Sarangyash

# 47_D6AD_YASH SARANG_EM4_ASSIGNMENT 2.

Q4.

$$\frac{1}{z^2+4}, \quad z=-i.$$

→

$$f(z) = \frac{1}{z^2+4}, \quad f(z) \text{ is not analytic at } z = 2i, -2i.$$

$$\frac{1}{(z-2i)(z+2i)} = \frac{A}{(z-2i)} + \frac{B}{(z+2i)}.$$

$$\therefore 1 = A(z+2i) + B(z-2i)$$

for $z = 2i$, $A = 1/4i$

$z = -2i$, $B = -1/4i$.

$$\therefore \frac{1}{(z-2i)(z+2i)} = \frac{1/4i}{(z-2i)} + \frac{-1/4i}{(z+2i)}.$$

$$\frac{1}{(z+i)+i} = \frac{1}{(z+i)\left[1+\frac{i}{z+i}\right]} = \frac{1}{z+i}\left[1-\left(\frac{i}{z+i}\right)+\left(\frac{i}{z+i}\right)^2-\dots\right]$$

where $i < |z+i|$ ——— ①

$$\frac{1}{z-2i} = \frac{1}{z+i-3i} = \frac{1}{z+i}\left[\frac{1}{1-\frac{3i}{z+i}}\right] = \frac{1}{z+i}\left[1+\left(\frac{3i}{z+i}\right)+\left(\frac{3i}{z+i}\right)^2+\dots\right]$$

$3i < |z+i|$ ——— ②

$$\frac{1}{(z+i)+i} = \frac{1}{i}\left(\frac{1}{1+\frac{z+i}{i}}\right) = \frac{1}{i}\left[1-\left(\frac{z+i}{i}\right)+\left(\frac{z+i}{i}\right)^2+\dots\right]$$

$|z+i| < i$ ——— ③

$$\frac{1}{-3i\left[1-\frac{z+i}{3i}\right]} = \frac{1}{-3i}\left[1+\left(\frac{z+i}{3i}\right)+\left(\frac{z+i}{3i}\right)^2+\dots\right] \quad |z+i| < 3i$$ ——— ④

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$\therefore$ for $|z+i| > 0$, from ① and ② $\left[\cancel{2} < \cancel{3}\cancel{2}\right]$

$$\frac{1}{(z-2i)(z+2i)} = \frac{1}{4i} \times \frac{-1}{(z+i)}\left[1 - \frac{i}{z+i} + \left(\frac{i}{z+i}\right)^2 - \dots\right]$$
$$+ \frac{1}{4i} \times \frac{1}{(z+i)}\left[1 + \frac{3i}{z+i} + \left(\frac{3i}{z+i}\right)^2 \dots\right]$$

för $|z+i| < 0$, from ③ and ④

$$\frac{1}{(z-2i)(z+2i)} = \frac{-1}{4i} \times \frac{1}{i}\left[1 - \frac{z+i}{i} + \left(\frac{z+i}{i}\right)^2 + \dots\right] + \frac{-1}{4i} \times \frac{1}{3i}\left[1 + \frac{z+i}{3i} + \left(\frac{z+i}{3i}\right)^2 \dots\right]$$

Q6. $I = \int \frac{z+5}{(z+1)^2(z-2)}$ $\quad f(3) = \frac{z+5}{(z+1)^2(z-2)}$

$f(3)$ is not analytic at $(z+1)^2(z-2) = 0$.
i.e $z = 2, -1$

$$\frac{z+5}{(z+1)^2(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z-2)}$$

$$z+5 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$$

for $z = 2$, $\quad 7 = 9C$, $\quad \therefore C = 7/9$.

$z = -1$, $\quad 4 = B(-3)$ $\quad \therefore B = -4/3$

for powers of $z^2$, $\quad 0 = A + C$, $\quad \therefore A = -7/9$.

$$\therefore \frac{z+5}{(z+1)^2(z-2)} = \frac{-7}{9(z+1)} + \frac{-4}{3(z+1)^2} + \frac{7}{9(z-2)}$$

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$$\frac{1}{(z+1)\left(1-\frac{3}{z+1}\right)} = \frac{1}{z+1}\left[1+\frac{3}{z+1}+\left(\frac{3}{z+1}\right)^2+\dots\right] \quad 3<|z+1| \text{ — ①}$$

$$\frac{1}{(z+1-3)} = \frac{1}{-3\left[1-\frac{z+1}{3}\right]} = \frac{-1}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\dots\right] \quad |z+1|<3 \text{ — ②}$$

i) $|z+1|<3$,

$$\therefore I = \frac{-7}{9(z+1)} + \frac{-4}{3(z+1)^2} + \frac{7\times(-1)}{9\times 3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\dots\right]$$

ii) $|z+1|>3$, $\frac{1}{z+1-3} = \frac{1}{(z+1)(1-3/z+1)} = \frac{1}{z+1}\left[1+\frac{3}{z+1}+\left(\frac{3}{z+1}\right)^2+\dots\right]$

for $|3/z+1|<1 \quad \therefore |z+1|>3$.

$$\therefore I = \frac{-7}{9(z+1)} + \frac{-4}{3(z+1)^2} + \frac{7}{9}\left(\frac{1}{z+1}\right)\left[1+\frac{3}{z+1}+\left(\frac{3}{z+1}\right)^2+\dots\right]$$

iii) $1<|z|<2$, $\frac{1}{z+1} = \frac{1}{1+z} = \left[1-z+z^2-\dots\right] \quad |z|<1$ — ①

$$\frac{1}{z[1+1/z]} = \frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\dots\right] \quad |z|>1 \text{ — ②}$$

$$\frac{1}{(z+1)^2} = \frac{1}{z^2[1+1/z]^2} = \frac{1}{z^2}\left[1+\frac{2}{z}+\frac{3}{z^2}+\dots\right] \quad 1<|z| \text{ — ③}$$

$$\frac{1}{(z+1)^2} = 1+2z+3z^2+\dots \quad |z|<1 \text{ — ④}$$

iv) $$\therefore I = \frac{-7}{9}\left[\frac{1}{z-2}\right]\left[1-\frac{3}{z-2}+\left(\frac{3}{z-2}\right)^2-\dots\right]+\frac{7}{9}\left(\frac{1}{z-2}\right)$$
$$-\frac{4}{3}\times\frac{1}{z^2}\left[1+\frac{2}{z}+\frac{3}{z^2}+\dots\right]$$

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$$\frac{1}{2(1-2/z)} = \frac{1}{2}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\dots\right] \qquad 2<|z| \quad \text{— (5)}$$

$$\frac{1}{-2[1-z/2]} = \frac{-1}{z}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right] \qquad |z|<2 \quad \text{— (6)}$$

iii) $$I = \frac{-7}{9z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\dots\right] + \frac{-4}{3}\times\frac{1}{z^2}\left[1-\frac{2}{z}+\frac{3}{z^2}-\dots\right]$$
$$+\frac{7}{9}\times\left(\frac{-1}{z}\right)\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right]$$

$$\frac{1}{z-1+2} = \frac{1}{(z-1)\left(1+\frac{2}{z-1}\right)} = \frac{1}{z-1}\left[1-\frac{2}{z-1}+\left(\frac{2}{z-1}\right)^2-\dots\right] \qquad 2<|z-1| \quad \text{— (7)}$$

$$\frac{1}{(z+1)^2} = \frac{1}{(z-1+2)^2} = \frac{1}{(z-1)^2\left[1+\frac{2}{z-1}\right]^2} = \frac{1}{(z-1)^2}\left[1-2\left(\frac{z}{z-1}\right)+\frac{3(z)^2}{(z-1)^2}-\dots\right]$$

$$|z|<|z-1| \quad \text{— (8)}$$

$$\frac{1}{(z-1+2)^2} = \frac{1}{z^2\left[1+\frac{z-1}{z}\right]^2} = \frac{1}{4}\left[1-2\left(\frac{z-1}{2}\right)+3\left(\frac{z-1}{2}\right)^2-\dots\right]$$

$$|z-1|<2 \quad \text{— (9)}$$

$$1/z-2 = 1/z-1-1 = \frac{1}{(z-1)}\times 1/\left[1-\frac{1}{z-1}\right] = \frac{1}{z-1}\left[1+\frac{1}{z-1}+\left(\frac{1}{z-1}\right)^2+\dots\right]$$

$$1<|z-1| \quad \text{— (10)}$$

v) $$I = \frac{-7}{9}\left[\frac{1}{2}\left\{1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\dots\right\}\right] - \frac{4}{3}\times\frac{1}{4}\left[1-\frac{2(z-1)}{2}+3\left(\frac{z-1}{2}\right)^2\right.$$
$$+\frac{7}{9}\left\{\frac{1}{z-1}\left[1+\frac{1}{z-1}+\left(\frac{1}{z-1}\right)^2+\dots\right]\right\}$$

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7) $\frac{1}{z-\sin z}$ at $z=0$.

→ $f(z) = 1/z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right]$

$= 1/\frac{z^3}{3!} - \frac{z^5}{5!} \dots = 1/z^3\left(\frac{1}{3!} - \frac{z}{5!} + \dots\right)$

$= 1/\frac{z^3}{3!}\left(1 - \frac{3!z}{5!} + \dots\right)$

$z=0$ is a pole of order 3.

$f(z) = 1/\frac{z^3}{3!}\left[\frac{6}{6} - \frac{z}{10\times 2} + \dots\right]$

Res $f(z)$ at $(z=0)$ coefficient of $\frac{1}{z} = \frac{3}{10}$.

$\int_c f(z).dz = 2\pi i(3/10) = \frac{6}{10}\pi i$.

15) $I = \int_c \frac{1}{z^3}.e^{1-\cos z}.dz$, $|z|=1$.

$f(z) = \frac{e^{1-\cos z}}{z^3}$, $f(z)$ is not analytic at $z^3=0$.
$\therefore z=0$.

Res $f(z) = e^{1-\cos z}/z^3$

$= \frac{1}{2!}\frac{d^2}{dz^2}\left(e^{1-\cos z}\right) = \frac{e}{2!}\frac{d^2}{dz^2}\left(e^{-\cos z}\right)$

$= \frac{e}{2!}\left[\sin z\,\frac{e^{-\cos z}}{\sin z} - e^{-\cos z}(\cos z)\right]_{z=0}$

$= \frac{e}{2!} \times e^{-\cos z}(1-\cos z) \Big/ \left[1 - \frac{z^2}{3!} + \dots\right]\left[1 - \frac{z^2}{3!} + \dots\right]_{z=0}$

$\underset{z=0}{\text{Res}} f(z) = 0$, $\int_c f(z).dz = 2\pi i \times 0$
$= 0$.

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18) → $I = \int_c \frac{\tan \pi z}{z^4} . dz$, $f(z) = \frac{\sin \pi z}{z^4 (\cos \pi z)}$.

$f(z)$ is not analytic at $z^4 (\cos \pi z) = 0$,
$z = 0$ or $\cos \pi z = 0$

$\cos \pi z = 0$,
$\pi z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

$z = \pm \frac{1}{2}, \frac{\pm 3}{2}, \frac{\pm 5}{2}$ & $z = \frac{-1}{2}, \frac{-3}{2}, \frac{-5}{2}, \ldots$ lies inside circle and a pole of order 1.

$h(z) = z^4 (\cos \pi z)$, $h'(z) = 4z^3 (\cos \pi z) - \pi z^5 \sin \pi z$.
$h'(z) = 4z^3 (\cos \pi z) + z^4 [\pi - \sin \pi^2]$

$h\left(\frac{-1}{2}\right) = 4 \left(\frac{-1}{2}\right)^3 \left[\cos\left(\frac{-\pi}{2}\right)\right] \neq 0.$

$\underset{z=-1/2}{\mathrm{Res}} f(z) = \underset{z=-1/2}{\mathrm{Res}} \frac{\sin \pi z}{4z^3 \cos \pi z + z^4 [\pi (-\sin \pi z)]}$

$= \frac{\sin(-\pi/2)}{4(-1/2)^3 \cos(-\pi/2) + \left(\frac{-1}{2}\right)^4 \left(\pi \sin\left(\frac{\pi}{2}\right)\right)}$

$= \frac{-1}{1/16 \times \pi} = \frac{-16}{\pi}$

$\underset{z=-3/2}{\mathrm{Res}} f(z) = \frac{\sin \pi z}{4z^3 \cos \pi z + z^4 [-\pi \sin \pi^2]}_{z=-3/2}$

$= \frac{\sin(-3\pi/2)}{4(-3/2)^3 \cos\left(\frac{-3\pi}{2}\right) + \left(\frac{-3}{2}\right)^4 \left[\pi + \sin\left(\frac{3\pi}{2}\right)\right]}$

$= \frac{-16}{81\pi}$.

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$$\operatorname{Res}_{z=-5/2} f(z) = \frac{\sin \pi z}{4z^3 \cos \pi z + z^4 \left[-\pi \sin \pi z\right]}\Bigg|_{z=-5/2}$$

$$= \frac{-\sin 5\pi/2}{4\left(\frac{-5}{2}\right)^3 \cos\left(\frac{-5\pi}{2}\right) + \left(\frac{-5}{2}\right)^4 \left[-\pi \sin \frac{-\pi 5}{2}\right]}\Bigg|_{z=-5/2}$$

$$= \frac{-16}{625\pi}$$

$\vdots$

$$\therefore \oint f(z)\,dz = 2\pi i\left[\frac{-16}{\pi} + \frac{-16}{81\pi} + \frac{-16}{625\pi} + \frac{-16}{2401\pi} + \frac{-16}{6561\pi}\right]$$

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