Linear Programming Problems

Formulation of LPP:

A Mutual Fund company has Rs 20 lakhs available for investment in government bonds, blue chip stocks, speculative stocks and short term bank deposits. The annual expected return and risk factor are given below:

Type of investment	Annual expected return (%)	Risk factor (0 to 100
Government bonds	14	12
Blue chip stocks	19	24
Speculative stocks	23	48
Short term deposits	12	6

Mutual fund is required to keep at least Rs 2 lakhs in short term deposits and not exceed an average risk factor of 42. Speculative stocks must be at most 20% of the total amount invested. How should mutual fund invest the funds so as to maximize its total expected annual return? Formulate the problem.

Linear Programming Problems (LPP)

Formulation of LPP: -

Let x_1 = Amount to be invested in Government Bonds

x_2 = —" —" — Blue chip stocks

x_3 = —" —" — Speculative stocks

x_4 = —" —" — Short term deposits

Maximize $Z = \frac{14}{100}x_1 + \frac{19}{100}x_2 + \frac{23}{100}x_3 + \frac{12}{100}x_4$

$\Leftrightarrow \quad Z = 14x_1 + 19x_2 + 23x_3 + 12x_4$

subject to $\quad x_1 + x_2 + x_3 + x_4 = 20, \quad \text{——(i)}$

$x_4 \geq 2 \quad \text{——(ii)}$

$$\frac{12x_1 + 24x_2 + 48x_3 + 6x_3}{x_1 + x_2 + x_3 + x_4} \leq 42$$

$\Leftrightarrow \quad 12x_1 + 24x_2 + 48x_3 + 6x_3 \leq 42 \times 20$

$\Leftrightarrow \quad 12x_1 + 24x_2 + 48x_3 + 6x_3 \leq 840 \quad \text{——(iii)}$

$x_3 \leq \frac{20}{100} \times 20$

$\Leftrightarrow \quad x_3 \leq 4 \quad \text{——(iv)}$

$x_1, x_2, x_3, x_4 \geq 0 \quad \text{——(v)}$

General Form of a LPP:

Maximize (or Minimize) $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \ (\leq, =, \geq) b_i, \quad i = 1, 2, ..., m$

$x_j \geq 0, \quad j = 1, 2, ..., n$

The function $Z = \sum_{j=1}^{n} c_j x_j$ is called **Objective function.**

Variables $x_1, x_2, ..., x_n$ are known as **Decision variables.**

Constants $c_1, c_2, ..., c_n$ are known as **Cost coefficients**

The Canonical Form of a LPP:

Maximize $Z = \sum_{j=1}^{n} c_j x_j$ or Minimize $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, ..., m$ Subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, 2, ..., m$

$x_j \geq 0, \quad j = 1, 2, ..., n$ $x_j \geq 0, \quad j = 1, 2, ..., n$

The Standard Form of a LPP:

Maximize (or Minimize) $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i,$

$x_j \geq 0, \ b_i \geq 0, \quad i = 1, 2, ..., m, \ j = 1, 2, ..., n$

Note:

1. The Inequality $\leq$ type is changed to equality by adding a non negative variable called
Slack variable or simply **Slack**

e.g. $a_1 x_1 + a_2 x_2 \leq b_1, \ b_1 \geq 0$ is changed to

$a_1 x_1 + a_2 x_2 + s_1 = b_1, \ b_1 \geq 0, \ s_1 \geq 0$

The new variable s_1 is a slack here.

2. The Inequality $\geq$ type is changed to equality by subtracting a non negative variable called
Surplus variable or negative Slack

e.g. $a_1 x_1 + a_2 x_2 \geq b_1, \ b_1 \geq 0$ is changed to

$a_1 x_1 + a_2 x_2 - s_2 = b_1, \ b_1 \geq 0, \ s_2 \geq 0$

The new variable s_2 is a surplus here.

Ex. Write the following problem in canonical form.

1. Max $Z = 3x_1 + x_2 + 4x_3 + x_4 + 9x_5$

Sub to $4x_1 - 5x_2 - 94x_3 + x_4 - 2x_5 \le 6$

$2x_1 + 3x_2 + 4x_3 - 5x_4 + x_5 = 9$

$-x_1 - x_2 + 5x_3 + 7x_4 - 11x_5 \ge -10$

$x_1, x_2, x_4, x_5 \ge 0$

Soln put $x_3 = x_3' - x_3''$

Max $Z = 3x_1 + x_2 + 4x_3' - 4x_3'' + x_4 + 9x_5$

sub to $4x_1 - 5x_2 - 94x_3' + 94x_3'' + x_4 - 2x_5 \le 6$

$2x_1 + 3x_2 + 4x_3' - 4x_3'' - 5x_4 + x_5 \le 9$

$2x_1 + 3x_2 + 4x_3' - 4x_3'' - 5x_4 + x_5 \ge 9$

$-x_1 - x_2 + 5x_3' - 5x_3'' + 7x_4 - 11x_5 \ge -10$

$x_1, x_2, x_3', x_3'', x_5 \ge 0$

$\Rightarrow$ Max $Z = 3x_1 + x_2 + 4x_3' - 4x_3'' + x_4 + 9x_5$

Sub to $4x_1 - 5x_2 - 94x_3' + 94x_3'' + x_4 - 2x_5 \le 6$

$2x_1 + 3x_2 + 4x_3' - 4x_3'' - 5x_4 + x_5 \le 9$

$-2x_1 - 3x_2 - 4x_3' + 4x_3'' + 5x_4 - x_5 \le -9$

$x_1 + x_2 - 5x_3' + 5x_3'' - 7x_4 + 11x_5 \le 10$

$x_1, x_2, x_3', x_3'', x_4, x_5 \ge 0$

E.x. Write the LPP in standard form.

$$\text{Max } Z = 3x_1 + x_2$$

$$\text{sub to } \quad 2x_1 - 5x_2 \le 6$$

$$2x_1 + 3x_2 \ge 9$$

$$-x_1 - x_2 \ge -10$$

$$x_2 \ge 0$$

Soln: put $x_1 = x_1' - x_1''$

$$\text{Max } Z = 3x_1' - 3x_1'' + x_2$$

$$\text{sub to } \quad 2x_1' - 2x_1'' - 5x_2 + S_1 = 6$$

$$2x_1' - 2x_1'' + 3x_2 - S_2 = 9$$

$$-x_1' + x_1'' - x_2 - S_3 = -10$$

$$x_1', x_1'', x_2, S_1, S_2, S_3 \ge 0$$

$$\text{Max } Z = 3x_1' - 3x_1'' + x_2$$

$$\text{sub to } \quad 2x_1' - 2x_1'' - 5x_2 + S_1 = 6$$

$$2x_1' - 2x_1'' + 3x_2 - S_2 = 9$$

$$x_1' - x_1'' + x_2 + S_3 = 10$$

$$x_1', x_1'', x_2, S_1, S_2, S_3 \ge 0$$

Definitions:

If in the standard form of a LPP, there are n variables and m equality constraints say

Maximize (or Minimize) $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i,$

$x_j \geq 0,\ b_i \geq 0, \qquad i = 1, 2, \ldots, m,\ \ j = 1, 2, \ldots, n$

Then

1. A solution of the problem can be obtained by equating $(n - m)$ variables to zero and solving for the rest m variables.
These m variables are known as **basic variables.**
The $(n - m)$ variables which are equated to zero are known as **non basic variables**
The solution so obtained is called a **basic solution**.

2. If the value of all variables in a solution is non negative (≥ 0) then the solution is a **Feasible solution.**
A basic solution which is also feasible is known as a **basic feasible solution.**

3. If the value of all the **basic variables** are greater than zero (> 0) in a solution then the solution is called **non degenerate** otherwise **degenerate**.

4. A basic solution is said to be **Optimal or Optimum** if it is feasible and optimizes the objective function.

Simplex Method for Maximization Problems:

Example 1:

Solve the LPP

Max $\quad Z = 3x_1 + 4x_2$

Sub to $\quad x_1 + x_2 \leq 450$

$2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0$

Solution:

Step1: Express the LPP in standard form

Max $\quad Z = 3x_1 + 4x_2$

Sub to $\quad x_1 + x_2 + s_1 = 450$

$2x_1 + x_2 + s_2 = 600$

$x_1, x_2, s_1, s_2 \geq 0$

Step2: Find the initial basic feasible solution

Put $x_1 = 0, x_2 = 0$ [Non basic variables]
Then $s_1 = 450, s_2 = 600$ [Basic variables]

Step3: Express the information in tabular form and perform optimality test.

c_B	c_j / Basis	3 / x_1	4 / x_2	0 / s_1	0 / s_2	Solution Value b	Minimum Ratio θ
0	s_1	1	(1)	1	0	450	450 $\rightarrow$
0	s_2	2	1	0	1	600	600
	z_j	0	0	0	0		
	$c_j - z_j$	3	4 $\uparrow$	0	0		
4	x_2	1	1	1	0	450	
0	s_2	1	0	-1	1	150	
	z_j	4	4	4	0		
	$c_j - z_j$	-1	0	-4	0		

Step4: Compute $z_j = \sum_i c_B a_{ij}$ and $(c_j - z_j)$ called Net evaluation row.

If all $(c_j - z_j) \leq 0$ then the **solution is optimum** and no further improvement is possible.

If $(c_j - z_j) > 0$ under at least one column, then the **solution is not optimum** and further improvement is required.

Step5: The variable heading the greatest positive value of $(c_j - z_j)$ is **Incoming variable** in the basis and the column in which it occurs is called **key column or pivot column** marked as $\uparrow$
If more than one variable appears with the equal greatest positive value in $(c_j - z_j)$ row then any one of these variables may selected as incoming variable, neglecting the slacks.

Step6: Compute the column θ which is the ratio between solution column (b) and key column **known as replacement ratio or minimum ratio.**
The row containing the least positive ratio in the column θ is known as **key row** marked as $\rightarrow$ and the variable in the basis heading the key row is the **outgoing variable.**
The element lying at the intersection of the key column and key row is called **key element** and marked as ( ).

Note: If all the ratios under the column θ are **negative or infinite** then the problem has an **Unbounded solution** and the iteration stops.

Step7: Improving upon the solution:
Make the key element unity by dividing the key row by key element.
Make the element of key column zero other than key element by using elementary row operation and key row.
Thus we will have new improved solution of the LPP. Performing the optimality test we can check whether optimality is achieved.

Thus in above example, after two iteration optimality is achieved since all $(c_j - z_j) \leq 0$
Optimal solution is $x_1 = 0, \ x_2 = 450$
and Optimal value is $Z_{max} = 1800$

Note: If the value of $(c_j - z_j)$ **under a non basic variable is zero** and optimality is achieved then there is **existence** of an **alternate optimal solution.**
Let X and Y are two optimal solution of given LPP
Then $(1-t)X + tY$ are also solution of the LPP for all $0 \leq t \leq 1$
Thus there exist infinite numbers of optimal solution of the given LPP.

Simplex Method for Minimization Problems:
Method1: The minimization problems can be solve by converting into maximization problems

Method2: To solve minimization problems directly without converting into maximization type, the incoming variable will be the one having the least negative value in the $(c_j - z_j)$ row.
The optimality is achieved if all $(c_j - z_j) \geq 0$.

Ex. Find the all basic solutions and ~~sp~~ classify them. Find the optimum solution if exist.

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Sub to
$$3x_1 + 5x_2 + 4x_3 = 7$$
$$6x_1 + x_2 + 3x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

Standard form of LPP is

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

sub to
$$3x_1 + 5x_2 + 4x_3 = 7$$
$$6x_1 + x_2 + 3x_3 + s_1 = 4$$
$$x_1, x_2, x_3, s_1 \geq 0$$

S. No.	Non Basic variables	Basic Equations	Solution of Basic variables	Value of Z	Classification
1.	$x_1 = 0$ $x_2 = 0$	$4x_3 = 7$ $3x_3 + s_1 = 4$	$x_3 = 7/4$ $s_1 = -\frac{5}{4}$	—	Infeasible solution
2.	$x_1 = 0$ $x_3 = 0$	$5x_2 = 7$ $x_2 + s_1 = 4$	$x_2 = 7/5$ $s_1 = \frac{13}{5}$	$-\frac{14}{5}$	Feasible, Non-degenrate solution
3.	$x_1 = 0$ $s_1 = 0$	$5x_2 + 4x_3 = 7$ $x_2 + 3x_3 = 4$	$x_2 = \frac{5}{11}$ $x_3 = \frac{13}{11}$	$\frac{42}{11}$	Feasible, Non-degerate solution
4.	$x_2 = 0$ $x_3 = 0$	$3x_1 = 7$ $6x_1 + s_1 = 4$	$x_1 = 7/3$ $s_1 = -10$	—	Infeasible solution

S.No.	Non-Basic variables	Basic Equations	Solution of Basic Variables	value of Z	classification
5.	$x_2=0$ $S_1=0$	$3x_1+4x_3=7$ $6x_1+3x_3=4$	$x_1=-1/3$ $x_3=2$	—	Infeasible solution
6.	$x_3=0$ $s_1=0$	$3x_1+5x_2=7$ $6x_1+x_2=4$	$x_1=\frac{13}{27}$ $x_2=\frac{10}{9}$	$-\frac{7}{9}$	Feasible, Non-degenerate solution

$\therefore$ optimal solution is

$x_1=0,\ x_2=7/5,\ x_3=0$

and $Z_{min}=\frac{-14}{5}$

Ex. Solve the LPP

$$\text{Max } Z=4x_1+3x_2+6x_3$$

Sub. to $2x_1+3x_2+2x_3\le 440$

$4x_1+3x_3\le 470$

$2x_1+5x_2\le 430$

$x_1, x_2, x_3\ge 0$

converting it into standard form and finding initial solution.

$\max Z = 4x_1 + 3x_2 + 6x_3$

sub to $\quad 2x_1 + 3x_2 + 2x_3 + S_1 = 440$

$\qquad 4x_1 + 3x_3 + S_2 = 470$

$\qquad 2x_1 + 5x_2 + S_3 = 430$

$\qquad x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

put $x_1 = 0, x_2 = 0, x_3 = 0$ (Non-basic variable)

$\Rightarrow S_1 = 440, S_2 = 470, S_3 = 430$ (Basic variables)

	c_j	4	3	6	0	0	0	Sol^n	Min. Ratio
C_B	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b	θ
0	S_1	2	3	2	1	0	0	440	220
0	S_2	4	0	(3)	0	1	0	470	470/3 →
0	S_3	2	5	0	0	0	1	430	∞
	Z_j	0	0	0	0	0	0		
	$C_j - Z_j$	4	3	6 ↑	0	0	0		
0	S_1	$-2/3$	(3)	0	1	$-2/3$	0	380/3	$\frac{380}{9}$ →
6	x_3	$4/3$	0	1	0	$1/3$	0	470/3	∞
0	S_3	2	5	0	0	0	1	430	86
	Z_j	8	0	6	0	2	0		
	$C_j - Z_j$	-4	3 ↑	0	0	-2	0		
3	x_2	$-2/9$	1	0	$1/3$	$-\frac{2}{9}$	0	$\frac{380}{9}$	
6	x_3	$4/3$	0	1	0	$1/3$	0	470/3	
0	S_3	$28/9$	0	0	$-5/3$	$\frac{10}{9}$	1	$\frac{1970}{9}$	
	Z_j	$22/3$	3	6	1	$4/3$	0		
	$C_j - Z_j$	$-\frac{10}{3}$	0	0	-1	$-\frac{4}{3}$	0		

$\therefore$ The optimal solution is

$x_1 = 0, \; x_2 = \frac{380}{9} \; ; \; x_3 = \frac{470}{3}$

& $Z_{max} = \frac{3200}{3}$

E-x. Show that the given LPP has an alternate solution.

$$\text{Max } Z = 4x_1 + 10x_2$$

$$\begin{aligned} \text{Sub to} \quad & 2x_1 + 5x_2 \leq 20 \\ & 2x_1 + x_2 \leq 10 \\ & 2x_1 + 3x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Soln standard form:

$$\text{Max } Z = 4x_1 + 10x_2$$

$$\begin{aligned} \text{sub to} \quad & 2x_1 + 5x_2 + s_1 = 20 \\ & 2x_1 + x_2 + s_2 = 10 \\ & 2x_1 + 3x_2 + s_3 = 18, \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

put $x_1 = 0, \; x_2 = 0$

$\Rightarrow \; s_1 = 20, \; s_2 = 10, \; s_3 = 18$

C_B	Basis	x_1 (4)	x_2 (10)	s_1 (0)	s_2 (0)	s_3 (0)	Sol^n	Min Ratio
	c_j	4	10	0	0	0		
0	s_1	2	(5)	1	0	0	20	4 →
0	s_2	2	1	0	1	0	10	10
0	s_3	2	3	0	0	1	18	6
	z_j	0	0	0	0	0		
	$c_j - z_j$	4	10 ↑	0	0	0		
10	x_2	2/5	1	1/5	0	0	4	10
0	s_2	(8/5)	0	-1/5	1	0	6	15/4 →
0	s_3	4/5	0	-3/5	0	1	6	15/2
	z_j	4	10	2	0	0		
	$c_j - z_j$	0 ↑	0	-2	0	0		

Since all $c_j - z_j \leq 0$; $\Rightarrow$ optimality is achieved &

$x_1 = 0$, $x_2 = 4$ is an optimal solution.

Since $c_j - z_j = 0$ under x_1 which is currently a non-basic variable
$\Rightarrow$ There exist an alternate optimal solution.

C_B	Basis	x_1	x_2	s_1	s_2	s_3	Sol^n	
10	x_2	0	1	1/4	-1/4	0	5/2	
4	x_1	1	0	-1/8	5/8	0	15/4	
0	s_3	0	0	-1/2	-1/2	1	3	
	z_j	4	10	2	0	0		
	$c_j - z_j$	0	0	-2	0	0		

$\Rightarrow$ $x_1 = \frac{15}{4}$, $x_2 = 5/2$ is another optimal solution.

Thus; $X = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $Y = \begin{bmatrix} \frac{15}{4} \\ 5/2 \end{bmatrix}$ are two optimal solution of given LPP.

$$\Rightarrow (1-t) X + t Y$$

$$= (1-t) \begin{bmatrix} 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} \frac{15}{4} \\ 5/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{15}{4} t \\ 4 - 3/2\, t \end{bmatrix}$$

i.e. $x_1 = \frac{15}{4} t$, $x_2 = 4 - 3/2\, t$ for $0 \le t \le 1$ are also optimal solutions

$\Rightarrow$ There are infinite numbers of optimal solutions.

& $Z_{max} = 40$.

Ex. Max $Z = 4x_1 + x_2 + 3x_3 + 5x_4$

sub to $4x_1 - 6x_2 - 5x_3 - 4x_4 \geq -20$

$-3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$-8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

Sol^n Standard form is

Max $Z = 4x_1 + x_2 + 3x_3 + 5x_4$

sub to $-4x_1 + 6x_2 + 5x_3 + 4x_4 + S_1 = 20$

$-3x_1 - 2x_2 + 4x_3 + x_4 + S_2 = 10$

$-8x_1 - 3x_2 + 3x_3 + 2x_4 + S_3 = 20$

$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$

$\therefore$ For $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$; $S_1 = 20, S_2 = 10, S_3 = 20$

	C_j	4	1	3	5	0	0	0		
C_B	Basis	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Sol^n	Min Ratio
0	S_1	-4	6	5	(4)	1	0	0	20	5 $\rightarrow$
0	S_2	-3	-2	4	1	0	1	0	10	10
0	S_3	-8	-3	3	2	0	0	1	20	10
	Z_j	0	0	0	0	0	0	0		
	$C_j - Z_j$	4	1	3	5 $\uparrow$	0	0	0		
5	x_4	-1	$3/2$	$5/4$	1	$1/4$	0	0	5	-5
0	S_2	-2	$-7/2$	$11/4$	0	$-1/4$	1	0	5	$-5/2$
0	S_3	-6	-6	$1/2$	0	$-1/2$	0	1	10	$-5/3$
	Z_j	-5	$15/2$	$25/4$	5	$5/4$	0	0		
	$C_j - Z_j$	9 $\uparrow$	$-13/2$	$-13/4$	0	$-5/4$	0	0		

Since all replacement ratios are Negative

$\Rightarrow$ x_1 can be increased infinitely without violating any constraints

$\Rightarrow$ Problem has an unbounded solution.

Artificial Variable Techniques:

There are many LPP where slack variables cannot provide the initial feasible solution easily. In these problems at least one of the constraints is of $=$ or $\geq$ type. Then we adopt the following method

The Big M Method:

1. Express the LPP in the standard form.
2. Add a non negative variable to left hand side of all the constraints of $=$ or $\geq$ type. These variables (say) $A_1, A_2, \ldots$ are called **artificial variables.**
3. Assign a large penalty $-M$ **for maximization problem** and $+M$ **for minimization problems.** $M > 0$ to these variables in the objective function.
4. Carry out the usual simplex method.

While making iteration

Case1. If **no artificial variable remains in the basis** and the optimality conditions is satisfied, then the **solution is an optimal feasible** solution.

Case2. If at least one **artificial variable appears in the basis with zero value in the solution column** and optimality condition is satisfied, then the **solution is an optimal solution and degenerate.**

Case3. If at least one **artificial variable appears in the basis with non zero value in the solution column** and optimality conditions is satisfied, then the LPP has **no optimal feasible solution**.

The solution satisfies the constraints but does no optimizes the objective function since it contains a very large penalty M and is called **Pseudo optimal solution.**

Note: An artificial variable once driven out of the solution can be omitted from further consideration in the succeeding tableaux.

The Big M-Method

$$\text{Max } Z = 2x_1 + 3x_2 + 4x_3$$

sub to
$$3x_1 + x_2 + 4x_3 \le 600$$
$$2x_1 + 4x_2 + 2x_3 \ge 480$$
$$2x_1 + 3x_2 + 3x_3 = 540$$
$$x_1, x_2, x_3 \ge 0$$

Sol^n ~~standard form is~~ Introducing slacks, & Artificial variable, Standard form is

$$\text{Max } Z = 2x_1 + 3x_2 + 4x_3 - MA_1 - MA_2$$

sub to
$$3x_1 + x_2 + 4x_3 + S_1 = 600$$
$$2x_1 + 4x_2 + 2x_3 - S_2 + A_1 = 480$$
$$2x_1 + 3x_2 + 3x_3 + A_2 = 540$$
$$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \ge 0$$

	C_j	2	3	4	0	0	$-M$	$-M$		
	Basis	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Sol^n	Min Ratio
0	S_1	3	1	4	1	0	0	0	600	600
$-M$	A_1	2	(4)	2	0	-1	1	0	480	120 →
$-M$	A_2	2	3	3	0	0	0	1	540	180
	Z_j	$-4M$	$-7M$	$-5M$	0	M	$-M$	$-M$		
	$C_j - Z_j$	$2+4M$	$3+7M$	$4+5M$	0	$-M$	0	0		
			↑							
0	S_1	$5/2$	0	$7/2$	1	$1/4$	—	0	480	$960/7$
3	x_2	$1/2$	1	$1/2$	0	$-1/4$	—	0	120	240
$-M$	A_2	$1/2$	0	$(3/2)$	0	$3/4$	—	1	180	120 →
	Z_j	$\frac{3}{2}-\frac{M}{2}$	3	$\frac{3}{2}-\frac{3M}{2}$	0	$-\frac{3}{4}-\frac{3M}{4}$	—	$-M$		
	$C_j - Z_j$	$\frac{1}{2}+\frac{M}{2}$	0	$\frac{5}{2}+\frac{3M}{2}$	0	$\frac{3}{4}+\frac{3M}{4}$	—	0		

		x_1	x_2	x_3	s_1	s_2	A_1	A_2	Soln
0	s_1	$4/3$	0	0	1	$-3/2$	—	—	60
3	x_2	$1/3$	1	0	0	$-1/2$	—	—	60
4	x_3	$1/3$	0	1	0	$1/2$	—	—	120
	z_j	$7/3$	3	4	0	$1/2$			
	$c_j - z_j$	$-1/3$	0	0	0	$-1/2$			

$\therefore$ optimal solution is

$x_1 = 0,\ x_2 = 60,\ x_3 = 120;$

$Z_{max} = 660$

(2) $\text{Max } Z = x_1 + 2x_2 + 3x_3 - x_4$

Sub to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Soln Introducing artificial variables. standard form is

$\text{Max } Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

Sub to $x_1 + 2x_2 + 3x_3 + A_1 = 15$

$2x_1 + x_2 + 5x_3 + A_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4, A_1, A_2 \geq 0$

C_j	Basis	1 x_1	2 x_2	3 x_3	-1 x_4	-M A_1	-M A_2	Soln	Min Ratio
-M	A_1	1	2	3	0	1	0	15	5
-M	A_2	2	1	(5)	0	0	1	20	4 →
-1	x_4	1	2	1	1	0	0	10	10
	Z_j	$-1-3M$	$-2-3M$	$-1-8M$	-1	-M	-M		
	C_j-Z_j	$1+3M$	$2+3M$	$1+8M$ ↑	0	0	0		
-M	A_1	$-1/5$	$(7/5)$	0	0	1	–	3	$15/7$ →
3	x_3	$2/5$	$1/5$	1	0	0	–	4	20
-1	x_4	$3/5$	$9/5$	0	1	0	–	6	$\frac{10}{3}$
	Z_j	$\frac{3}{5}+\frac{M}{5}$	$\frac{6}{5}-\frac{7}{5}M$	3	-1	-M			
	C_j-Z_j	$-\frac{2}{5}-\frac{M}{5}$	$-\frac{6}{5}+\frac{7}{5}M$ ↑	0	0	0	–		
2	x_2	$-1/7$	1	0	0	–	–	$15/7$	∞
3	x_3	$3/7$	0	1	0	–	–	$27/7$	∞
-1	x_4	$(6/7)$	0	0	1	–	–	$15/7$	$\frac{15}{7}$ →
	Z_j	$1/7$	2	3	-1	–	–		
	C_j-Z_j	$\frac{6}{7}$ ↑	0	0	0				
2	x_2	0	1	0	$1/6$	–	–	$5/2$	
3	x_3	0	0	1	$-1/2$	–	–	$5/2$	
1	x_1	1	0	0	$7/6$	–	–	$5/2$	
	Z_j	1	2	3	0				
	C_j-Z_j	0	0	0	-1				

$\Rightarrow x_1 = 5/2,\ x_2 = 5/2,\ x_3 = 5/2,\ x_4 = 0$

is the optimal solution.

$Z_{max} = 15$

E-x. Min $Z = x_1 + 2x_2 + x_3$

sub to $2x_1 + x_2 + x_3 \le 2$

$3x_1 + 4x_2 + 2x_3 \ge 16$

$x_1, x_2, x_3 \ge 0$

Introducing the slacks and Artificial variable; standard form is

Min $Z = x_1 + 2x_2 + x_3 + M A_1$

sub to $2x_1 + x_2 + x_3 + s_1 = 2$

$3x_1 + 4x_2 + 2x_3 - s_2 + A_1 = 16$

$x_1, x_2, x_3, s_1, s_2, A_1 \ge 0$.

put $x_1 = 0, x_2 = 0, x_3 = 0, s_2 = 0,$

$\Rightarrow s_1 = 2, A_1 = 16$

C_j		1	2	1	0	0	M	Sol^n	Min Ratio
	Basis	x_1	x_2	x_3	S_1	S_2	A_1		
0	S_1	2	(1)	1	1	0	0	2	$2 \rightarrow$
M	A_1	3	4	2	0	-1	1	16	4
	Z_j	3M	4M	2M	0	-M	M		
	$C_j - Z_j$	1-3M	2-4M	1-2M	0	M	0		
			$\uparrow$						
2	x_2	2	1	1	1	0	0	2	
M	A_1	-5	0	-2	-4	-1	1	8	
	Z_j	4-5M	2	2-2M	2-4M	-M	M		
	$C_j - Z_j$	-3+5M	0	-1+2M	-2+4M	M	0		

Since all $C_j - Z_j \geq 0$; optimality is reached for minimization problem.

But A_1 appears in the Basis with positive value $A_1 = 8$; the ~~the~~ problem has <u>infeasible solution</u> or <u>pseudo optimal solution</u>.