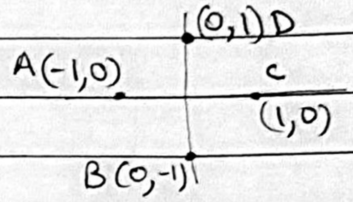
47_D6AD_YASH SARANG_EM4_ASSIGNMENT 1

Q 3) $\int_C |z|^2 . dz$, C: Square with vertices $(-1,0), (0,-1), (1,0), (0,1)$.

$\rightarrow$ Let, $A \equiv (-1,0)$, $B \equiv (0,-1)$, $C \equiv (1,0)$, $D \equiv (0,1)$.



$I = \int_C |z|^2 . dz$

$I = I_1 + I_2 + I_3 + I_4$, $\quad z = x + iy$, $dz = dx + i dy$

i) I_1 is $A \rightarrow B$. $\quad I_1 = \int_{(-1,0)}^{(0,-1)} |z|^2 . dz$

$\therefore \dfrac{y-0}{0-(-1)} = \dfrac{x-(-1)}{-1-0}$

$\therefore y = -x-1$, $\quad z = x + i(-x-1) = x - i(x+1)$

$\therefore dy = -dx$, $\quad dz = dx - i dx$.

$\therefore I_1 = \int_{-1}^{0} \left[x^2 - (x+1)^2 - 2x(x+1)i\right]\left[1-i\right] dx$

$= \int_{-1}^{0} \left[-2x-1-(2x^2+2x)i\right](1-i) . dx$

$= \int_{-1}^{0} 2x^2 \cancel{+2x} \cancel{-2x} -1 \quad -2x(x+1)i + 1(2x+1)i$

$= \int_{-1}^{0} 2x^2 - 1 + (-2x^2+1) i \quad dx$

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$$= (1-i) \left[-\frac{2x^2}{2} - x - \frac{2x^3 i}{3} - \frac{2x^2 i}{2} \right]_{-1}^{0}$$

$$= (1-i) \; -1 \left(\cancel{-1} \cancel{-(-1)} - \frac{2}{3}(-1)i - i \right)$$

$$= (i-1)\left(\frac{2}{3}i - i\right) \quad = \frac{-1}{3}(-1) + \frac{1}{3}i$$

$$\boxed{I_1 = 2(i+1)/3}$$

Similarly, for $B \to C$

$$I_2 = \frac{2-2i}{3}$$

for $C \to D$

$$I_3 = -2\left(\frac{1+i}{3}\right)$$

for $D \to A$

$$I_4 = 2\left(\frac{-1+i}{3}\right)$$

$$\therefore I = I_1 + I_2 + I_3 + I_4 = 0.$$

Q.11 $\int_{C:|z|=2} \frac{1}{z(z-1)^2(z+3)} \cdot dz$

→

$$f(z) = \frac{1}{z(z-1)^2(z+3)}, \quad f(z) \text{ is non-analytic at } z = 0, 1, -3.$$

$z = 0, 1$ lies inside the circle.

$$\frac{1}{z(z-1)^2(z+3)} = \frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} + \frac{D}{(z+3)}.$$

$$\therefore 1 = A(z-1)^2(z+3) + \ldots\ldots + D(z-1)^2 z.$$

for $z=0$, $A(-1)^2(3) = 1$ $\quad \therefore A = 1/3.$

$z=1$, $C(1)(4) = 1$ $\quad \therefore C = 1/4.$

$z=-3$, $D(-4)^2(-3) = 1.$ $\quad \therefore D = -1/48.$

co-efficient of z^3 : $A + B + D = 0$

$B = 1/48 - 1/3 = -5/16.$

$$\therefore f(z) = \frac{1}{3z} + \frac{-5/16}{(z-1)} + \frac{1/4}{(z-1)^2} + \frac{-1/48}{(z+3)}.$$

$$\int_C f(z) \cdot dz = \frac{1}{3} \times 2\pi i - \frac{5}{16} \times 2\pi i + 0 + 0$$

$$= \frac{\pi i}{24}$$

$$\boxed{\therefore \int_{C:|z|=2} \frac{1}{z(z-1)^2(z+3)} \cdot dz = \frac{\pi i}{24}}$$

Sarangapath

Q14) If $f(a) = \int_C \frac{z^2+z+1}{z-a} . dz$, where C is $|z| = 2$.

find the values of $f(1)$, $f(3i)$, $f'(i)$, $f''(2.5)$, $f''(-1)$.

→ $f(z)$ is not analytic at $z=a$.

i) $f(1) = \int_C \frac{z^2+z+1}{z-1} . dz$. non analytic at $z=1$, which lies inside the circle.

$$= 2\pi i \left[z^2+z+1\right]_{z=1}$$

$$\underline{f(1) = 6\pi i}$$

Similarly,

ii) $f(3i) = 2\pi i \left[z^2+z+1\right]_{z=3i}$

$$= 2\pi i \left[9(-1)+3i+1\right] = -6\pi - 16\pi i$$

$$f(3i) = -\pi\left(6+16i\right)$$

but $3i$ lies outside the circle $|z|=2$.

$\therefore$ $\underline{f(3i) = 0}$

iii) $f'(i)$, $f'(a) = -\int_C \frac{z^2+z+1}{(z-a)^2} . dz$

$f'(i) = \int_C \frac{z^2+z+1}{(z-i)^2} . dz$ $z_0 = i$ lies inside the circle.

$$= \frac{2\pi i}{1!} \frac{d}{dz}\left(z^2+z+1\right) = 2\pi i \left(2z+1\right)_{z_0=i}$$

$$\boxed{f'(i) = -4\pi + 2\pi i}$$

Sarangyash

$$f''(a) = -1 \times \cancel{4} - 2 \times \int_C \frac{z^2+z+1}{(z-a)^3} \cdot dz$$

iv) $$f''(2.5) = 2 \times \int_C \frac{z^2+z+1}{(z-2.5)^3} \cdot dz$$

Since, 2.5 lies outside the circle $|z|=2$,

$$\boxed{\therefore f''(2.5) = 0.}$$

v) $$f''(-1) = 2 \int_C \frac{z^2+z+1}{(z+1)^3} \cdot dz$$

$$= \frac{2\pi i}{2!} \frac{d^2}{dz^2} \left[z^2+z+1\right]_{-1}$$

$$= \pi i \times 2$$

$$\boxed{\therefore f''(-1) = 2\pi i}$$

Sarangyash

Q 13) Evaluate $\int_C \frac{2z^3+z^2+4}{z^4+4z^2}.dz$, where ~~C is |z|=2~~ $C : |z-2-2i| = 3$.

→

$$I = \int_C \frac{2z^3+z^2+4}{z^2(z^2+4)}.dz$$

$f(z) = \frac{2z^3+z^2+4}{z^2(z^2+4)}$, $f(z)$ is not analytic at $0, 2i, -2i$.

$$\frac{2z^3+z^2+4}{z^2(z-2i)(z+2i)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-2i} + \frac{D}{z+2i}$$

for $z=0$, $\cancel{2z^3}+\cancel{z^2}+4 = 4B$, $\therefore B = 1$.

$z = 2i$, $2z^3+z^2+4 = (-4)(4i)C$.

$-16i \cancel{-4} \cancel{+4} = -16iC$, $\therefore C = 1$.

$z = -2i$, $2(-8)(-i) + \cancel{(-4)} \cancel{+4} = D(-4i)(-4)$

$16i = 16iD$. $\therefore D = 1$.

Coefficient of z^3 : $A+C+D = 2$.

$\therefore A = 0$.

$$\therefore f(z) = \frac{1}{z^2} + \frac{1}{z-2i} + \frac{1}{z+2i}$$

$$\therefore I = \int_C f(z).dz = \int_C \frac{1}{z^2}.dz + \int_C \frac{1}{z-2i}.dz + \int_C \frac{1}{z+2i}.dz$$

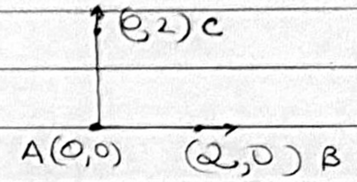
$$= 0 + 2\pi i + 2\pi i$$

$$= 4\pi i$$

$$\boxed{\therefore I = \int_C \frac{2z^3+z^2+4}{z^4+4z^2}.dz = 4\pi i}$$

Q7) Verify Cauchy's Integral theorem for $\int_C z^2 \, dz$ where C is the boundary of the triangle with vertices $A \equiv (0,0)$, $B \equiv (2,0)$, $C \equiv (0,2)$. i.e $0, 2, 2i$.

→



$I = I_1 + I_2 + I_3$, $\quad z = x + iy$

$dz = dx + i\,dy$.

$I_1 = \int_A^B z^2 . dz.$

for $A \to B$, $y = 0$. $\therefore z = x$.

$dy = 0 \quad \therefore dz = dx$

$= \int_0^2 x^2 . dx = \left[\frac{x^3}{3}\right]_0^2$

$\boxed{I_1 = 8/3}$

$I_2 = \int_B^C z^2 . dz$ for $B \to C$, $\frac{y-0}{0-2} = \frac{x-2}{2-0} \quad \therefore y = 2 - x$

$\therefore dy = -dx.$

$= \int_B^C [x + i(2-x)][dx + i(-dx)]$ as x goes from B to C, x goes from 2 to 0.

$= (1-i) \int_2^0 (x - xi + 2i) . dx.$

$= (1-i) \left[\frac{x^2}{2} - \frac{x^2 i}{2} + 2ix\right]_2^0 = (1-i)(-1)\left[\frac{\cancel{4}^2}{\cancel{2}} - \frac{\cancel{4}^2}{\cancel{2}} i + 4i\right]$

$= (-2)(1-i)(1+i) = -8 + 8i/3$

$\boxed{\therefore I_2 = (-8+8i)/3}$

$I_3 = \int_C^A z^2 . dz$ for $C \to A$, $x = 0 \quad dx = 0$

$z = y \quad dz = i\,dy.$

$= i \int_2^0 y^2 . dy = i\left[\frac{y^3}{3}\right]_2^0 = \frac{-8i}{3}$

$I_3 = -8i/3$

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$\therefore I = I_1 + I_2 + I_3$
$= 0.$

∴ Since the given curve is a contour analytic at all points on and inside the curve, the integral of $f(z)$ will always be 0.

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