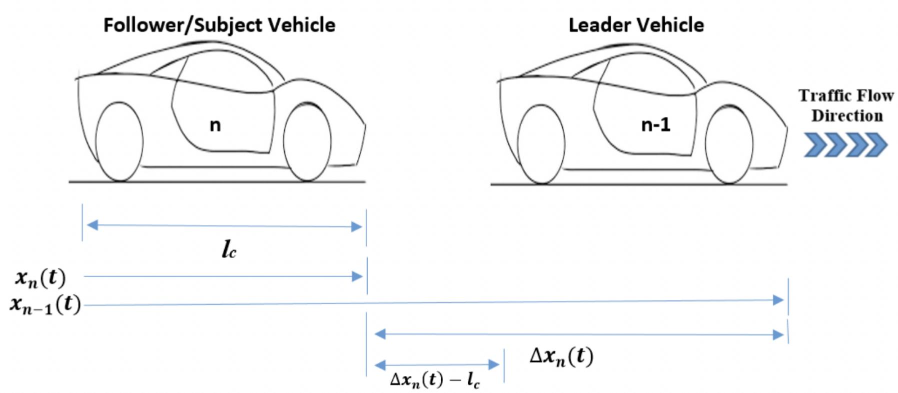

Figure 2: Graphical representation of OVM model [3]

4 Microscopic Models

Microscopic traffic flow models are mathematical models used to simulate and analyze the behavior of individual vehicles in traffic. These models aim to capture the complex dynamics of traffic flow and provide insights into various traffic phenomena. We will discuss two microscopic models namely OVM and our model.

4.1 OVM

A dynamic equation serves as the model's foundation. It is a straightforward yet accurate model of traffic flow that considers the dynamics of traffic congestion. OVM has nonlinear behavior, which means that the relationship between elements like acceleration, velocity, and distance is not proportional. This nonlinearity makes it possible to describe complicated traffic interactions, such as the effects of unexpected speed changes and different driving styles, which results in a more accurate depiction of traffic flow dynamics. The OVM's dynamic equation at time t is shown below [3]:

$$\frac{d^2x_i(t)}{dt^2} = a(V(\Delta x_i(t)) - \frac{dx_i(t)}{dt})$$

$$\Delta x_i(t) = x_{i-1}(t) - x_i(t)$$

$$V(\Delta x_i) = V_1 + V_2.\tanh[C_1(\Delta x - l_c) - C_2]$$

Where i represents the vehicle number and a is a constant which represents the driver's sensitivity. Here we have assumed that the sensitivity of the driver is independent of i. $x_n(t)$ is the position of the i-th car at time t.
$v_i(t) = \frac{dx_i(t)}{dt}$ is the velocity of the i-th car at time t.
$\Delta x_i(t)$ is the space headway.
l_c is the length of the car.
$V(\Delta x_i)$ is the optimal velocity function of the i-th car.

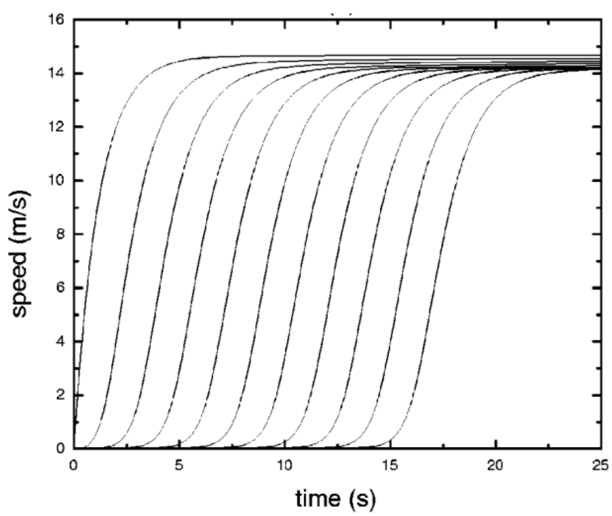

Figure 3: Variation of velocity with time in OVM Model [4]

The following distance of the prior car number $(n-1)$ determines the ideal velocity function. The velocity must be decreased and made modest enough to avoid colliding with the vehicle in front of the subject when the distance between them gets too close. On the other hand, as the headway lengthens, the car can proceed at a faster speed while still staying inside the maximum speed limit.

4.2 Our Model

According to Newton's Mechanics where the acceleration may be regarded as the response of a matter to the stimulus it receives in the form of the force it receives from the interaction with other particles in the system.[2] Hence, the basic philosophy of car-following theories can be summarized by the following equation:

$$[Response]_n \propto [stimulus]_n$$

Where the proportionality constant is known as sensitivity. The sensitivity characterizes how the driver responds to any change in the velocity of the car moving in front of him / her. The value of sensitivity varies with circumstances and model. Here, the Response is the acceleration / deceleration of the nth vehicle. Stimulus is the relative speed of the follower and leader or difference in speed of following car and the speed of the leading car or only the speed of the leading car.
According to the above theory, we tried to create a model of our own, by fusing few concepts from some microscopic models of traffic flow, like Optimal Velocity Model and Classical Car-Following Model.

1. According to our model, the response is given according to the acceleration of n-th vehicle.

2. Stimulus is given by the difference in speed of the following car and the speed of the leading car.

3. Sensitivity is given by the Sigmoid function of difference between the velocity of following car and the leading car.

The Sigmoid function is given as:

$$\mathrm{f}(x) = \frac{1}{1 - e^{-x}}$$

Following is the graph for Sigmoid function.

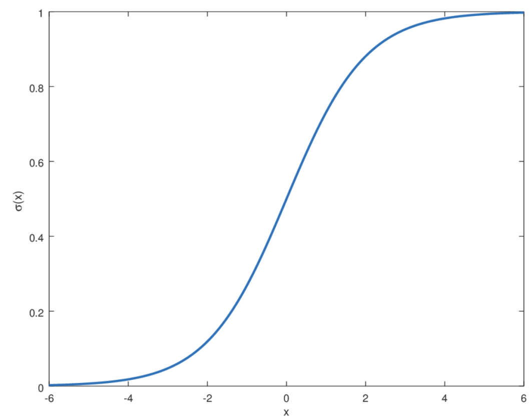

Figure 4: Sigmoid function [1]

We represent our traffic model as the following equation:

$$\frac{dv_n}{dt} = -\frac{v_n - v_{n-1}}{1 + e^{v_{n-1} - v_n}}$$

where v_i is velocity of i-th vehicle. Here we have taken sensitivity coefficient to be variable unlike in the OVM microscopic model. The sensitivity coefficient varies as Sigmoid function of difference in velocities (following-leading).

5 Governing Equations and Parameters

5.1 OVM

The vehicle j is affected only by the vehicle ahead j+1, called the leading vehicle. The equation of motion for vehicle j is given as [5]:

$$\frac{dx_j(t+\tau)}{dt} = V(\Delta x_j(t))$$

Here, $x_j(t)$: Position of vehicle j at time t. τ: Delay time. $\Delta x_j(t) = x_{j+1}(t) - x_j(t)$: Headway of vehicle j at time t. $V(\Delta x_j(t))$: Optimal velocity.

Using Taylor expansion for the above equation, we get:

$$\frac{d^2x_j(t)}{dt^2} = \frac{1}{\tau}\left(V(\Delta x_j(t)) - \frac{dx_j(t)}{dt}\right)$$

The optimal velocity function is given by:

$$V(\Delta x_j(t)) = \frac{v_{\max}}{2}\left(\tanh(\Delta x_j(t) - x_c) + \tanh(x_c)\right)$$

where x_c is a constant representing the safety distance, when $\Delta x_j \to \infty$ and $x_c > 0$, $V(\infty) \approx v_{\max}$.

The use of the hyperbolic tangent function (tanh) in the Optimal Velocity Model (OVM) and similar traffic flow models serves important mathematical purposes:

1. Smooth Transition: The tanh function ensures a smooth and continuous transition in vehicle velocity with changing relative distances $(\Delta(xj(t)))$. Abrupt velocity changes can cause unrealistic and unstable traffic flow.
2. Saturation Effect: Tanh exhibits a saturation effect, approaching ± 1 as the argument grows. This mirrors real-world behavior, where vehicles reach maximum and minimum relative velocities as they get far apart or close.
3. Modeling Driver Behavior: Tanh aligns with driver behavior by reflecting acceleration with comfortable gaps and deceleration to maintain safe distances from the vehicle ahead.

5.2 Our Model

Let Δx_{n+1} is the gap available for (n + 1)th vehicle, and let Δx_{safe} is the safe distance, v_{n+1} and v_n are the velocities, the gap required is given by[2],

$$\Delta x_{n+1} = \Delta x_{safe} + k.v_{n+1}$$

k is the sensitivity coefficient. The above equation can be written as:

$$x_n - x_{n+1} = \Delta x_{safe} + k.v_{n+1}$$

Differentiating the above equation with respect to time, we get,

$$v_n - v_{n+1} = k.a_{n+1}$$

$$a_{n+1} = \frac{1}{k}.v_n - v_{n+1}$$

When a vehicle is standing still at a signal and the signal turns green, the vehicle typically undergoes an initial acceleration to start moving. A commonly used mathematical model to describe this initial acceleration is the "ramp-up" or "sigmoid" function. This function starts from zero acceleration, increases gradually, and then levels off as the vehicle reaches a constant speed. We are taking our sensitivity coefficient to be sigmoid of difference in velocities(following - leading).

So we represent the velocity equation of our model as follows:

$$\frac{dv_n}{dt} = -\frac{v_n - v_{n-1}}{1 + e^{v_{n-1} - v_n}} \quad - (1)$$

In order to find the position of the leader vehicle, we define the position equation:

$$x_{n+1} = x_n + h.(v_n) \quad - (2)$$

where x_{n+1} represents the position of the leading vehicle x_n is the position of the nth vehicle, h denotes the discrete time step, and v_n signifies the velocity of the nth car.

6 Assumptions

The key assumptions in the optimal velocity model include:

1. Homogeneity: The model assumes that all vehicles in the traffic flow are identical and have the same characteristics. This assumption simplifies the analysis by considering a homogeneous traffic stream without accounting for variations in vehicle types, sizes, or driver behaviors.

2. One-dimensional traffic: The model assumes that traffic flow occurs in a single lane or on a one-dimensional road. This assumption allows for a simplified representation of the traffic dynamics, as it eliminates the complexities associated with multi-lane traffic or road networks.

3. Continuous traffic flow: The model assumes a continuous flow of vehicles without any interruptions or disruptions. This assumption implies that there are no traffic signals, intersections, or other factors that may cause vehicles to stop or change their behavior. It allows for a smooth and uninterrupted flow of traffic.

4. Optimal velocity function: The model assumes that each vehicle adjusts its velocity to maintain an optimal distance from the preceding vehicle. This optimal distance is determined by a specific velocity function, which depends on factors such as the desired speed, the distance to the preceding vehicle, and the reaction time of the driver. The model assumes that vehicles strive to maintain this optimal velocity to ensure smooth and efficient traffic flow.

5. No external influences: The model assumes that there are no external influences or disturbances affecting the traffic flow. This means that factors such as weather conditions, road conditions, or external events are not considered in the model. It simplifies the analysis by focusing solely on the internal dynamics of the vehicles.

6. No overtaking : This model assumes no driver attempts to overtake. This assumption simplifies predictions, aligns with regulations, and streamlines simulations.

7. Response to Stimulus : This model takes into consideration the fact that each driver response similarly to any change in the velocity of the car moving ahead of him/her.

These assumptions may not hold true in all real-world traffic scenarios. However, they provide a simplified framework for understanding and analyzing traffic flow dynamics.

7 Boundary conditions

In order to solve these differential equations, boundary conditions need to be specified.

1. Initially, the distance between all the vehicles waiting at the signal are equal and the initial velocities of all the vehicles is zero, i.e. at $t = 0$, $v_i = 0$ and $x_i = 0 \; \forall i = [1, 2, ..., n]$.

2. Velocity of first vehicle is taken to be a Sigmoid function of time.
   $v_1 = W(\frac{1}{1-e^{-t}}) - b$.
   Here we are taking W as 40 m/s and b as 20 m/s.
   So our equation becomes:
   $v_1 = 40(\frac{1}{1-e^{-t}}) - 20$.

3. Assume the leader changes the velocity according to $v_l = v_0(t)$ and the follower duplicates the leader's velocity but with some delay time, that is, $v_f = v_0(t - \delta t)$. We define the delay time of car motion by δt which is taken to be $1.1s$ in our model. Here v_l is velocity of leader and v_f is velocity of follower.

8 Methodology

We solve the differential equations obtained in our model by employing suitable numerical methods in the following procedure.

First, we initialize the velocity of the first vehicle v0, it is calculated based on the sigmoid function with time. This represents the velocity of the leader vehicle. We then proceed to calculate the velocities and positions of the vehicles over time. It is done by implementing a fourth-order Runge Kutta method on the velocity equation (1) to calculate the velocities of the vehicles. This method is applied from the first vehicle to the end of the time interval. Then using Euler's method on the position equation (2), we calculate the positions of the vehicles based on the new velocities.

We plot the velocities and positions of the vehicles, it helps to visualize how the positions and velocities of the vehicles evolve over time to compare both the models.

9 Numerical Solution

Our model consists of two first order ordinary differential equations representing the velocities and positions of vehicles over time. To solve the above system of equations, we use a discrete-time approach in which we update the velocities and positions of vehicles at different time steps.

$$\frac{dv_n}{dt} = -\frac{v_n - v_{n-1}}{1 + e^{v_{n-1} - v_n}}$$

$$x_{n+1} = x_n + h.v_n$$

The velocity equation in our traffic flow model is a nonlinear differential equation. Nonlinear equations often benefit from higher-order numerical methods like Runge-Kutta, as they can capture nonlinear behavior more accurately. Moreover,it maintains a consistent order of accuracy across multiple time steps. This ensures that the errors introduced by the numerical integration method do not accumulate rapidly. The position equation is essentially a first-order differential equation in terms of position, and Euler's method is well-suited for solving first-order equations. It provides a straightforward way to update the position over time.

10 Algorithm

10.1 Using Runge-Kutta Method for Velocity Equation:

1. Initialize the velocity of the leader vehicle v(0) at t = 0.
2. Define the time step h.
3. Create an array or list to store the values of v(n) over time.
4. Use the Runge-Kutta formula (fourth-order) to update the velocity at each time step, starting from time t = 0 with initial velocity v(0).

   $k_1 = h.\frac{dv_n}{dt}$
   $k_2 = h.\frac{d(v_n)+0.5.k_1}{dt}$
   $k_3 = h.\frac{d(v_n)+0.5.k_2}{dt}$
   $k_4 = h.\frac{d(v_n)+k_3}{dt}$
   $v_{n+1} = v_n + \frac{(k_1+2k_2+2k_3+k_4)}{6}$

   This formula computes v(n+1) based on the current velocity v(n) and its rate of change dv(n)/dt at each time step.
5. Repeat the above steps for each time step until you reach your desired time interval.

10.2 Using Euler's Method for Position Equation:

1. Initialize the position of the leader vehicle x(0) at t = 0.

2. Define the time step h.
3. Create an array or list to store the values of xn over time.
4. Use Euler's method to update the position at each time step using the position update equation, starting from t = 0 with initial position x(0).

   $x_{n+1} = x_n + h.v_n$

   Where v(n-1) is the velocity at the previous time step.
5. Repeat the above steps for each time step until you reach your desired time interval.

11 Real Life Application

We observed that our model describes the behaviour of the vehicles on a signal. So, this can be very helpful in fixing the traffic signal controlling mechanism. The plots which we have obtained can be used to determine the amount of time for which the signal should be kept open to allow n vehicles to pass in the mean time.

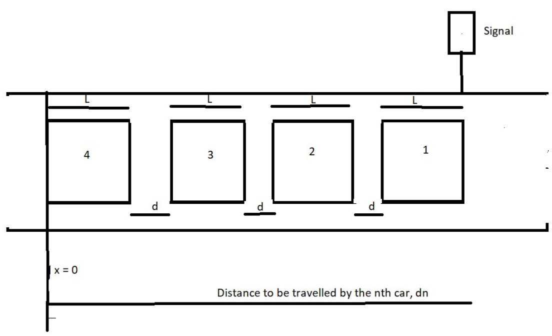

Figure 5: Traffic jam

According to above diagram, we observe that the distance that nth vehicle must traverse is equal to

$$x_n = n.L + (n-1).d$$

Therefore, using the graph of position versus time, we can get the time corresponding to the distance that the last vehicle must traverse so as to cross the signal, i.e dn.

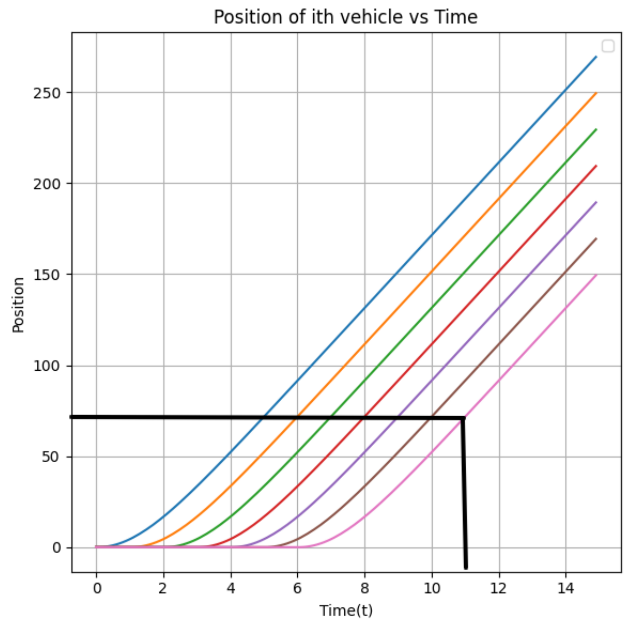

Figure 6: Correspondence between distance and time at the signal

12 Discussions and the future scope

1. Our model very well describes the motion of multiple vehicles on a highway starting from a traffic signal. But it only accounts for single lane linear motion with no overtaking.
2. This model can be further extended to develop another model that incorporates overtaking and multi lane motion, to make it more realistic.
3. Also, we can also incorporate properties such as sudden change in the density of the traffic on that particular lane due to the entry of one more vehicle in the lane from the adjacent lane

13 Results

1. After the signal turns green at t = 0, the vehicles start moving with a slight delay between starting time of 2 consecutive vehicles. But we observe that with time, the velocity of all the vehicles converges to the maximum speed of the first vehicle.
2. After all the vehicles attain the maximum speed, we observe that the distance between any two consecutive vehicles becomes constant.

3. This model can play a significant role in setting the traffic signal parameters.

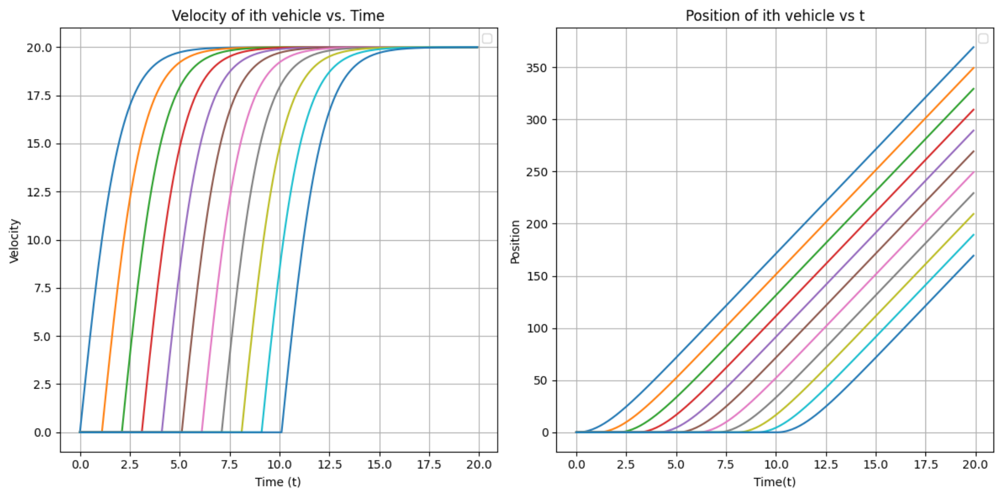

Figure 7: Plotting velocity and position as a function of time

References

[1] BotPenguin AI. Sigmoid function. https://botpenguin.com/glossary/sigmoid-function, Accessed: 2021.

[2] IIT Bombay. Car following models. https://www.civil.iitb.ac.in/ vmtom/nptel/533$_{C}arFol/web/web.html, A$ $MonthDay, Year$.

[3] Md Rafiul Islam. Comparison of vehicle dynamics of microscopic car following models: Optimal velocity and intelligent driver model. 2018.

[4] Rui Jiang, Qing-Song Wu, and Zhi-Xi Zhu. Full velocity difference model for a car-following theory. *Physical Review E*, 64(1):017101, 2001.

[5] Takashi Nagatani. The physics of traffic jams. *Reports on Progress in Physics*, 65(9):1331–1386, 2002.