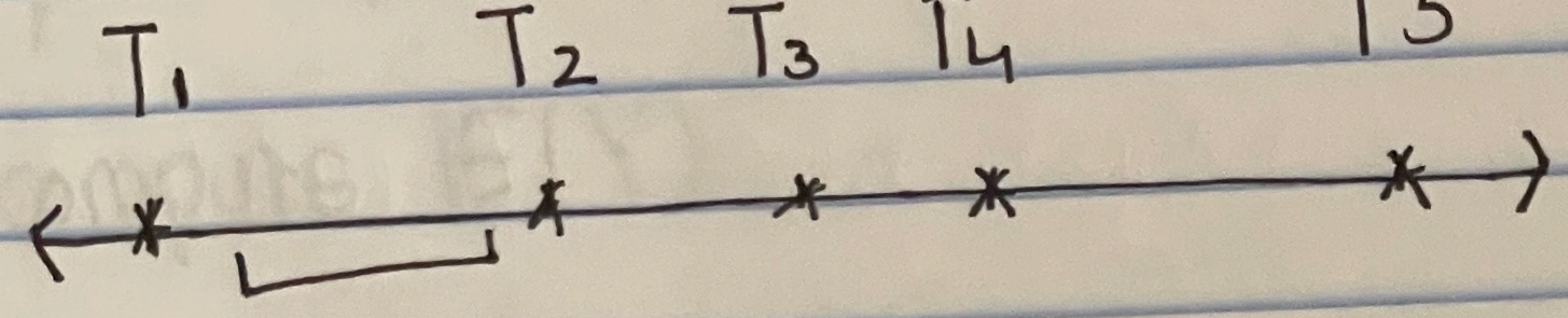
Homework 8.

9.1.) Find the joint distribution of $(T_1, T_2, T_5)$




$T_i$ = arrival times

seperation times are $S_i = T_i - T_{i-1}$

$\therefore T_i = \sum_{i=1}^{n} S_i$

$\therefore U = T_1$, $V = T_2 - T_1$, $W = T_5 - T_2$

$= S_1$, $= S_2$, $= S_3 + S_4 + S_5$

$\therefore (U, V, W) =$ product of individual densities

$W = \text{gamma}(3, \lambda)$, $U, V = \text{exponential}(\lambda)$

$\therefore U \cdot V \cdot W = \lambda e^{-\lambda u} \lambda e^{-\lambda v} \dfrac{\lambda^3 w^2 e^{-\lambda w}}{2!}$

we know $T_1 = U$, $T_1 + T_2 = V$, $T_1 + T_2 + T_5 = U + V + W$

$\therefore p(T_1 \cdot T_2 \cdot T_5) = \dfrac{\lambda e^{-\lambda T_1} \lambda e^{-\lambda T_2 T_1} \lambda^3 (T_1 + T_2 + T_5)^2 e^{-\lambda (T_1 + T_2 + T_5)}}{2!}$

9.9.) $\lambda \sim$ geometric$(p)$

$Y|\lambda \sim$ possion $(\lambda)$

compute $\mathbb{E}(Y)$

we know expected value of $\lambda = \frac{1-p}{p}$

$\cancel{P(y|\lambda)} = \frac{e^{-\lambda}\lambda^{y}}{y!}$ expected value $= \lambda$

possion $=$

$\mathbb{E}[Y] = E_{\lambda}[\underbrace{\mathbb{E}[Y|\lambda]}_{\lambda}]$ $\therefore E[\lambda] = \frac{1-p}{p}$ QED

This is because possion's mean & variance is $\lambda$ itself!

9.10.) $B_n$ = black ball in urn before $n^{th}$ draw

1 Black 2 white balls at start!

① $\mathbb{E}(B_{n+1} | B_n)$

{next page}

let's see → $\mathbb{E}$z just what you want / what you have

① $\mathbb{E}(B_{n+1} | B_n)$

let $T_n$ = total balls
$W_n$ = white balls
$B_n$ = Black balls

$\therefore \frac{B_n}{T_n}$ = probability of drawing black ball

$\therefore$ if black ball is drawn:

$B_{n+1} = B_n + 1$ else $B_n = B_n$

$\therefore \mathbb{E}(B_{n+1} | B_n) = P(\text{Black}) B_n + 1 + P(\text{White}) \cdot B_n$

$= \frac{B_n}{T_n}(B_n + 1) + \frac{W_n}{T_n}(B_n)$

$= \frac{B_n(B_n+1) + B_n(T_n - B_n)}{T_n}$

$= \frac{B_n(T_n+1)}{T_n} = \frac{B_n(n+3)}{n+2}$

note on $T_n$ ⇒ always increases by 1 regardless of B, W

$\therefore T_n = n+2$

② Law of conditional Expected Value

WTF: $\mathbb{E}(G_{n+1})$ know $\mathbb{E}(G_{n+1} | G_{n+1})$

$\therefore \mathbb{E}(B_{n+1}) = \mathbb{E}[\mathbb{E}(B_{n+1} | B_n)]$

$= \mathbb{E}\left(\frac{B_n(n+3)}{n+2}\right)$ $\therefore \mathbb{E}(B_{n+1}) = \mathbb{E}(B_n)\frac{n+3}{n+2}$

Induction $\mathbb{E}(B_1)=1$ let's see fro $B_2$ we get

$\frac{1(1+3)}{1+2} = \frac{4}{3}$, $B_3 = \frac{2+3}{2+2} \cdot \frac{4}{3} = \frac{5}{3}$,

$B_4 = \frac{3+3}{3+2}\left(\frac{5}{3}\right) = 2$, $2\left(\frac{4+3}{4+2}\right) = \frac{7}{3}$

$\therefore$ obsereve that increases by $\frac{n+2}{3}$

$\therefore$ IH $= \mathbb{E}(B_{n+1}) = \frac{(n+1)+2}{3} = \frac{n+3}{3}$

given $E(B_n) = \frac{n+2}{3}$

$\therefore$ by formula $= \frac{n+3}{3} = \frac{n+2}{3} \cdot \left(\frac{n+3}{n+2}\right)$

$= \frac{n+3}{3}$ ✓ QED

9.3.) Exeped # of black balls before $n^{th}$ draw → gut says that should be 1/3 let's see

$$\cancel{\mathbb{E}(P_n)} \quad P_n = \frac{B_n}{T_n} = \frac{B_n}{n+2}$$

$$\therefore E(P_n) = \mathbb{E}\left(\frac{B_n}{n+2}\right) = \frac{\mathbb{E}(B_n)}{n+2}$$

$$= \frac{\frac{n+2}{3}}{n+2} = \frac{1}{3} \quad \therefore \frac{1}{3} \text{ indeed QED}$$

9.4) $f(x,y) = e^{-y}$, for $0<x<y$ and $f(x,y)=0$ else

① $X$ and $y$ independent if $P(X,y) = P(X) \cdot P(y)$ or $P(y|X) = Py \parallel P(x|y) = P(X)$ since $f(x,y)$ depends on $x$ and $y$ over the plane; they are not independent. Getting information about $x$ changes my belief on $y$. if $x$ is (−) for example; I know what $y$ is; however I dont when $x$ is (+).

2.) marginal density over $y$ since $0<x<y$

$\therefore f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{x=0}^{y} e^{-y}\,dx = ye^{-y}$

only valid for all $y>0$ $f_Y(y) = ye^{-y}$

$\square$ QED

3.) $f(x|y) = \frac{f(x,y)}{f(y)}$ ← given, ← part 2 $= \frac{\cancel{e^{-y}}}{y\cancel{e^{-y}}} = \frac{1}{y}$

$\therefore f_{X|Y}(x|y) = \frac{1}{y}$ for $0<x<y$ QED $\square$

4.) $\mathbb{E}(X|Y=y)$ we know from 3 that $f(x|y) = 1/y$

$\therefore \int_{x=0}^{y} x\, f(x|y)\,dx = \int_{x=0}^{y} x\,\frac{1}{y}\,dx = \frac{1}{y}\left[\frac{x^2}{2}\right]\Big|_{x=0}^{x=y}$

$= \frac{1}{y}\left[\frac{y^2}{2} - 0\right] = \frac{y^2}{2y} = \frac{y}{2}$, QED $\square$

5.) $E(X) = \mathbb{E}_Y[E(X|Y)] = \cancel{\mathbb{E}_Y[y/2]}$

$\int_{y=0}^{\infty} \mathbb{E}(X|Y=y)\cdot f_Y(y)\,dy = \int_0^{\infty} \frac{y}{2}\cdot ye^{-y}\,dy$

= definition of gamma integral → for $n=2$ evaluates to 2 $\therefore \frac{1}{2}(2)=1$ $\therefore \mathbb{E}(X)=1$, QED $\square$