Fourier Transformation

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Fourier Transform

Every function, whether continuous or discontinuous, can be expanded in a summation series of sines of different frequency and amplitude. Whether the waveform of the function is periodic or random, there will always exist one distinct set of sines to produce the required waveform.

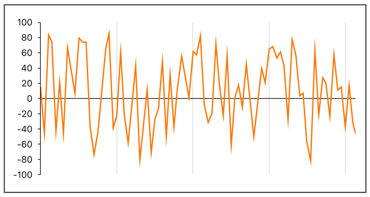

Figure 1: Random Waveform

Fourier Transform is nothing but an algorithm to find out which sine waves have been added up to make the resultant waveform.

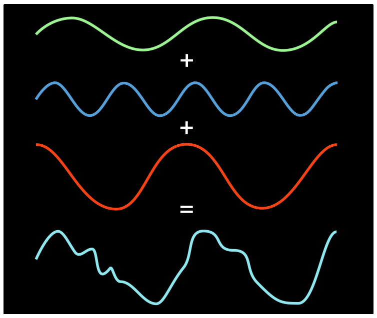

Figure 2: Wave as a sum of sine waves

It also finds out the frequency of the sine waves that the original waveform comprises of and the frequency is plotted on something called as a frequency domain.

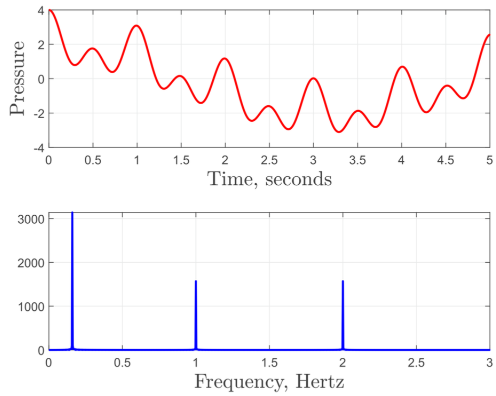

Figure 3: Frequency domain of a wave

So, Fourier Transform can be formally defined as the improper Reimann Integral. It decomposes the function into its constituents frequency.

Definition

The Fourier transformation of any function f(x) is given by:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{1}$$

This equation signifies that the transform of the function f(x) at the angular frequency ω is denoted by a complex number $\widehat{f}(\omega)$, which gives a frequency domain function for all the values of ω.

f and $\widehat{f}(\omega)$ are called Fourier Transform Pair.

Fourier Inversion Integral is defined as:

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{-i\omega t} d\omega \tag{2}$$

This transformation gives us the time domain of a function from a known frequency domain.

Frequency Domain: Refers to the analysis of mathematical functions with respect to frequency and not time.

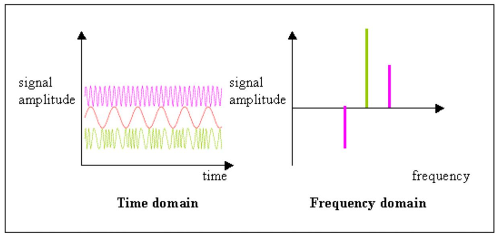

Figure 4: Time and Frequency Domain

Although the inherent nature of this transformation is to decompose waves into constituent sine waves, the usage of this extend far and wide, from quantum mechanics to harmonic oscillators, signal and data processing and much more.

Calculating Fourier Transform

By Euler's Formula we have:

$$e^{ix} = cos(x) + i \cdot sin(x) \tag{3}$$

So, in the Fourier integral (1):

$$e^{-i\omega t} = cos(\omega t) - i \cdot sin(\omega t) \tag{4}$$

By putting equation (4) in equation (1), we get:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)(cos(\omega t) - i \cdot sin(\omega t)) \tag{5}$$

By distributing the f(t) inside, we get:
a real part

$$\int_{-\infty}^{\infty} f(t) \cdot cos(\omega t)dt$$

and an imaginary part

$$-\int_{-\infty}^{\infty} f(t) \cdot sin(\omega t)dt$$

And since the real and complex plane are mutually perpendicular, the magnitude of the Fourier transform is the magnitude of the the integral which can

be found by using basic Pythagorus Theorem.
Phase is the angle made by the magnitude in the complex plane. Both the magnitude and the phase change by changing the angular frequency ω

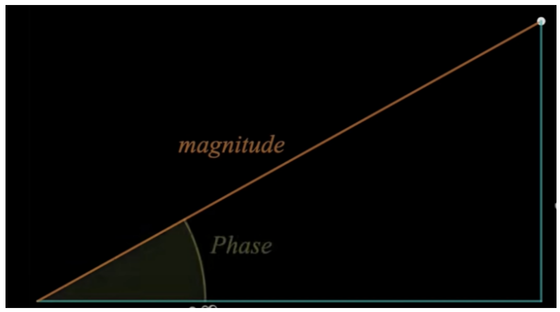

Figure 5: Magnitude and Phase of the Fourier Transform

Working of Fourier Transform

A sine or a cosine curve is periodic in nature, so it has equal number of positive and negative area. Thus, we can safely assume that their area oscillate between equal positive and negative, and by extension, we can also assume that the area of an infinite sine curve is 0.

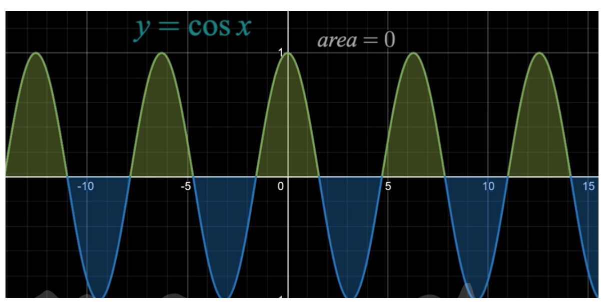

Figure 6: Area of an Infinite cosine Curve

When we multiply two cosine curves, we get a resultant waveform whose area will again oscillate between two finite values. Moreover, looking at the larger portion of the graph, we see that sometimes, the positive area is larger than the negative area and vice-versa.
Thus, we can again safely say, that the area under the infinite graph of most of such functions will always be zero.
However, once the period of the parent equation and the equation multiplied

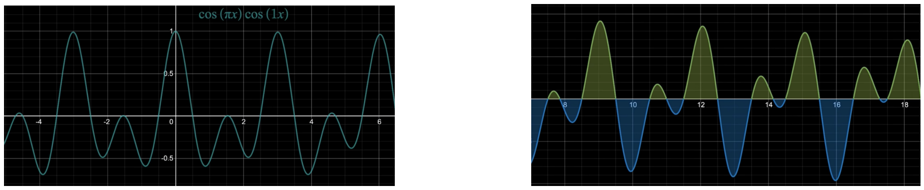

Figure 7: Area under the graph cos(πx ·cos(x))=0

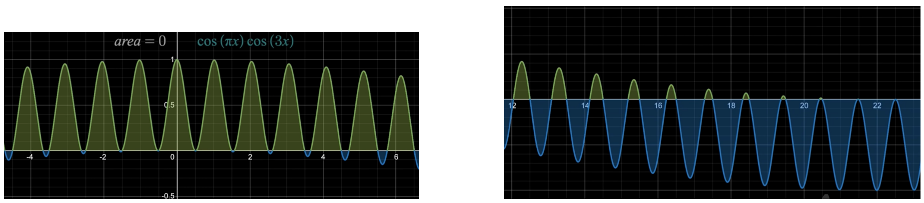

Figure 8: Area under the graph cos(πx)·cos(3x))=0

is equal, the function is squared, making all the area segments positive, consequently making the area infinity.

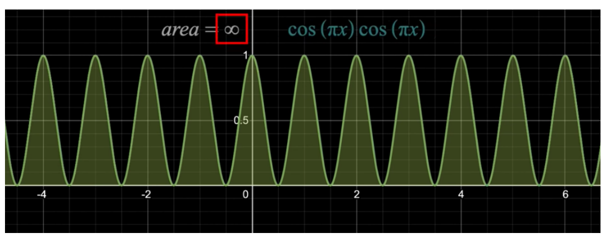

Figure 9: Area = ∞ when the period of both the equations is equal.

The sine and the cosine term of the integral (equation (5)), give us the constituent sine and cosine curves of the waveform, and by making use of the imaginary plane, we also get the magnitude and the phase difference between them.

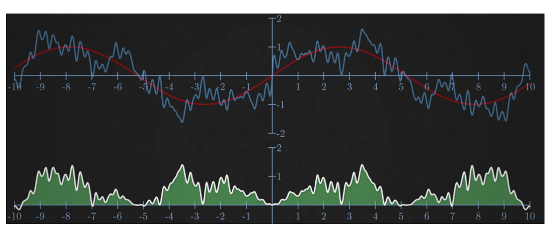

Figure 10: Random waveform and a correlated sine wave, giving non-zero(∞) area

Properties of Fourier Transform

$f(x), g(x)$ and $h(x)$ are integrable functions then $\widehat{f}(\omega), \widehat{g}(\omega)$ and $\widehat{h}(\omega)$ are their respective Fourier Transforms.

1 Linearity

It states that the FT of a weighted sum of two signals is equal to the weighted sum of their individual FT.
For any complex number a and b, if $h(x) = af(x) + bg(x)$, then,

$$\widehat{h}(\omega) = a\widehat{f}(\omega) + b\widehat{g}(\omega)$$

2 Frequency Shifting

For any real number γ_0, if $h(x) = e^{i2\pi\gamma_0 x} f(x)$, then:

$$\widehat{h}(\gamma) = \widehat{f}(\gamma - \gamma_0)$$

3 Time Shifting

For any real number x_0, if $h(x) = f(x - x_0)$, then

$$\widehat{h}(\gamma) = e^{-i2\pi x_0 \gamma} \widehat{f}(\gamma)$$

4 Time Scaling

For any non zero real number a, if $h(x) = f(ax)$, then

$$\widehat{h}(\gamma) = \frac{1}{|a|} \widehat{f}(\frac{\gamma}{a})$$

Discrete Fourier Transform

During the practical application of Fourier Transform, we generally don't get a continuous function. Rather, we get a waveform which seems to be continuous but is made up a many many discrete points.
Applying Fourier Transform to this type of data set is called Discrete Fourier Transform, and rather than integration, we use summation:

$$\widehat{f}(\omega) = \sum_{x=0}^{N-1} f(x) e^{i\omega \frac{x}{N}} \tag{6}$$

(This equation is just a substitution for integral, used for a discrete function, also called numerical integration)

In DFT, we don't get one value of ω, rather, we get a band with multiple values.

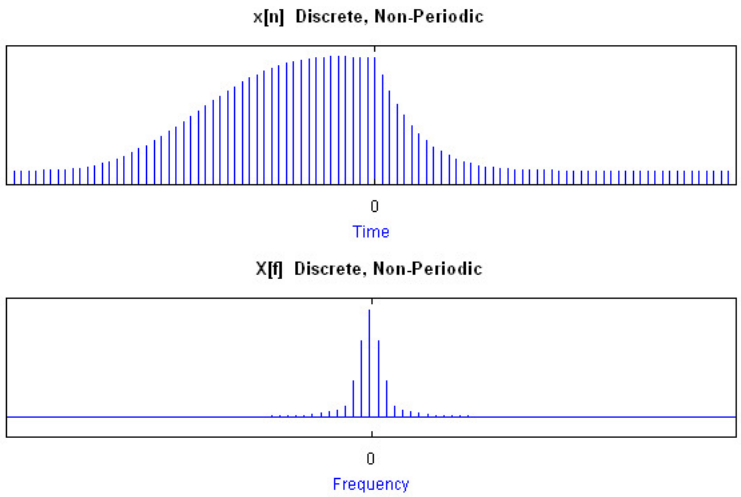

Figure 11: Discrete waveform

Fast Fourier Transform

Fourier Transformation requires immense computation. To find out one particular ω in the sea of infinite ω's requires alot of power and time.
However, Fast Fourier Transform, considered to be the most important algorithm of all time, makes use of the inherent symmetry of sinusoids to calculate FT and exponentially reduce the number of calculations required.

It reduces the number of calculations by a factor of $\mathrm{N}\log_2 N$.

It was originally discovered by the German mathematician Carl Friedrich Gauss in the 18^{th} century but lost in his manuscripts. It was rediscovered by James Cooley and John Tukey who published it in 1965.

Applications of FT and FFT

FT is used in so many diverse ways that it's an integral part of both Quantum Mechanics and Signal Processing.

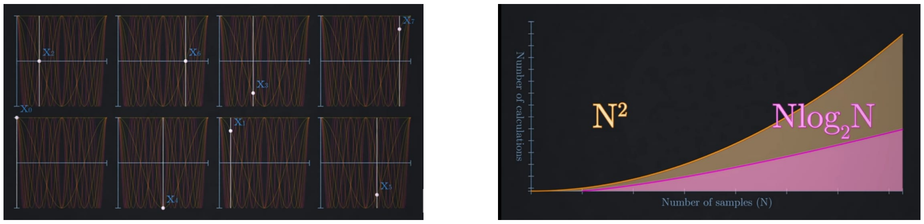

Figure 12: FFT making use of symmetry and exponentially reducing the number of calculations required.

5 Signal Processing

In the electrical world, the use of FT is a must, it simplifies your electrical waveform into constituent waveform in the frequency domain.

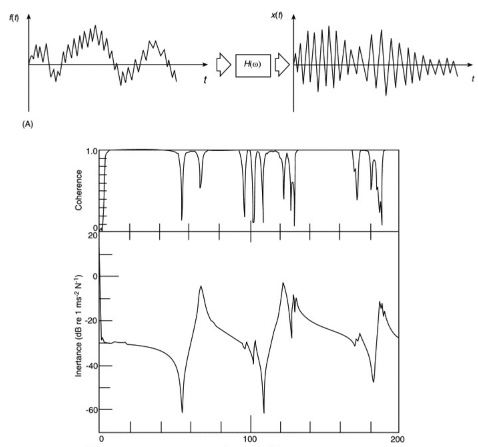

Figure 13: signal processing

6 Heisenberg Uncertainty Principle

A signal, concentrated in time, must have a spread out Fourier Transform.
This is true due to the fact longer waves provide more data.
This forms the basis of uncertainty principle:

$$p = h\gamma \tag{7}$$

where p is the momentum, h is the planck's constant and γ is called as the spatial frequency (i.e. how many times the wave cycles in a unit distance.)

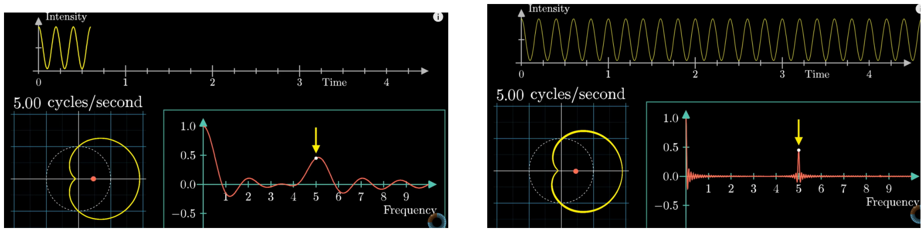

Figure 14: Spread out signal has More precise FT

So, if we precisely measure the position of a particle (which in quantum mechanics is nothing but a wave-packet), the spatial frequency, which indicates the momentum of the particle must be spread out. In other words, if we measure the position precisely, the momentum becomes uncertain and vice-versa.

7 Doppler Radar

Doppler radar is a radar which measures the position and the velocity of objects. However, Fourier Trade-off takes place here as well. We need to send a longer pulse to properly measure the position of the object, but this results in extensive echoes received by the radar, which results in a very ambiguous waveform. However, by using a shorter pulse, the echoes might be small but the precision in measurement will also reduce drastically.

8 Spectography

Spectography refers to the study of spectrums. Spectrums provide us vital info about the constituents of the wave. The spectrum of a light reflected from a planet will give us information about the elements and other constituents in the planet's atmosphere. The spectrum is decomposed into several different waves by using FFT, providing us invaluable details about the universe.

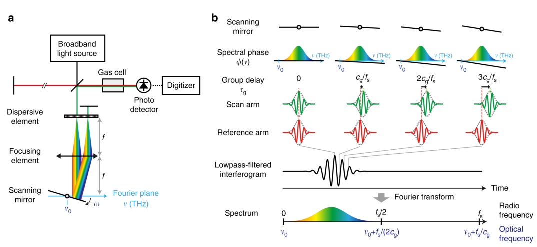

Figure 15: Spectography using FFT