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TUTORIAL 7.5 — Series

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(a)

$$\sum_{n=1}^{\infty} \pi^{1-3n}$$
$$= \sum_{n=1}^{\infty} \frac{\pi}{\pi^{3n}}$$
$$= \sum_{n=1}^{\infty} \pi\left(\frac{1}{\pi^3}\right)^n$$

The above is a geometric series with common ratio $\frac{1}{\pi^3}$. Since $\left|\frac{1}{\pi^3}\right| \leq 1$, the series converges.

The first couple of terms are: $\{\frac{1}{\pi^2}, \frac{1}{\pi^5}, \frac{1}{\pi^8}\}$.

Re-indexing: $\sum_{n=0}^{\infty} \frac{1}{\pi^2}\left(\frac{1}{\pi^3}\right)^n$, where $a = \frac{1}{\pi^2}$ and $r = \frac{1}{\pi^3}$

$\frac{a}{1-r} = \frac{1/\pi^2}{1-1/\pi^3}$ converges to $\frac{\pi}{\pi^3-1}$, approximately 0.1047.

$\implies \sum_{n=1}^{\infty} \pi^{1-3n}$ converges to approximately 0.1047.

(b)

$$\sum_{n=1}^{\infty} \frac{3+4^n}{5^n}$$

Applying ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{3+4^{n+1}}{5^{n+1}}\right)\left(\frac{5^n}{3+4^n}\right)\right| = \lim_{n\to\infty}\left|\frac{3+4^{n+1}}{5(3+4^n)}\right| = \lim_{n\to\infty}\left|\frac{3/4^n+4}{5(3/4^n+1)}\right| = \frac{4}{5}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq 1$$

Hence, the series is absolutely convergent.

2 More proofs about p and q

Type $Ty := AA||BB$
$p, q : Ty \to \mathbb{B}$

$\vdash (\forall x : Ty.p(x) \to \neg q(x)) \to$
$\quad (\exists x : Ty.p(x)) \to$
$\quad p(AA) \to$
$\quad \exists y : Ty.q(y)$

By impl-elims on goal:
1) $\forall x : Ty.p(x) \to \neg q(x)$
2) $\exists x : Ty.\neg p(x)$
3) $p(\text{AA})$

$\vdash \exists y : Ty.q(y)$

Ty	p	q
AA	T	F
BB	F	T/F

The implication allows $q(BB)$ to be false while allowing all assumptions to be true. Hence, by counterexample the goal is false.

3 Proofs are contrary to fun

Type Ty
$foo : Ty$
$p, q : Ty \to \mathbb{B}$

$\vdash (\forall w : Ty.p(w) \implies \forall x : Ty.\neg q(x)) \implies$
$(\exists y : Ty.q(y)) \implies$
$p(foo) \implies$
$(\forall z : Ty.q(z))$

By impl-elims on goal:
1) $\forall w : Ty\ (p(w) \implies \forall x : Ty.\neg q(x))$
2) $\exists y : Ty.q(y)$
3) $p(foo)$

$\vdash \forall z : Ty.q(z)$

By forall-elim on 1 using $w = foo$
4) $p(foo) \implies \forall x : Ty.\neg q(x)$
By impl-elim on 4 using 3
5) $\forall x : Ty.\neg q(x)$
By exists-elim on 2:
6) $y : Ty$
7) $q(y)$
By forall-elim on 5 using $x = y$
8) $\neg q(y)$

Assumption 5 contradicts 8, therefore the goal is false.

4 Simple proofs can be sick

Type person, location, liquid
visited: (person, location) $\to \mathbb{B}$
sick: person $\to \mathbb{B}$
ooj: liquid (old orange juice)
beach: location
drank: (person, liquid) $\to \mathbb{B}$
Marat: person

1. Everyone who drank old orange juice got sick.
   $\forall p : \text{person} \ . \ \text{drank}(p, ooj) \implies \text{sick}(p)$

2. Everyone who drank old orange juice went to the beach.
   $\forall p$: person . drank$(p, ooj) \implies$ visited(p, beach)

3. Marat did not get sick.
   $\neg$ sick(Marat)

4. To prove that the three statements imply that Marat did not go to the beach:

p	sick(p)	visited(p, beach)	drank(p, ooj)
Marat	F	T	F

   The above environment illustrates a case where all assumptions hold, yet the goal is not satisfied. Therefore, by counterexample, the statements are insufficient to prove the goal.

Submitted by Yashashwin Karthikeyan on March 28, 2024.