Random Numbers
AI1110: Probability and Random Variables

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1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution: Download the C source code by executing the following commands

```
wget https://raw.githubusercontent.com/
    gadepall/AI1110/main/sim/codes/exrand.c
wget https://raw.githubusercontent.com/
    gadepall/AI1110/main/sim/codes/coeffs.h
```

Compile and run the C program by executing the following

```
gcc exrand.c
./a.out
```

1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: Download the following Python code that plots Fig. 1.2

```
wget https://raw.githubusercontent.com/
    YashRRamteke/Random−numbers/main/
    Code/cdf_plot.py
python3 cdf_plot.py
```

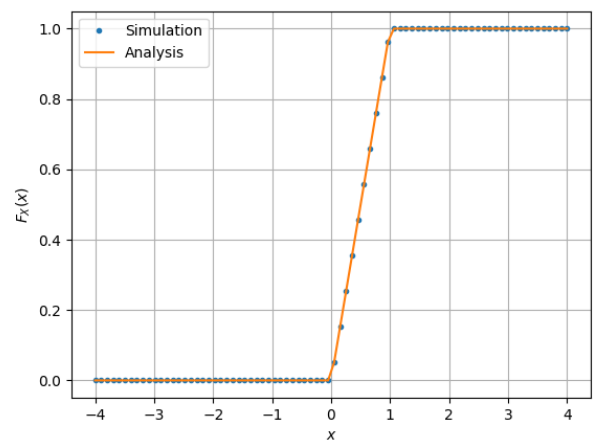

Fig. 1.2. The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: U is given by

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.2}$$

Therefore, we have:

$$F_U(x) = \int_0^x U(x)dx \tag{1.3}$$

Computing the integral, we get:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution: Add the following function to coeffs.h

```
double variance(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;
fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x*x;
}
double mn = mean(str);
fclose(fp);
temp = temp/(i-1);
return temp - mn*mn ;
}
```

Following the steps mentioned below gives the required result:

```
gcc exrand.c
./a.out
mean = 0.500031
variance = 0.083247
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution: Since

$$dF_U(x) = p_U(x)dx \tag{1.8}$$

we have:

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \tag{1.9}$$

Also,

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.10}$$

Therefore, from Equations 1.9 and 1.10, we have:

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x)dx \tag{1.11}$$

$$= \int_0^1 x^2 dx \tag{1.12}$$

$$= \frac{1}{3} \tag{1.13}$$

Similarly,

$$E[U] = \int_{-\infty}^{\infty} x p_U(x)dx \tag{1.14}$$

$$= \int_0^1 x dx \tag{1.15}$$

$$= \frac{1}{2} \tag{1.16}$$

Therefore, the mean is $\frac{1}{2}$, and the variance equals:

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.17}$$

$$= \frac{1}{12} \tag{1.18}$$

2. Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Add the following line to **exrand.c** and execute the code:

```
gaussian("gau.dat", 1000000);
gcc exrand.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

```
wget https://raw.githubusercontent.com/
    YashRRamteke/Random-numbers/main/
    Code/cdf_plot.py
```

and execute it with

```
python3 cdf_plot.py
```

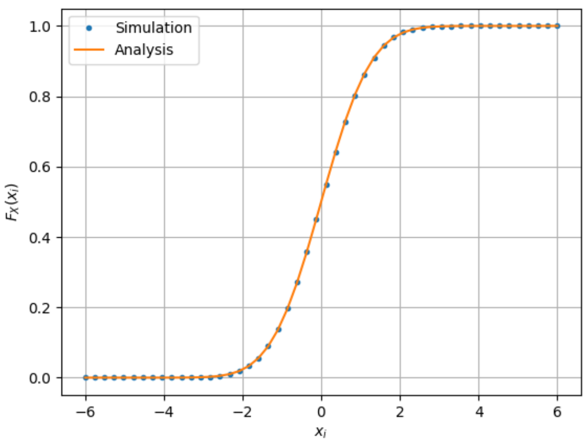

Fig. 2.2. The CDF of X

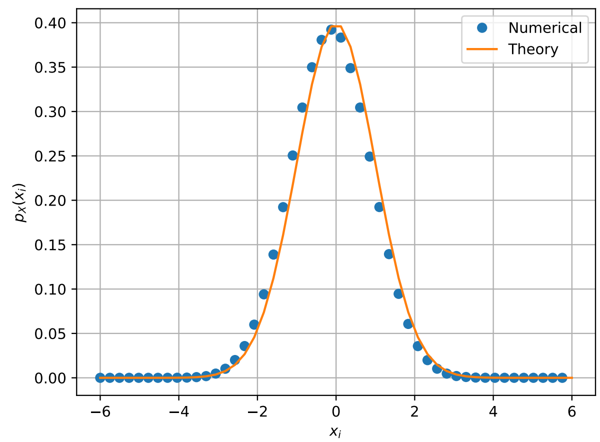

Fig. 2.3. The PDF of X

The CDF of a probability distribution has the following properties:

a) It is non-decreasing
b) It is right-continuous
c) $\lim_{x\to-\infty} F_X(x) = 0$
d) $\lim_{x\to\infty} F_X(x) = 1$

The CDF of the normal distribution is expressed in terms of the Q-function as $F_X(x) = 1 - Q(x)$.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://raw.githubusercontent.com/
    YashRRamteke/Random−numbers/main/
    Code/pdf_plot.py
python3 pdf_plot.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution: Use the main and variance functions in **coeffs.h**, and execute the code below

```
gcc exrand.c
./a.out
```

We get

```
mean = 0.000685
variance = 1.000025
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: We have:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.4}$$
$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} \tag{2.5}$$
$$= 0 \tag{2.6}$$

Also,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.7}$$
$$= -\frac{x}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} \tag{2.8}$$
$$= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \tag{2.9}$$
$$= 1 \tag{2.10}$$

Hence,

$$\mathrm{var}(X) = E[X^2] - E[X]^2 \tag{2.11}$$
$$= 1 \tag{2.12}$$

Therefore, the mean is 0 and the variance is 1. Running the empirical code in **./Code/exrand.c**, we get mean = 0.000685 and variance = 1.000025, which closely matches the theoretical values.

2.6 Find the theoretical CDF of X

Solution: To find the theoretical CDF, consider:

$$Q_X(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.13}$$

$$= \frac{\operatorname{erfc}(\frac{x}{\sqrt{2}})}{2} \tag{2.14}$$

The CDF is then:

$$F_X(x) = 1 - Q_X(x) \tag{2.15}$$

$$= 1 - \frac{\operatorname{erfc}(\frac{x}{\sqrt{2}})}{2} \tag{2.16}$$

3. From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF. **Solution:** Add the following function to **coeffs.h**:

```
void logarithmic(char *str){
t i=0,c;
 *fp, *fp2;
le x, temp=0.0;
 fopen(”uni.dat”,”r”);
= fopen(str, ”w”);
t numbers from file
e(fscanf(fp,”%lf”,&x)!=EOF)

mp = −2*log(1−x);
rintf(fp2,”%lf\n”,temp);

se(fp);
se(fp2);
rn ;
```

Using this function in **exrand.c** prints the numbers in **log.dat**

3.2 Find a theoretical expression for $F_V(x)$.

Solution: We have:

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$

$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

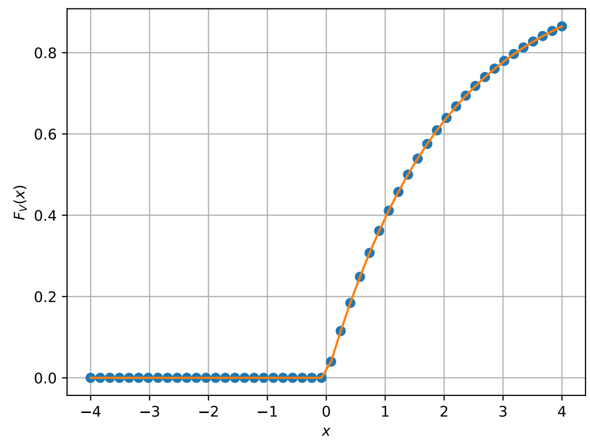

Fig. 3.6. The CDF of V

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases} \tag{3.7}$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \tag{3.8}$$

The CDF of V is plotted in Fig. 3.2