HW 1

Q.3) From discussion we know the change of variable formula, which is

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}(g^{-1}(y))$$

$U_1 \sim U(0,1)$ and $U_2 \sim U(0,1)$

$\Theta = 2\pi U_1$ $\quad R = \sqrt{-2\ln(U_2)}$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \end{pmatrix}$$

PDF of R is $f_R(r) = re^{-\frac{r^2}{2}}$

PDF of θ is just uniform from $[0, 2\pi]$

$\therefore$ Joint distribution of z_1, z_2 =

$\cancel{f(z_1,z_2) = f_R(r) \cdot f_\theta(\theta)}$.

Redid next page for easy understanding

$R = \sqrt{z_1^2 + z_2^2}$ $\quad \theta = \arctan\left(\frac{z_2}{z_1}\right)$

$\cancel{f_{z_1,z_2}(z_1,z_2) = f_{R,\theta}(R,\theta) \cdot \frac{1}{|R|}}$

Jacobian computed next page ← determinant of Jacobian

$$J = \begin{array}{cc} \cos\theta & -R\sin\theta \\ \multicolumn{2}{c}{\begin{pmatrix} \frac{\partial z_1}{\partial R} & \frac{\partial z_1}{\partial \theta} \\ \frac{\partial z_2}{\partial R} & \frac{\partial z_2}{\partial \theta} \end{pmatrix}} \\ \sin\theta & R\cos\theta \end{array}$$

$\therefore J = \begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix}$ we know $\sin^2 + \cos^2 = 1$

$\therefore$ $\det(J) = R(\cos^2 + \sin^2) = R$

$\therefore$ Based on the discussion formula

$$f_{z_1,z_2}(z_1, z_2) = f_R(r) \cdot f_{(\Theta)}(\theta) \left| \frac{1}{R} \right| \rightarrow \text{abs val}$$

$$= r e^{-\frac{r^2}{2}} \cdot \frac{1}{2\pi} \cdot \frac{1}{R} \quad \therefore (r) = R = \sqrt{z_1^2 + z_2^2}$$

$$= \frac{1}{2\pi} e^{-\frac{z_1^2 + z_2^2}{2}}$$ → That's bivariate normal distribution formula!

QED!