Minimal colorings for properly colored subgraphs in complete graphs

Chunqiu $\operatorname{Fang}^{*1,2,4}$, Ervin Győri^{†2}, and Jimeng Xiao^{‡2,3,5}

¹Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

²Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u.13-15, 1053 Budapest, Hungary

³Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, China

⁴Yau Mathematical Sciences Center, Tsinghua University, Beijing 10084, China

⁵Xi'an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi'an, China

Abstract

Let $pr(K_n, G)$ be the maximum number of colors in an edge-coloring of K_n with no properly colored copy of G. In this paper, we show that $pr(K_n, G) - ex(n, G') = o(n^2)$, where $G' = \{G - M : M \text{ is a matching of } G\}$. Furthermore, we determine the value of $pr(K_n, P_l)$ for $l \geq 27$ and $n \geq 2l^3$ and the exact value of $pr(K_n, G)$, where G is C_5, C_6 and K_4^- , respectively. Also, we give an upper bound and a lower bound of $pr(K_n, K_{2,3})$.

Keywords: Properly colored subgraph, Turán number, Anti-Ramsey number.

1. Introduction

^{*}fcq15@mails.tsinghua.edu.cn, supported in part by CSC(No. 201806210164).

[†]gyori.ervin@renyi.mta.hu, supported in part by the National Research, Development and Innovation Office NKFIH, grants K116769, K117879 and K126853.

[‡]xiaojimeng@mail.nwpu.edu.cn, supported in part by CSC(No. 201706290171).

We call a subgraph of an edge-coloring graph rainbow, if all of its edges have different colors. While a subgraph is called $properly\ colored\ (also\ can\ be\ called\ locally\ rainbow)$, if any two adjacent edges receive different colors. The $anti-Ramsey\ number$ of a graph G in a complete graph K_n , denoted by $ar(K_n, G)$, is the maximum number of colors in an edge-coloring of K_n with no rainbow copy of G. Namely, $ar(K_n, G) + 1$ is the minimum number k of colors such that any k-edge-coloring of K_n contains a rainbow copy of G. In this paper, we let $pr(K_n, G)$ be the maximum number of colors in an edge-coloring of K_n with no properly colored copy of G. Namely, $pr(K_n, G) + 1$ is the minimum number k of colors such that any k-edge-coloring of K_n contains a properly colored copy of G.

Given a family \mathcal{F} of graphs, we call a graph G an \mathcal{F} -free graph, if G contains no graph in \mathcal{F} as a subgraph. The $Tur\acute{a}n$ number $ex(n,\mathcal{F})$ is the maximum number of edges in a graph G on n vertices which is \mathcal{F} -free. Such a graph G is called an extremal graph, and the set of extremal graphs is denoted by $EX(n,\mathcal{F})$. The celebrated result of Erdős-Stone-Simonovits Theorem [7, 5] states that for any \mathcal{F} we have

$$ex(n, \mathcal{F}) = (\frac{p-1}{2p} + o(1))n^2,$$
 (*)

where $p = \Psi(\mathcal{F}) = \min\{\chi(F) : F \in \mathcal{F}\} - 1$, is the subchromatic number.

The anti-Ramsey number was introduced by Erdős, Simonovits and Sós in [6]. There they showed that $ar(K_n, G) - ex(n, \mathcal{G}) = o(n^2)$, where $\mathcal{G} = \{G - e : e \in E(G)\}$ and by (*), they showed that $ar(K_n, G) = (\frac{d-1}{2d} + o(1))n^2$, where $d = \Psi(\mathcal{G})$. This determined $ar(K_n, G)$ asymptotically when $\Psi(\mathcal{G}) \geq 2$. In case $\Psi(\mathcal{G}) = 1$, the situation is more complex. Already the cases when G is a tree or a cycle are nontrival. For a path P_k on k vertices, Simonovits and Sós [20] proved $ar(K_n, P_{2t+3+\epsilon}) = tn - {t+1 \choose 2} + 1 + \epsilon$, for large n, where $\epsilon = 0$ or 1. Jiang [10] showed $ar(K_n, K_{1,p}) = \lfloor \frac{n(p-2)}{2} \rfloor + \lfloor \frac{n}{n-p+2} \rfloor$ or possibly this value plus one if certain conditions hold. For a general tree T of k edges, Jiang and West [11] proved $\frac{n}{2} \lfloor \frac{k-2}{2} \rfloor + O(1) \leq ar(K_n, T) \leq ex(n, T)$ for $n \geq 2k$ and conjectured that $ar(K_n,T) \leq \frac{k-2}{2}n + O(1)$. For cycles, Erdős, Simonovits and Sós [6] conjectured that for every fixed $k \geq 3$, $ar(K_n, C_k) = (\frac{k-2}{2} + \frac{1}{k-1})n + O(1)$, and proved that for k=3. Alon [1] proved this conjecture for k=4 and gave some upper bounds for $k\geq 5$. Finally, Montellano-Ballesteros and Neumann-Lara [17] completely proved this conjecture. For cliques, Erdős, Simonovits and Sós [6] showed $ar(K_n, K_{p+1}) = ex(n, K_p) + 1$ for $p \geq 3$ and sufficiently large n. Montellano-Ballesteros and Neumann-Lara [16] and independently Schiermeyer [18] showed that $ar(K_n, K_{p+1}) = ex(n, K_p) + 1$ holds for every $n \geq p \geq 3$. For complete bipartite graphs $K_{s,t}, s \leq t$, Axenovich and Jiang [2] showed that $ar(K_n, K_{2,t}) = ex(n, K_{2,t-1}) + O(n)$. Krop and York [12] showed that $ar(K_n, K_{s,t}) = ex(n, K_{s,t-1}) + O(n)$. Also, there are many other results about anti-Ramsey number. We mention the excellent survey by Fujita, Magnant, and Ozeki [8] for more conclusions on this topic.

The minimum number of colors guaranteeing the existence of properly colored subgraphs in an edge-colored complete graph was studied by Manoussakis, Spyratos, Tuza and Voigt in [14]. For cliques, they showed that

Theorem 1. ([14]) For $t \geq 3$, let $b = \lfloor \frac{t-1}{2} \rfloor$, we have $pr(K_n, K_t) = (\frac{b-1}{2b} + o(1))n^2$.

For paths and cycles, they showed [14] that $pr(K_n, P_n) = \binom{n-3}{2} + 1$ for large n and $pr(K_n, C_n) = \binom{n-1}{2} + 1$. Also, they gave a conjecture on cycles as follows.

Conjecture 1. ([14]) Let $n > l \ge 4$. Assume that K_n is colored with at least k colors, where

$$k = \begin{cases} \frac{1}{2}l(l+1) + n - l + 1, & \text{if } n \le \frac{10l^2 - 6l - 18}{6(l-3)}, \\ \frac{1}{3}ln - \frac{1}{18}l(l+3) + 2, & \text{if } n \ge \frac{10l^2 - 6l - 18}{6(l-3)}. \end{cases}$$

Then, K_n admits a properly colored cycle of length l+1.

In this paper, we generalize Theorem 1 to arbitrary graph G which shows that $pr(K_n, G)$ is related to the Turán number like the anti-Ramsey number:

Theorem 2. Let G be a graph and $\mathcal{G}' = \{G - M : M \text{ is a matching of } G\}$, then $pr(K_n, G) \geq ex(n, \mathcal{G}') + 1$ and $pr(K_n, G) = (\frac{d-1}{2d} + o(1))n^2$, where $d = \Psi(\mathcal{G}')$.

We will prove Theorem 2 in Section 2 by the method used in the proof of Theorem 1 in [14]. Theorem 2 determines $pr(K_n, G)$ asymptotically when $\Psi(\mathcal{G}') \geq 2$. As the anti-Ramsey number, the case $\Psi(\mathcal{G}') = 1$ is more complex.

In Section 3, we will determine $pr(K_n, P_l)$ for $l \geq 27$ and large n by proving the following theorem.

Theorem 3. Let P_l be a path with $l \geq 27$ and $l \equiv r_l \pmod{3}$, where $0 \leq r_l \leq 2$. For $n \geq 2l^3$, we have $pr(K_n, P_l) = (\lfloor \frac{l}{3} \rfloor - 1)n - (\lfloor \frac{l}{3} \rfloor) + 1 + r_l$.

Although we just prove Theorem 3 for $l \geq 27$, we are sure that it is also true for $l \leq 26$.

For cycles, we slightly improved the lower bound of Conjecture 1 (See Proposition 4.1). Also, We modify Conjecture 1 as follows.

Conjecture 2. Let C_k be a cycle on k vertices and $(k-1) \equiv r_{k-1} \pmod{3}$, where $0 \le r_{k-1} \le 2$, for $n \ge k$,

$$pr(K_n, C_k) = \max \left\{ \binom{k-1}{2} + n - k + 1, \lfloor \frac{k-1}{3} \rfloor n - \binom{\lfloor \frac{k-1}{3} \rfloor + 1}{2} + 1 + r_{k-1} \right\}.$$

It is easy to see that $pr(K_n, C_3) = ar(K_n, C_3) = n - 1$. Li, Broersma and Zhang [13], and later Xu, Magnant, and Zhang[21] showed that $pr(K_n, C_4) = n$ for $n \ge 4$. We consider C_5 and C_6 in section 4.

Theorem 4. (a) $pr(K_n, C_5) = n + 2$ for $n \ge 5$;

(b)
$$pr(K_n, C_6) = n + 5 \text{ for } n \ge 6.$$

In Section 5, we consider two graphs K_4^- and $K_{2,3}$, where K_4^- is the graph K_4 minus one edge of it.

Theorem 5. For $n \geq 3$, $pr(K_n, K_4^-) = \lfloor \frac{3(n-1)}{2} \rfloor$.

Theorem 6. For $n \geq 5$, $\frac{7}{4}n + O(1) \leq pr(K_n, K_{2,3}) \leq 2n - 1$.

Notations: Let G be a simple undirected graph. For $x \in V(G)$, we denote the neighborhood and the degree of x in G by $N_G(x)$ and $d_G(x)$, respectively. The maximum degree of G is denoted by $\Delta(G)$. The common neighborhood of $U \subset V(G)$ is the set of vertices in $V(G) \setminus U$ that are adjacent to every vertex in U. We will use G - x to denote the graph that arises from G by deleting the vertex $x \in V(G)$. For $\emptyset \neq X \subset V(G)$, G[X] is the subgraph of G induced by G[X] is the subgraph of G[X] induced by G[X] induced by G[X] induced by G[X] is the subgraph of G[X] induced by G[X] is the subgraph of G[X] induced by G[X]. Given a graph G[X] induced by G[X] is the subgraph of G[X] induced by G[X] induc

Given an edge-coloring c of G, we denote the color of an edge uv by c(uv). A color a is starred (at x) if all the edges with color a induce a star $K_{1,r}$ (centered at the vertex x). We let $d^c(v) = |\{a \in C(v)| a \text{ is starred at } v\}|$. For a subgraph H of G, we denote $C(H) = \{c(uv)| uv \in E(H)\}$. A representing subgraph in an edge-coloring of K_n is a spanning subgraph containing exactly one edge of each color.

2. Proof of Theorem 2

In this section, we will prove Theorem 2 by a similar argument used in the proof of Theorem 1 in [14].

Theorem 2. Let G be a graph and $\mathcal{G}' = \{G - M : M \text{ is a matching of } G\}$, then $pr(K_n, G) \geq ex(n, \mathcal{G}') + 1$ and $pr(K_n, G) = (\frac{d-1}{2d} + o(1))n^2$, where $d = \Psi(\mathcal{G}')$.

Proof. Let F be a graph in $EX(n, \mathcal{G}')$. We color the edges of K_n as follows. Take a subgraph F of K_n , and assign distinct colors to all of E(F) and a new color c_0 to all the remaining edges. Suppose there is a properly colored G, then $M = \{e \in E(G), e \text{ is colored with } c_0\}$ is a matching of G, and $G - M \subset F$. By the definition of G', we have $G - M \in G'$, and this is a contradiction with F being G'-free. Thus we have $P(K_n, G) \geq ex(n, G') + 1 = (\frac{d-1}{2d} + o(1))n^2$ by (*).

Let $G_0 = G - M_p$, where M_p is a p-matching of G and $\chi(G_0) = d + 1$. We prove that for every fixed $\varepsilon > 0$, and for n large enough with respect to $n_0 = |V(G)|$ and ε , there is a properly colored copy of G in any $(\frac{d-1}{2d} + \varepsilon)n^2$ -edge coloring of K_n . In a representing subgraph of K_n with $(\frac{d-1}{2d} + \varepsilon)n^2$ edges, for an arbitrarily fixed s, and for n sufficiently large, by (*), there exists a complete (d+1)-partite subgraph $K_{s,s,\dots,s}$ with s vertices in each class. We take $s = 2^{n_0 + d + 1}$.

Denote by V the vertex set of $K_{s,s,\dots,s}$ and by V_1,V_2,\dots,V_{d+1} its vertex classes. We apply the procedure that follows.

For each $i=1,2,\cdots,d+1$ do sequentially the following:

- (1) Select arbitrarily $2^{n_0+d+1-i}$ pairwise disjoint pairs $\{u_{ij}, v_{ij}\}$ in $V_i, j = 1, 2 \cdots, 2^{n_0+d+1-i}$.
- (2) For $j = 1, 2, \dots, 2^{n_0 + d + 1 i}$, delete from $K_{s,s,\dots,s}$ the (at most one) vertex $z \in V \setminus V_i$ for which either $c(zu_{ij}) = c(u_{ij}v_{ij})$ or $c(zv_{ij}) = c(u_{ij}v_{ij})$, and if z has already been selected in a previous pair $\{u_{i'j'}, v_{i'j'}\}$, for some i' < i, then also delete the other member of its pair.

Claim 1. It is possible to carry out the above procedure and that at the end of the execution, in each V_i , at least 2^{n_0} pairs remains undeleted.

The proof of Claim 1. In the beginning, V_i contains 2^{n_0+d+1} vertices, $i=1,2,\cdots,d+1$. In the first iteration, i=1, we can carry out (1) and (2) easily. Suppose we have carried out up to the (i-1)-st iteration. Before executing the i-th iteration observe that at most $\sum_{1\leq j\leq i-1} 2^{n_0+d+1-j} = 2^{n_0+d+1} - 2^{n_0+d+2-i}$ vertices have been deleted from V_i . Thus, V_i contains at least $2^{n_0+d+2-i}$ vertices and it is enough to execute instruction (1) in the i-th iteration.

On the other hand, for any $i = 1, 2, \dots, d$, from the (i+1)-st iteration up to the end, due to instructions of type (2), at most $\sum_{i+1 \leq j \leq d+1} 2^{n_0+d+1-j} = 2^{n_0+d+1-i} - 2^{n_0}$ pairs in V_i have been delete and thus at least 2^{n_0} pairs in V_i remains undeleted. Note also that V_{d+1} contains 2^{n_0} pairs of vertices and there is no deletion of pair in V_{d+1} .

For $1 \leq i \leq d+1$, let $\{x_{ij}y_{ij}: 1 \leq j \leq 2^{n_0}\}$ be the 2^{n_0} pairs in V_i which remains undeleted and $V_i' = \{x_{ij}, y_{ij}: 1 \leq j \leq 2^{n_0}\}$. Let

$$H = K_{s,s,\dots,s}[V_1' \cup \dots \cup V_{d+1}'] \bigcup (\bigcup_{i=1}^{d+1} \{x_{ij}y_{ij} : 1 \le j \le 2^{n_0}\}).$$

H is properly colored, by Claim 1. Since $G_0 = G - M_p$ and $\chi(G_0) = d + 1$, we have $H \supset G$. Thus $pr(K_n, G) \leq (\frac{d-1}{2d} + o(1))n^2$.

3. Paths

In this section, we consider the minimum number of colors guaranteeing the existence of properly edge-colored paths in an edge-colored complete graph, and prove Theorem 3. Before doing so, we will determine $pr(K_n, P_l)$ for small l.

Proposition 3.1 (a) $pr(K_n, P_3) = 1$, for $n \ge 3$.

- (b) $pr(K_n, P_4) = 2$, for $n \ge 4$.
- (c) $pr(K_n, P_5) = 3$, for $n \ge 5$.
- (d) $pr(K_n, P_6) = n$, for $n \ge 6$.

Proof. (b) Choose a vertex v of K_n , color all edges incident to v with color c_1 and color all the remaining edges with color c_2 . We use two colors and there is no properly colored P_4 . Hence $pr(K_n, P_4) \geq 2$.

For $n \geq 5$, we have $pr(K_n, P_4) \leq ar(K_n, P_4) = 2$ (see [3]). Consider a 3-edge-coloring of K_4 . Let $V(K_4) = \{u, v, x, y\}$. Then there is at least one edge in $E(\{u, v\}, \{x, y\})$,

say ux such that $c(ux) \neq c(uv)$ and $c(ux) \neq c(xy)$. Thus vuxy is a properly colored P_4 . Hence $pr(K_n, P_4) \leq 2$.

(c) Choose $u, v \in V(K_n)$, assign c_1 to all edges incident with u, c_2 to all edges incident with v (except the edge uv) and c_3 to all the remaining edges. We use three colors and there is no properly colored P_5 . Hence $pr(K_n, P_5) \geq 3$.

Consider a 4-edge-coloring of K_n , $n \ge 5$, there is always a rainbow $P_4 = u_1 u_2 u_3 u_4$ since $ar(K_n, P_4) = 2$ when $n \ge 5$. Since $|C(P_4)| = |E(P_4)| = 3$, there is a color $c_0 \notin C(P_4)$.

Suppose there is no properly colored P_5 in the 4-edge-coloring of K_n . Then for all $u \in V(K_n) \setminus V(P_4)$, it must be $c(uu_1) = c(u_1u_2)$, $c(uu_4) = c(u_3u_4)$, $c(uu_2) \in \{c(u_1u_2), c(u_2u_3)\}$ and $c(uu_3) \in \{c(u_2u_3), c(u_3u_4)\}$. If $c(u_1u_4) = c_0$, then $uu_1u_4u_3u_2$ is a properly colored P_5 , a contradiction. If $c(u_1u_3) = c_0$ or $c(u_2u_4) = c_0$, say $c(u_1u_3) = c_0$, then $u_4uu_1u_3u_2$ is a properly colored P_5 , a contradiction. So there are two vertices $x, y \in V(K_n) - V(P_4)$ such that $c(xy) = c_0$. In this case, we have $u_4yxu_2u_1$ or $u_4yxu_2u_3$ is a properly colored P_5 , a contradiction. Hence $pr(K_n, P_5) \leq 3$.

(d) Choose a vertex v of K_n , assign distinct colors to all the edges incident with vertex v and a new color to all the remaining edges. We use n colors and there is no properly colored P_6 . Hence $pr(K_n, P_6) \geq n$.

Consider a (n+1)-edge-coloring of K_n . Then there is always a rainbow $P_5 = u_1u_2u_3u_4u_5$ since $ar(K_n, P_5) = n$ (see [3]).

Suppose there is no properly colored P_6 in the (n + 1)-edge-coloring of K_n . Then for all $u \in V(K_n) \setminus V(P_5)$, it must be $c(uu_1) = c(u_1u_2)$, $c(uu_5) = c(u_4u_5)$, $c(uu_2) \in \{c(u_1u_2), c(u_2u_3)\}$ and $c(uu_4) \in \{c(u_3u_4), c(u_4u_5)\}$. If there is a vertex $u \in V(K_n) \setminus V(P_5)$ such that $c(uu_2) = c(u_1u_2)$ and $c(uu_4) = c(u_4u_5)$, then at least one of $uu_2u_3u_4u_5u_1$ and $uu_4u_3u_2u_1u_5$ is a properly colored P_6 whatever $c(u_1u_5)$ is, a contradiction. Hence for all $u \in V(K_n) \setminus V(P_5)$, if $c(uu_2) = c(u_1u_2)$ (resp. $c(uu_4) = c(u_4u_5)$), then $c(uu_4) \neq c(u_4u_5)$ (resp. $c(uu_2) \neq c(u_1u_2)$).

If $c(u_1u_5) \notin C(P_5)$, take $u \in V(K_n)\backslash V(P_5)$, then $uu_1u_5u_4u_3u_2$ is a properly colored P_6 , a contradiction.

If $c(u_1u_4) \notin C(P_5)$ or $c(u_2u_5) \notin C(P_5)$, say $c(u_1u_4) \notin C(P_5)$, take $u \in V(K_n) \setminus V(P_5)$, then $u_5uu_1u_4u_3u_2$ is a properly colored P_6 , a contradiction.

Suppose $c(u_1u_3) \notin C(P_5)$ or $c(u_3u_5) \notin C(P_5)$, say $c(u_1u_3) \notin C(P_5)$. Take $u \in V(K_n) \setminus V(P_5)$. If $c(uu_2) = c(u_2u_3)$, then $uu_2u_1u_3u_4u_5$ is a properly colored P_6 , a contradiction. So $c(uu_2) = c(u_1u_2)$ which implies $c(uu_4) = c(u_3u_4)$. Hence, $u_1u_3u_2uu_4u_5$ is a properly colored P_6 , a contradiction.

If $c(u_2u_4) \notin C(P_5)$, take $u \in V(K_n) \setminus V(P_5)$, then at least one of $u_1uu_3u_2u_4u_5$ and $u_1u_2u_4u_3uu_5$ is a properly colored P_6 whatever $c(u_1u_3)$ is, a contradiction.

Suppose there is $u \in V(K_n) \setminus V(P_5)$ such that $c(uu_3) \notin C(P_5)$. Then at least one of $u_1u_2uu_3u_4u_5$ and $u_1u_2u_3uu_4u_5$ is a properly colored P_6 , a contradiction.

Since $n \geq 6$, there are two vertices $x, y \in V(K_n) - V(P_5)$ such that $c(xy) \notin C(P_5)$. Note that $c(xu_3) \in C(P_5)$. Then at least one of $u_1u_2u_3xyu_5$ and $u_1yxu_3u_4u_5$ is a properly colored P_6 whatever $c(xu_3)$ is, a contradiction. Hence $pr(K_n, P_6) \leq n$.

Here, we give the lower bound of $pr(K_n, P_l)$ by the following proposition.

Proposition 3.2 Let $n \ge l$ and P_l be a path with $l \equiv r_l \pmod{3}, 0 \le r_l \le 2$. We have

$$pr(K_n, P_l) \ge \max \left\{ \binom{l-3}{2} + 1, (\lfloor \frac{l}{3} \rfloor - 1)n - \binom{\lfloor \frac{l}{3} \rfloor}{2} + 1 + r_l \right\}.$$

Proof. We color the edges of K_n as follows. For the first lower bound, we choose a K_{l-3} and color it rainbow, and use one extra color for all the remaining edges. In such way, we use exactly $\binom{l-3}{2} + 1$ colors and do not obtain a properly colored P_l .

For the second lower bound, we partition K_n into two graphs $K_{\lfloor \frac{l}{3} \rfloor - 1} + \overline{K}_{n - \lfloor \frac{l}{3} \rfloor + 1}$ and $K_{n - \lfloor \frac{l}{3} \rfloor + 1}$. First we color $K_{\lfloor \frac{l}{3} \rfloor - 1} + \overline{K}_{n - \lfloor \frac{l}{3} \rfloor + 1}$ rainbow and color $K_{n - \lfloor \frac{l}{3} \rfloor + 1}$ with $(1 + r_l)$ new colors without producing a properly colored P_{3+r_l} (See the proof of Proposition 3.1). In such way, we use exactly $(\lfloor \frac{l}{3} \rfloor - 1)n - {\lfloor \frac{l}{3} \rfloor \choose 2} + 1 + r_l$ colors and do not obtain a properly colored P_l .

The proof of the following lemma is trivial. We will use it to prove Theorem 3.

Lemma 3.3 Let P_l be a path with l vertices, and $l \equiv r_l \pmod{3}, 0 \le r_l \le 2$. If an edge-coloring of K_n contains a rainbow copy of $K_{\lfloor \frac{l}{3} \rfloor - 1, 2 \lfloor \frac{l}{3} \rfloor + 3}$ but does not contain a properly colored P_l , then it is the following coloring: denote by Q the vertices of $K_n - K_{\lfloor \frac{l}{3} \rfloor - 1, 2 \lfloor \frac{l}{3} \rfloor + 3}$, by X the smaller class of $K_{\lfloor \frac{l}{3} \rfloor - 1, 2 \lfloor \frac{l}{3} \rfloor + 3}$ and by Y the other one. Then $|C(K_n[Y])| \le 1 + r_l$. Also, we have $|C(K_n[Y]) \cup C(E_{K_n}(Y,Q))| \le 1 + r_l$ and $|C(K_n[Y \cup Q])| \le 1 + r_l$. We get the most colors if the colors of all the edges between X and $Y \cup Q$ and all the edges in X are different, they differ from all the other edges and we use exactly $1 + r_l$ colors in $Y \cup Q$ such that there is no properly colored P_{3+r_l} in $Y \cup Q$. Then the number of colors is

$$(\lfloor \frac{l}{3} \rfloor - 1)n - {\lfloor \frac{l}{3} \rfloor \choose 2} + 1 + r_l.$$

Now, we will prove Theorem 3, and the idea comes from [20].

Theorem 3. Let P_l be a path with $l \geq 27$. Let $l \equiv r_l \pmod{3}, 0 \leq r_l \leq 2$. For $n \geq 2l^3$, we have $pr(K_n, P_l) = (\lfloor \frac{l}{3} \rfloor - 1)n - {\lfloor \frac{l}{3} \rfloor \choose 2} + 1 + r_l$.

Proof. We just need prove the upper bound. We shall use the following results of Erdős and Gallai (see [4]):

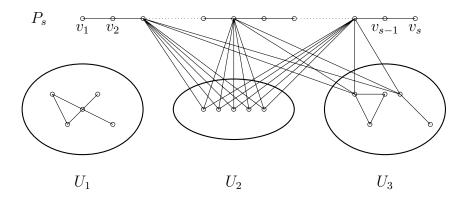
(a)
$$ex(n, P_r) \le \frac{r-2}{2}n;$$

(b)
$$ex(n, \{C_{r+1}, C_{r+2}, \dots\}) \le \frac{r(n-1)}{2}.$$

Consider an edge-coloring of K_n using $pr(K_n, P_l)$ colors without producing a properly colored P_l . Take a longest properly colored path $P_s = v_1 v_2 \cdots v_s$, where $s \leq l-1$. Denote by G the graph obtained by choosing one edge from each remaining color such that the number of edges joining P_s to the remaining n-s vertices as large as possible. We would partition $V(G)\backslash V(P_s)$ into three sets U_1, U_2 and U_3 as follows:

 U_1 is the vertex set of $V(K_n)\backslash V(P_s)$ not jointed to P_s at all: neither by edges nor by paths;

 U_2 is the set of isolated vertices of $V(K_n)\backslash V(P_s)$ jointed to P_s by edges; $U_3 = V(K_n)\backslash (V(P_s) \cup U_1 \cup U_2)$.



Claim 1. $G[U_1]$ contains no $P_{\lceil \frac{s+1}{2} \rceil}$, and $|E(G[U_1])| \leq \frac{l-3}{4} |U_1|$.

Proof of Claim 1 Suppose $P_{\lceil \frac{s+1}{2} \rceil} = u_1 u_2 \cdots u_{\lceil \frac{s+1}{2} \rceil}$ is a path in $G[U_1]$. By the constructing of G, $c(u_{\lceil \frac{s+1}{2} \rceil}v_{\lceil \frac{s}{2} \rceil}) \notin C(P_{\lceil \frac{s+1}{2} \rceil})$. Since $c(v_{\lceil \frac{s}{2} \rceil}-1v_{\lceil \frac{s}{2} \rceil}) \neq c(v_{\lceil \frac{s}{2} \rceil}v_{\lceil \frac{s}{2} \rceil+1})$, at most one of $c(v_{\lceil \frac{s}{2} \rceil}-1v_{\lceil \frac{s}{2} \rceil})$ and $c(v_{\lceil \frac{s}{2} \rceil}v_{\lceil \frac{s}{2} \rceil+1})$ is the same as $c(u_{\lceil \frac{s+1}{2} \rceil}v_{\lceil \frac{s}{2} \rceil})$. So at least one of $u_1u_2\cdots u_{\lceil \frac{s+1}{2} \rceil}v_{\lceil \frac{s}{2} \rceil}\cdots v_1$ and $u_1u_2\cdots u_{\lceil \frac{s+1}{2} \rceil}v_{\lceil \frac{s}{2} \rceil}\cdots v_s$ is a properly colored path, a contradiction to the maximality of P_s . Hence, $G[U_1]$ contains no $P_{\lceil \frac{s+1}{2} \rceil}$. By (a), we have

$$|E(G[U_1])| \le \frac{1}{2}(\lceil \frac{s+1}{2} \rceil - 2)|U_1| \le (\frac{1}{2}\lceil \frac{l}{2} \rceil - 1)|U_1| \le \frac{l-3}{4}|U_1|.$$

Claim 2. $E_G(U_2 \cup U_3, \{v_1, v_2, v_{s-1}, v_s\}) = \emptyset$.

Proof of Claim 2 It is obvious that $E_G(U_2 \cup U_3, \{v_1, v_s\}) = \emptyset$ by the maximality of P_s . Suppose that there is a vertex $u \in U_2 \cup U_3$ such that $uv_2 \in E(G)$ or $uv_{s-1} \in E(G)$, we say $uv_2 \in E(G)$, then at least one of $uv_1v_2 \cdots v_s$ and $v_1uv_2 \cdots v_s$ is a properly colored path of order s+1, a contradiction to the maximality of P_s .

Claim 3.
$$|E_G(U_2, P_s)| \leq (\lfloor \frac{l}{3} \rfloor - 1)|U_2|$$
.

Proof of Claim 3 For $v \in U_2$ and every three consecutive vertices $\{v_i, v_{i+1}, v_{i+2}\} \subset V(P_s)$, we claim that $|E_G(v, \{v_i, v_{i+1}, v_{i+2}\})| \leq 1$. Otherwise, at least two of v_i, v_i, v_{i+1}, v_i are edges of G. Then whatever $c(v_i)$ is, at least one of $v_1 \cdots v_i v_{i+1} v_{i+2} \cdots v_s$ and $v_1 \cdots v_i v_{i+1} v_{i+2} \cdots v_s$ is a properly colored path of order s+1, a contradiction to the

maximality of P_s . By Claim 2, we have $|E_G(v, P_s)| \leq \lceil \frac{s-4}{3} \rceil \leq \lceil \frac{l-5}{3} \rceil = \lfloor \frac{l}{3} \rfloor - 1$ and $|E_G(U_2, P_s)| \leq (\lfloor \frac{l}{3} \rfloor - 1)|U_2|$.

Claim 4. $|E_G(U_3, P_s)| + |E(G[U_3])| \leq \frac{l+2}{4}|U_3|$.

Proof of Claim 4 Take a component H of $G[U_3]$ and let r be the length of its longest cycle. If H contains no cycles, then write r=2. For each vertex $u \in V(H)$, we can find a path $P_r \subset H$ starting from it. Hence, $E_G(u, \{v_1, \cdots, v_r, v_{s-r+1}, \cdots, v_s\}) = \emptyset$. Otherwise, we can find a properly colored path of order at least s+1. For any four consecutive vertices $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$, there are no two independent edges between $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ and V(H) in G. Otherwise, suppose xv_i and $yv_j, j \in \{i+1, i+2, i+3\}$ are two independent edges between $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ and V(H) in G, where $x, y \in V(H)$. Take a path P_{xy} of H which connect x and y. Then whatever $c(xv_{i+1})$ is, at least one of $v_1 \cdots v_i x v_{i+1} \cdots v_s$ and $v_1 \cdots v_i v_{i+1} x P_{xy} y v_j \cdots v_s$ is a prorperly colored path of order at least s+1, a contradiction to the maximality of P_s . Hence, by (b), we have

$$|E_G(V(H), P_s)| + |E(H)| \le \lceil \frac{s - 2r}{4} \rceil |V(H)| + \frac{r|V(H)| - r}{2} \le \frac{s + 3}{4} |V(H)| \le \frac{l + 2}{4} |V(H)|.$$

By adding this up, we get

$$|E_G(U_3, P_s)| + |E(G[U_3])| \le \frac{l+2}{4}|U_3|.$$

By Claims 1, 3 and 4, the number of colors is

$$pr(K_n, P_l) = |C(K_n)| \le |C(P_s)| + |E(G)|$$

$$\le {s \choose 2} + |E(G[U_1])| + |E_G(U_2, P_s)| + |E_G(U_3, P_s)| + |E(G[U_3])|$$

$$\le {s \choose 2} + \frac{l-3}{4}|U_1| + (\lfloor \frac{l}{3} \rfloor - 1)|U_2| + \frac{l+2}{4}|U_3|.$$

Since $l \geq 27$, we have $\frac{l+2}{4} \leq \lfloor \frac{l}{3} \rfloor - 1 - \frac{1}{4}$. Let $U^* = \{u \in U_2 : d_G(u) = \lfloor \frac{l}{3} \rfloor - 1\}$. Then

$$(\lfloor \frac{l}{3} \rfloor - 1)n - \binom{\lfloor \frac{l}{3} \rfloor}{2} + 1 + r_l \leq pr(K_n, P_l) \leq \binom{s}{2} + (\lfloor \frac{l}{3} \rfloor - 1 - \frac{1}{4})(n - s - |U^*|) + (\lfloor \frac{l}{3} \rfloor - 1)|U^*|.$$

Hence for $n \geq 2l^3$, we have $|U^*| \geq l^3$ and we can get at least $2\lfloor \frac{l}{3} \rfloor + 3$ vertices $u_1, u_2, \dots, u_{2\lfloor \frac{l}{3} \rfloor + 3} \in U^*$ which have a common neighborhood of size $\lfloor \frac{l}{3} \rfloor - 1$ in G. By Lemma 3.3, the proof is completed.

4. Cycles

The lower bound of $pr(K_n, C_k)$ was given roughly by Manoussakis, Spyratos, Tuza and Voigt in [14]. Here we prove the lower bound precisely again.

Proposition 4.1 Let C_k be a cycle on k vertices and $k-1 \equiv r_{k-1} \pmod{3}$, where $0 \leq r_{k-1} \leq 2$. For $n \geq k$, we have

$$pr(K_n, C_k) \ge \max \left\{ \binom{k-1}{2} + n - k + 1, \lfloor \frac{k-1}{3} \rfloor n - \binom{\lfloor \frac{k-1}{3} \rfloor + 1}{2} + 1 + r_{k-1} \right\}.$$

Proof. We color the edges of K_n as follows. For the first lower bound, we choose a K_{k-1} and color it rainbow, and use one extra color for all the remaining edges. In such way, we use exactly $\binom{k-1}{2} + 1$ colors and do not obtain a properly colored C_k .

For the second lower bound, we partition K_n into two graphs $K_{\lfloor \frac{k-1}{3} \rfloor} + \overline{K}_{n-\lfloor \frac{k-1}{3} \rfloor}$ and $K_{n-\lfloor \frac{k-1}{3} \rfloor}$. First we color $K_{\lfloor \frac{k-1}{3} \rfloor} + \overline{K}_{n-\lfloor \frac{k-1}{3} \rfloor}$ rainbow and color $K_{n-\lfloor \frac{k-1}{3} \rfloor}$ with $(1+r_{k-1})$ new colors without producing a properly colored $P_{3+r_{k-1}}$ (See the proof of Proposition 3.1). In such way, we use exactly $(\lfloor \frac{k-1}{3} \rfloor)n - (\lfloor \frac{k-1}{3} \rfloor + 1 + r_{k-1}) + 1 + r_{k-1}$ colors and do not obtain a properly colored C_{k-1} .

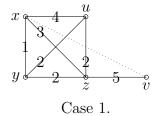
Conjecture 2. Let C_k be a cycle on k vertices and $(k-1) \equiv r_{k-1} \pmod{3}$, where $0 \leq r_{k-1} \leq 2$, for $n \geq k$,

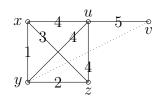
$$pr(K_n, C_k) = \max \left\{ \binom{k-1}{2} + n - k + 1, \lfloor \frac{k-1}{3} \rfloor n - \binom{\lfloor \frac{k-1}{3} \rfloor + 1}{2} + 1 + r_{k-1} \right\}.$$

Although Li et al. [13] and later Xu et al. [21] have got $pr(K_n, C_4) = n$ for $n \ge 4$, here we will use a different method to prove it again. We denote a cycle C_k with a pendant edge by C_k^+ . Gorgol [9] showed that $ar(K_n, C_k^+) = ar(K_n, C_k)$ for $n \ge k + 1 \ge 4$. We will use this result to prove the following proposition.

Proposition 4.2 ([13, 21]) For $n \ge 4$, $pr(K_n, C_4) = n$.

Proof. By Proposition 4.1, we have $pr(K_n, C_4) \ge n$ for $n \ge 4$. We will prove $pr(K_n, C_4) \le n$ by induction on n. The base case n = 4 is obvious. For $n \ge 5$, consider an (n+1)-coloring c of K_n . If there is a vertex v such that $d^c(v) \le 1$, then $|C(K_n-v)| \ge n+1-1 = (n-1)+1$ and there is a properly colored C_4 in K_n-v by induction. Thus we assume that $d^c(v) \ge 2$, for all $v \in V$. Since $ar(K_n, C_3^+) = ar(K_n, C_3) = n-1$, there is a rainbow C_3^+ . Let the triangle be xyzx and the pendant edge be xu. Let the edges xy, yz, xz, xu have colors 1, 2, 3, 4 respectively. We may assume $c(zu) \in \{2, 4\}$; otherwise xyzu is a properly colored C_4 .





Case 2.

Case 1. c(zu) = 2.

We may assume that c(yu) = 2; otherwise at least one of xyuzx and xuyzx is a properly colored C_4 . Since $d^c(z) \geq 2$, there is a vertex $v \in V(K_n) \setminus \{x, y, z, u\}$ such that c(zv) is starred at z and $c(zv) \neq 3$. Let c(zv) = 5. Note that $c(xv) \neq 5$. Then at least one of xyzvx and xuzvx is a properly colored C_4 .

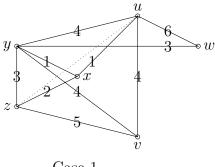
Case 2.
$$c(zu) = 4$$
.

We may assume that c(yu) = 4; otherwise at least one of xyuzx and xuyzx is a properly colored C_4 . Since $d^c(u) \geq 2$, there is a vertex $v \in V(K_n) \setminus \{x, y, z, u\}$ such that c(uv) is starred at u and $c(uv) \neq 4$. Let c(zv) = 5. Note that $c(yv) \neq 5$. Thus at least one of xuvyx and zuvyz is a properly colored C_4 .

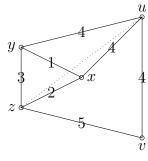
Now, we will use the same idea to prove Conjecture 1 for k=5. Let B be the bull graph, the unique graph on 5 vertices with degree sequence (1,1,2,3,3). Schiermeyer and Soták [18] showed that $ar(K_5, B) = 5$ and $ar(K_n, B) = n + 1$ for $n \ge 6$. We will use this result to prove the following proposition.

Proposition 4.3 For $n \ge 5$, $pr(K_n, C_5) = n + 2$.

Proof. By Proposition 4.1, we have $pr(K_n, C_5) \geq n+2$ for $n \geq 5$. We will prove $pr(K_n, C_5) \leq n+2$ by induction on n. The base case n=5 is easy since $pr(K_n, C_n) =$ $\binom{n-1}{2}+1$. For $n\geq 6$, consider an (n+3)-edge-coloring c of K_n . If there is a vertex v such that $d^c(v) \leq 1$, then $|C(K_n-v)| \geq n+3-1 = (n-1)+3$ and there is a properly colored C_4 by induction. Thus we assume that $d^c(v) \geq 2$, for all $v \in V$. Since $ar(K_n, B) = n + 1$ for $n \geq 6$, there is a rainbow B. Let $E(B) = \{xy, xz, yz, yu, zv\}$ and the edges xy, xz, yz, yu, zv have colors 1, 2, 3, 4, 5 respectively. We can assume that $c(uv) \in \{4,5\}$; otherwise xyuvzx is a properly colored C_5 . Assume, without loss of generality, that c(uv) = 4. We may assume that $c(xu) \in \{1, 4\}$; otherwise xyzvux is a properly colored C_5 .



Case 1.



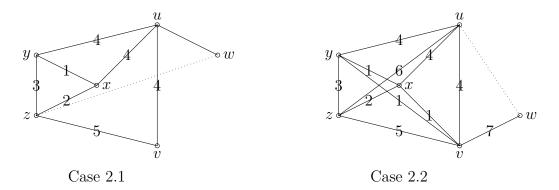
Case 2.

Case 1. c(xu) = 1.

We may assume that c(yv) = 4; otherwise at least one of yvzxuy and yvuxzy is a properly colored C_5 . Since $d^c(u) \geq 2$, there is a vertex $w \in V(K_n) \setminus \{x, y, z, u, v\}$ such that c(uw) is starred at u and $c(uw) \neq c(uz)$. Let c(uw) = 6. Note that $c(yw) \neq 6$. We may assume that c(yw) = 3; otherwise yzxuwy is a properly colored C_5 . Since $c(zu) \neq 6$, at least one of uzvywu and uzxywu is a properly colored C_5 .

Case 2. c(xu) = 4.

Since $d^c(u) \geq 2$, we consider the following two subcases.



Case 2.1 There is a vertex $w \in V(K_n) \setminus \{x, y, z, u, v\}$ such that c(uw) is starred at u and $c(uw) \neq 4$.

In this case, $c(uw) \neq c(zw)$. Then at least one of zwuxyz and zwuyxz is a properly colored C_5 .

Case 2.2 c(uz) is starred at u and $c(uz) \neq 4$.

Let c(uz) = 6. We can assume that c(xv) = 1; otherwise at least one of xvuzyx and xvzuyx is a properly colored C_5 . Also, we can assume that c(yv) = 1; otherwise at least one of yvuzxy and yvzuxy is a properly colored C_5 . Since $d^c(v) \ge 2$, there is a vertex $w \in V(K_n) \setminus \{x, y, z, u, v\}$ such that c(vw) is starred at v and $c(vw) \ne 5$. We assume that c(vw) = 7. Note that $c(uw) \ne 7$. Then at least one of wuyzvw and wuzxvw is a properly colored C_5 .

For C_6 , we consider more cases to prove it.

Proposition 4.4 For $n \geq 6$, $pr(K_n, C_6) = n + 5$.

Proof. By Proposition 4.1, we have $pr(K_n, C_6) \geq n+5$ for $n \geq 6$. We will prove $pr(K_n, C_6) \leq n+5$ by induction on n. For n=6, $pr(K_6, C_6) = \binom{6-1}{2} + 1 = 11$. For n=7, $pr(K_7, C_6) \leq ar(K_7, C_6) = 12$ (see [17]). For $n \geq 8$, consider an (n+6)-edge-coloring c of K_n . If there is a vertex v such that $d^c(v) \leq 1$, then $|C(K_n - v)| \geq n+6-1 = (n-1)+6$ and there is a properly colored C_6 by induction. Thus we assume that $d^c(v) \geq 2$ for all $v \in V(K_n)$. Let G be a subgraph of K_n such that $e \in E(G)$ if and only if the color c(e) appears only once in K_n . We have $|E(G)| \geq 2n - (n+6) = n-6 \geq 2$.

Case 1. $\Delta(G) \geq 2$.

In this case, G contains a path of order 3. Let $P_3 = xyz$ and $U = V(K_n) \setminus \{x, y, z\}$. For all $v \in U$ and all starred color c_v at v, we take an edge with color c_v to obtain a

subgraph H of K_n . Choose H such that $|E_H(\{x,y,z\},U)|$ as large as possible.

Case 1.1 $|E(H[U])| \ge 2$.

Let $u_1u_2, v_1v_2 \in E(H[U])$. If $u_1 \in \{v_1, v_2\}$ or $u_2 \in \{v_1, v_2\}$, say $u_2 = v_1$, then $c(xu_1) \neq c(u_1v_1)$ and $c(zv_2) \neq c(v_1v_2)$ by the choice of H. Thus $xyzv_2v_1u_1x$ is a properly colored C_6 . Now suppose u_1u_2 and v_1v_2 are two independent edges of H. Assume that $c(u_1u_2)$ and $c(v_1v_2)$ are starred at u_1, v_1 respectively. Thus $c(u_2v_2) \neq c(u_1u_2)$ and $c(u_2v_2) \neq c(v_1v_2)$. By the choice of H, we have $c(xu_1) \neq c(u_1u_2)$ and $c(yv_1) \neq c(v_1v_2)$. Thus, $xyv_1v_2u_2u_1x$ is a properly colored C_6 .

Case 1.2 |E(H[U])| = 1.

Assume $uv \in E(H[U])$ and c(uv) is starred at u. Then we have $c(xu) \neq c(uv)$. Also, $c(vz) \neq c(uv)$. Take a vertex $w \in U \setminus \{u, v\}$. Since $d^c(w) \geq 2$, we have $|E_H(w, \{x, y, z\})| \geq 2$. At least one of $\{x, z\}$, say x, such that c(wx) is starred at w and $c(wx) \neq c(wy)$. Also, we have $c(wx) \neq c(ux)$. Thus wxuvzyw is a properly colored C_6 .

Case 1.3 $E(H[U]) = \emptyset$.

For all $v \in U$, since $d^c(v) \geq 2$, we have $|E_H(v, \{x, y, z\})| \geq 2$. Notice that $|U| \geq n-3 \geq 5$. If there are three vertices in U, say $u_1, u_2, u_3 \in U$, such that they have a common neighborhood $\{x, z\}$ in H, then at least one of $\{u_1x, u_1z\}$, say u_1x , such that $c(u_1y) \neq c(u_1x)$. Also, at most one edge of $\{u_2x, u_2z, u_3x, u_3z\}$ has the same color as $c(u_2u_3)$. Thus, at least one of $\{xu_1yzu_3u_2x, xu_1yzu_2u_3x\}$ is a properly colored C_6 .

Now we assume that there are at least two vertices in U, say u_1, u_2 , such that they have a common neighborhood $\{x,y\}$ or $\{y,z\}$, say $\{x,y\}$ in H. If there is a vertex $u_3 \in U \setminus \{u_1,u_2\}$ such that $u_3y,u_3z \in E(H)$, we have $c(zx) \notin \{c(xu_1),c(xu_2),c(zu_3)\}$ and at most one edge of $\{u_1x,u_1y,u_2x,u_2y\}$ has the same color as $c(u_1u_2)$. Thus, at least one of $xu_1u_2yu_3zx$ and $xu_2u_1yu_3zx$ is a properly colored C_6 . If there is a vertex $u_3 \in U \setminus \{u_1,u_2\}$ such that $u_3x,u_3z \in E(H)$, at least one of $xu_1u_2yzu_3x$ and $xu_2u_1yzu_3x$ is a properly colored C_6 . We may assume that U has a common neighborhood $\{x,y\}$ in H. Take four distinct vertices $u_1,u_2,u_3,u_4 \in U$. At most one edge of $\{u_1x,u_1y,u_2x,u_2y\}$ has the same color as $c(u_1u_2)$ and at most one edge of $\{u_3x,u_3y,u_4x,u_4y\}$ has the same color as $c(u_3u_4)$. Thus there is a properly colored C_6 in $\{u_1u_2,u_3u_4,xu_i,yu_i:1\leq i\leq 4\}$.

Case 2. $\Delta(G) = 1$.

Note that if G has three independent edges, then we can find a properly colored C_6 . Recall that $|E(G)| \ge n - 6 \ge 2$. We have n = 8 and |E(G)| = 2. Let $E(G) = \{xy, zw\}$ and $U = V(K_8) \setminus \{x, y, z, w\} = \{u_1, u_2, u_3, u_4\}$.

Case 2.1 There is an edge u_iu_j such that $c(u_iu_j)$ is starred at u_i , say $c(u_1u_2)$ is starred at u_1 .

If there is one vertex in $\{x, y, z, w\}$, say x, such that $c(u_1x) \neq c(u_1u_2)$, then $u_1xyzwu_2u_1$ is a properly colored C_6 . We assume that $c(u_1x) = c(u_1y) = c(u_1z) = c(u_1w) = c(u_1u_2)$. Since $d^c(u_1) \geq 2$, we can assume that $c(u_1u_3)$ is starred at u_1 and $c(u_1u_3) \neq c(u_1u_2)$. Thus

 $u_1xyzwu_3u_1$ is a properly colored C_6 .

Case 2.2 For all edge $u_i u_j$, $c(u_i u_j)$ is not starred at u_i or u_j .

Since $d^c(u_1) \geq 2$ and $d^c(u_2) \geq 2$, we can find two distinct vertices $v_1, v_2 \in \{x, y, z, w\}$ such that $c(u_1v_1)$ is starred at u_1 and $c(u_2v_2)$ is starred at u_2 . If $v_1 = x$ and $v_2 = y$, then $u_1xzwyu_2u_1$ is a properly colored C_6 . If $v_1 = x$ and $v_2 = z$, then $u_1xywzu_2u_1$ is a properly colored C_6 .

5. K_4^- and $K_{2,3}$

In this section, we will prove Theorems 5 and 6. First, we will determine the exact value of $pr(K_n, K_4^-)$.

Theorem 5. For
$$n \ge 4$$
, $pr(K_n, K_4^-) = \lfloor \frac{3(n-1)}{2} \rfloor$.

Proof. The lower bound: Consider an edge-coloring of K_n as follows. Take a triangle $C_3 = xyz$ of K_n and a maximum matching $M = \{x_1y_1, x_2y_2, \cdots, x_{\lfloor \frac{n-3}{2} \rfloor}y_{\lfloor \frac{n-3}{2} \rfloor}\}$ of $K_n - \{x, y, z\}$. There is one vertex w in $V(K_n) \setminus (V(M) \cup \{x, y, z\})$ when n is even. For $1 \le i \le \lfloor \frac{n-3}{2} \rfloor$, color all the edges of $\{ux_i : u \in \{x, y, z, x_j, y_j, 1 \le j \le i-1\}\}$ with color c_{1i} and all the edges of $\{uy_i : u \in \{x, y, z, x_j, y_j, 1 \le j \le i-1\}\}$ with color c_{2i} . If n is even, color all edges of $\{uw : u \in V(K_n - w)\}$ with a new color. Finally, assign distinct new colors to all edges of $C_3 \cup M$. In such a coloring, there is no properly colored K_4^- , and the number of colors is $\lfloor \frac{3(n-1)}{2} \rfloor$.

The upper bound: We will prove that for any $\lfloor \frac{3n-1}{2} \rfloor$ edge-coloring of K_n , there is a properly colored K_4^- by induction on n. The base case n=4 is trivial. Consider a $\lfloor \frac{3n-1}{2} \rfloor$ edge-coloring of K_n . If there is a vertex v such that $d^c(v) \leq 1$, then $|C(K_n-v)| \geq \lfloor \frac{3n-1}{2} \rfloor - 1 \geq \lfloor \frac{3(n-1)-1}{2} \rfloor$, and there is a properly colored K_4^- in $K_n - v$ by induction. We may assume that $d^c(v) \geq 2$ for all $v \in V(K_n)$. Let G be a subgraph of K_n , such that $e \in E(G)$ if and only if the color c(e) appears only once in K_n . Since $d^c(v) \geq 2$ for all $v \in V(K_n)$, we have $|E(G)| \geq 2n - \lfloor \frac{3n-1}{2} \rfloor = \lceil \frac{n+1}{2} \rceil$ which implies there is a path $P_3 = xyz$ in G. By the construction of G, if $e = uv \in E(G)$, the c(e) is starred at u and v. We consider the following two cases.

Case 1. $xz \notin E(G)$.

In this case, c(xz) is not starred at x or z, say x. Since $d^c(x) \ge 2$, there is a vertex $w \notin \{x, y, z\}$ such that c(xw) is starred at x. Then $c(xz), c(yw) \notin \{c(xy), c(yz), c(xw)\}$ and $\{xy, yz, xz, xw, yw\}$ is a properly colored K_4^- .

Case 2. $xz \in E(G)$.

In this case, we can assume c(ux) = c(uy) = c(uz) for all $u \in V(K_n) \setminus \{x, y, z\}$; otherwise we easily have a properly colored copy of K_4^- in $K_n[x, y, z, u]$. Thus we have

$$|C(K_n - \{x, y\})| \ge \left\lfloor \frac{3n-1}{2} \right\rfloor - 3 \ge \left\lfloor \frac{3(n-2)-1}{2} \right\rfloor$$

and there is a properly colored K_4^- in $K_n - \{x, y\}$ by the induction hypothesis.

Now we prove the lower bound and upper bound of $pr(K_n, K_{2,3})$. We conjecture that the exact value is closer to the lower bound.

Theorem 6. For $n \geq 5$, $\frac{7}{4}n + O(1) \leq pr(K_n, K_{2,3}) \leq 2n - 1$.

Proof. Lower bound: Let n = 4k + r, where $1 \le r \le 4$. Set $V(K_n) = V_1 \cup \ldots \cup V_k \cup V_{k+1}$ such that $V_i \cap V_j = \emptyset$ for $i \ne j$, $|V_i| = 4$ for $1 \le i \le k$ and $|V_{k+1}| = r$. We color the edges with endpoints in the same set with $6k + \binom{r}{2}$ distinct colors and color the remaining edges with k addition colors c_1, c_2, \ldots, c_k such that all edges with endpoints in V_i and V_j are colored with $c_{\min\{i,j\}}$, where $i \ne j$. The total colors are $\frac{7}{4}n + O(1)$ and there is no properly colored $K_{2,3}$.

The upper bound: We will prove that for any 2n edge-coloring of K_n , there is a properly colored $K_{2,3}$ by induction on n. The base case n=5 are trivial. Consider a 2n edge-coloring of K_n . If there is a vertex v such that $d^c(v) \leq 2$, then $|C(K_n - v)| \geq 2n - 2$ and there is a properly colored $K_{2,3}$ in $K_n - v$ by induction. We may assume that $d^c(v) \geq 3$ for all $v \in V(K_n)$. Let G be a subgraph of K_n where $e \in E(G)$ if and only if the color c(e) appears only once in K_n . Since $d^c(v) \geq 3$ for all $v \in V(K_n)$, we have $|E(G)| \geq 3n - 2n = n$. Note that for $n \geq 4$, $ex(n, P_4) \leq n$ where equality holds for the graph of disjoint copies of C_3 (see [4]). So we will consider the following two cases.

Case 1. G contains a $P_4 = xyzw$.

If $G[V(P_4)] \cong K_4$, then we can assume c(ux) = c(uy) = c(uz) = c(uw) for all $u \in V(K_n) \setminus \{x, y, z, w\}$; otherwise we easily have a properly colored copy of $K_{2,3}$. Therefore

$$|C(K_n - \{x, y, z\})| \ge 2n - 6 = 2(n - 3)$$

and there is a properly colored copy of $K_{2,3}$ in $K_n - \{x, y, z\}$ by the induction hypothesis.

Now we consider the case $G[V(P_4)] \not\cong K_4$. Since $d^c(x) \geq 3$ and $d^c(w) \geq 3$, there is a vertex $u \in V(K_n) \setminus \{x, y, z, w\}$ such that c(xu) or c(wu), say c(xu) is starred at x and $c(xu) \notin \{c(xy), c(xw)\}$. Therefore, $\{xy, yz, zw, xw, xu, zu\}$ is a properly colored $K_{2,3}$.

Case 2. G is the graph of disjoint copies of C_3 .

Take a triangle $T_1 = xyzx$ of G. Since $d^c(x) \geq 3$, there is a vertex $u \in V(K_n) \setminus \{x, y, z\}$ such that c(xu) is starred at x and $c(xu) \notin \{c(xy), c(xz)\}$. Suppose u belong to the triangle $T_2 = uvwu$ of G. Then $\{xy, xu, zy, zu, vy, vu\}$ is a properly colored $K_{2,3}$.

6. Open problems

Although the topic of this paper has been proposed by Manoussakis, Spyratos, Tuza and Voigt [14] about twenty years ago, there are a few results about it. In this paper,

we get the relationship of $pr(K_n, G)$ and $ex(n, \mathcal{G}')$ by Theorem 1. Many problems on $pr(K_n, G)$ are worth being studied and we state four problems here.

Problem 1. Recall that Conjecture 2 is still open in the range $k \geq 7$.

Problem 2. By Theorem 2, we have $pr(K_n, K_4) \ge ex(n, C_4) + 1$. Also, it is easy to see that $pr(K_n, K_4) = O(n^{\frac{3}{2}})$. It is natural to ask for the exact upper bound, i.e.

$$pr(K_n, K_4) \le (1 + o(1))ex(n, C_4)$$
?

Problem 3. By Theorem 6, we have $\frac{7}{4}n + O(1) \leq pr(K_n, K_{2,3}) \leq 2n - 1$. It is natural to ask for the exact value of $pr(K_n, K_{2,3})$. Furthermore, it is interesting to determine $pr(K_n, K_{s,t})$.

Problem 4. Let T_k be a tree of k edges. The famous conjecture of Erdős and Sós says that

$$ex(n, T_k) \le \frac{(k-1)n}{2}.$$

Jiang and West [11] conjectured that

$$ar(K_n, T_k) \le \frac{(k-2)n}{2} + O(1).$$

Notice that $pr(K_n, T_k) \leq ar(K_n, T_k)$ and $pr(K_n, K_{1,k}) = ar(K_n, K_{1,k}) = \frac{(k-2)n}{2} + O(1)$. It is natural to conjecture that

$$pr(K_n, T_k) \le \frac{(k-2)n}{2} + O(1).$$

Furthermore, it is interesting to investigate the upper bound of $pr(K_n, T_k)$ when $T_k \neq K_{1,k}$.

References

- [1] N. Alon. On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey theorems. J. Graph Theory, 7(1):91-94, 1983.
- [2] M. Axenovich and T. Jiang. Anti-Ramsey numbers for small complete bipartite graphs. Ars Combinatoria, 73: 311-318, 2004.
- [3] A. Bialostocki, S. Gilboa, Y. Roditty. Anti-Ramsey numbers of small graphs. Ars Combinatoria 123: 41-53, 2015.
- [4] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. Acta Math. Acad. Sci. Hungar 10: 337-356, 1959.
- [5] P. Erdős and M. Simonovits. A limit theorem in graph theory. Studia Sci Math Hungar 1: 51-57, 1966.

- [6] P. Erdős, M. Simonovits and V. T. Sós. Anti-Ramsey theorems. In Infinite and finite sets, Vol. II, pages 633-643. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.
- [7] P. Erdős and A. H. Stone. On the structure of linear graphs. Bull Amer Math Soc 52: 1087-1091, 1946.
- [8] S. Fujita, C. Magnant and K. Ozeki. Rainbow generalizations of Ramsey theory: A survey. Graphs Combin. 26: 1-30, 2010.
- [9] I. Gorgol. Rainbow numbers for cycles with pendant edges. Graphs Combin. 24(4): 327-331, 2008.
- [10] T. Jiang. Edge-coloring with no large polychromatic stars. Graphs Combin.18(2): 303-308, 2002.
- [11] T. Jiang and D. B. West. Edge colorings of complete graphs that avoid polychromatic trees, Discrete Math. 274: 137-147, 2004.
- [12] E. Krop and M. York. On anti-Ramsey numbers for complete bipartite graphs and the Turán function. manuscript.
- [13] R. Li, H. Broersma, and S. Zhang. Properly edge-colored theta graphs in edge-colored complete graphs, Graphs Combin., 35: 261-286, 2019.
- [14] Y. Manoussakis, M. Spyratos, Zs. Tuza and M. Voigt. Minimal colorings for properly colored subgraphs. Graphs Combin., 12:345-360, 1996.
- [15] J. J. Montellano-Ballesteros. An anti-Ramsey theorem on diamonds. Graphs Combin., 26(2): 283-291, 2010.
- [16] J. J. Montellano-Ballesteros and V. Neumann-Lara. An anti-Ramsey theorem. Combinatorica 22: 445-449, 2002.
- [17] J. J. Montellano-Ballesteros and V. Neumann-Lara. An anti-Ramsey theorem on cycles. Graphs Combin., 21(3):343-354, 2005.
- [18] I. Schiermeyer. Rainbow numbers for matchings and complete graphs. Discrete Math 286(1-2): 157-162, 2004.
- [19] I. Schiermeyer and R. Soták. Rainbow numbers for graphs containing small cycles. Graphs Combin. 31(6):1985-1991, 2015.
- [20] M. Simonovits and V. T. Sós. On restricted colorings of K_n . Combinatorica 4(1), 101-110, 1984.
- [21] C. Xu, C. Magnant, and S. Zhang. Properly colored C_4 's in edge-colored graphs. arXiv:1905.10584.