CS 725: Quiz 0 Solution

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Question 1

$\frac{1}{2} \times 0.2 + \frac{1}{4} \times 0.3 + \frac{1}{4} \times 0.5 = 0.3$

Question 2

$\text{rank}(\mathbf{P}^T\mathbf{P}) \leq \min\{\text{rank}(\mathbf{P}^T), \text{rank}(\mathbf{P})\} \leq 2$ which is less than full rank, hence not invertible.

Question 3

Given, $g(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{P}(0.3 \leq X \leq 0.8) = \int_{0.3}^{0.8} 3x^2 \, dx = 0.8^3 - 0.3^3 = 0.485$

Question 4

$$
\begin{aligned}
&\mathbb{E}[y] = 3 \\
&\Rightarrow 0 \times \frac{1}{5} + 2 \times \frac{1}{5} + 4 \times \frac{1}{5} + 6 \times \frac{1}{5} + q \times \frac{1}{5} = 3 \\
&\Rightarrow \frac{1}{5} \times (2 + 4 + 6 + q) = 3 \\
&\Rightarrow 12 + q = 15 \\
&\Rightarrow q = 3
\end{aligned}
$$

Question 5

$$
\begin{aligned}
P(X, Y) &= P(X|Y)P(Y) \\
&= P(X)P(Y|X) \\
P(Y|X) &= \frac{P(X|Y)P(Y)}{P(X)} \\
P(Y|X) &= \frac{P(Y|X)P(X)}{P(Y)}
\end{aligned}
$$

If X and Y are independent, then $P(X|Y) = P(X)$.

Question 6

With $\frac{1}{2}$ probability, X is doubled and with $\frac{1}{2}$ probability, X is halved. Since X is initially 2 and consequently, doubled or halved, the smallest value of X which is ≥ 1000 is 1024.
The problem is similar to Gambler's Ruin (refer Random Walks).
Let P_n be the probability that X reaches 1024 before we reach 1, given that we started with $n = 2$.
For different values of n as starting value, we have:

$$\begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n = 1024 \\ \frac{1}{2}P_{2n} + \frac{1}{2}P_{\frac{n}{2}} & \text{if } 0 \leq n \leq 1024 \end{cases}$$

Since n is a power of 2, we can represent $n = 2^m$. Rewriting the above equation, we get:

$$\begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m = 10 \\ \frac{1}{2}P_{m+1} + \frac{1}{2}P_{m-1} & \text{if } 0 \leq m \leq 10 \end{cases}$$

We solve case 3 as a recurrence relation and obtain:

$$\frac{1}{2}P_{m+1} + \frac{1}{2}P_{m-1} - P_m = 0 \qquad \text{where } P_0 = 0 \text{ and } P_{10} = 10.$$

We solve this linear homogeneous equation by solving the characteristic equation:

$$\begin{aligned} \frac{1}{2}r^2 - r + \frac{1}{2} &= 0 \\ (r-1)^2 &= 0 \end{aligned}$$

The solution to the above characteristic equation is a double root $(r = 1)$, hence

$$P_m = Am(1)^m + B(1)^m = Am + B \tag{1}$$

We know that, $P_0 = 0$ and when $m = 0$, $P_0 = B$, hence $B = 0$. Also,

$$P_{10} = 1 = 10A + B = 10A$$

This gives us $A = \frac{1}{10}$. Putting values of A and B in equation 1, we get $P_m = \frac{1}{10} * m + 0$. Since we started with $n = 2$ i.e. $m = 1$, we get $P_m = \frac{1}{10}$.

Question 7

$$\begin{aligned} A^2 &= A \\ A^2 - A &= 0 \\ A(A - I) &= 0 \end{aligned}$$

Question 8

$$\begin{aligned}
\text{rank}(NM) &= \text{rank}((NM)^T \\
&= \text{rank}(M^T N^T) \\
&= \text{rank}(MN)
\end{aligned}$$

Question 9

Vector space for b is $\mathbb{R}^7$ while $rank(A) < 7$. So, there exists b having no solution.

Question 10

$$\begin{aligned}
(AA^{-1})^T &= I \\
(A^{-1})^T A^T &= I \\
(A^{-1})^T &= (A^T)^{-1}
\end{aligned}$$