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ASSIGNMENT 3 — **Proofs by Elimination**

1 Proof with quantifiers and elimination

Type Ty
$p, q : Ty \to \mathbb{B}$

$\vdash (\forall x : Ty.p(x) \to q(x)) \to (\forall y : Ty.p(y) \to \exists z : Ty.q(z))$

By impl-elim on goal:
  1) $\forall x : Ty.p(x) \to q(x)$

$\vdash \forall y : Ty \ (p(y) \to \exists z : Ty.q(z))$

By forall-elim on goal:
  2) $y : Ty$

$\vdash p(y) \to \exists z : Ty.q(z)$

By impl-elim on goal:
  3) $p(y)$

$\vdash \exists z : Ty.q(z)$

By forall-elim on assumption 1 using $x = y$ from 2.
  4) $p(y) \to q(y)$
By impl-elim on assumption 4 using 3
  5) $q(y)$
By exists-elim on goal using $z = y$
$\vdash q(y)$

QED by assumption 5.

2 More proofs about p and q

Type $Ty := AA || BB$
$p, q : Ty \to \mathbb{B}$

$\vdash (\forall x : Ty.p(x) \to \neg q(x)) \to$
$\quad (\exists x : Ty.p(x)) \to$
$\quad p(AA) \to$
$\quad \exists y : Ty.q(y)$

By impl-elims on goal:
$\quad$ 1) $\forall x : Ty.p(x) \to \neg q(x)$
$\quad$ 2) $\exists x : Ty.\neg p(x)$
$\quad$ 3) $p(\text{AA})$

$\vdash \exists y : Ty.q(y)$

Ty	p	q
AA	T	F
BB	F	T/F

The implication allows $q(BB)$ to be false while allowing all assumptions to be true. Hence, by counterexample the goal is false.

3 Proofs are contrary to fun

Type Ty
$foo : Ty$
$p, q : Ty \to \mathbb{B}$

$\vdash (\forall w : Ty.p(w) \implies \forall x : Ty.\neg q(x)) \implies$
  $(\exists y : Ty.q(y)) \implies$
  $p(foo) \implies$
  $(\forall z : Ty.q(z))$

By impl-elims on goal:
  1) $\forall w : Ty \ (p(w) \implies \forall x : Ty.\neg q(x))$
  2) $\exists y : Ty.q(y)$
  3) $p(foo)$

$\vdash \forall z : Ty.q(z)$

By forall-elim on 1 using $w = foo$
  4) $p(foo) \implies \forall x : Ty.\neg q(x)$
By impl-elim on 4 using 3
  5) $\forall x : Ty.\neg q(x)$
By exists-elim on 2:
  6) $y : Ty$
  7) $q(y)$
By forall-elim on 5 using $x = y$
  8) $\neg q(y)$

Assumption 5 contradicts 8, therefore the goal is false.

4 Simple proofs can be sick

Type person, location, liquid
visited: (person, location) $\to \mathbb{B}$
sick: person $\to \mathbb{B}$
ooj: liquid (old orange juice)
beach: location
drank: (person, liquid) $\to \mathbb{B}$
Marat: person

1. Everyone who drank old orange juice got sick.
   $\forall p : \text{person} \ . \ \text{drank}(p, ooj) \implies \text{sick}(p)$

2. Everyone who drank old orange juice went to the beach.
$\forall p : \text{person} \;.\; \text{drank}(p, ooj) \implies \text{visited}(p, \text{beach})$

3. Marat did not get sick.
$\neg \, \text{sick}(\text{Marat})$

4. To prove that the three statements imply that Marat did not go to the beach:

p	sick(p)	visited(p, beach)	drank(p, ooj)
Marat	F	T	F

The above environment illustrates a case where all assumptions hold, yet the goal is not satisfied. Therefore, by counterexample, the statements are insufficient to prove the goal.

Submitted by Yashashwin Karthikeyan, Roozbeh Ali on January 30, 2024.