Poisson Distribution

a) $p(X/\lambda) = \dfrac{e^{-\lambda} \times \lambda^{x}}{x!}$

$p(y;\eta) = b(y) \exp(\eta^T T(y) - a(\eta))$

$p(X/\lambda) = \dfrac{e^{\ln \lambda^{x}} \cdot e^{-\lambda}}{x!}$

$= \dfrac{1}{x!} \exp(x \ln \lambda - \lambda)$

$b(x) = \dfrac{1}{x!}$, $T(x) = x$, $\eta^T = \ln \lambda$, $a(\eta) = \lambda$

$\eta = \ln \lambda \rightarrow \lambda = e^{\eta}$

$a(\eta) = \lambda = e^{\eta}$

Cannonical Response Function $g(\eta) = a'(\eta) = e^{\eta}$

$E(T(y)) = g(\eta) = e^{\eta} = \lambda$

b) $y = \{1, 2 \cdots k\}$

$$T(y)_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad T(y)_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \cdots \quad T(y)_{k-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$p(y=1; \phi) = \phi_1 \quad , p(y=2; \phi) = \phi_2 \cdots p(y=k-1; \phi) = \phi_{k-1}$$

$$p(y=k; \phi) = 1 - \sum_{i=1}^{k-1} \phi_i$$

$$p(y; \phi) = \phi_1^{T(y)_1} \cdot \phi_2^{T(y)_2} \cdot \phi_3^{T(y)_3} \cdots \phi_k^{1 - \sum_i^{k-1} T(y)_i}$$

$$= \exp\left(T(y)_1 \ln \phi_1 + T(y)_2 \ln \phi_2 \cdots \left(1 - \sum_1^{k-1} T(y)_i\right) \ln \phi_k \right)$$

$$= \exp\left(T(y)_1 \ln\left(\frac{\phi_1}{\phi_k}\right) + T(y)_2 \ln\left(\frac{\phi_2}{\phi_k}\right) \cdots + \ln \phi_k \right)$$

$$= b(y) \exp(\eta^T T(y) - a(\eta))$$

$$b(y) = 1 \quad , \eta^T = \left[\ln\left(\frac{\phi_1}{\phi_k}\right) \quad \ln\left(\frac{\phi_2}{\phi_k}\right) \cdots \ln\left(\frac{\phi_{k-1}}{\phi_k}\right) \right]$$

$$a(\eta) = -\ln(\phi_k)$$

$$\eta_i = \ln\left(\frac{\phi_i}{\phi_k}\right) \rightarrow \phi_i = \phi_k e^{\eta_i}$$

$$\phi_k \sum_1^k e^{\eta_i} = \sum_1^k \phi_i = 1$$

$$\phi_k = \frac{1}{\sum_1^k e^{\eta_i}}$$

So, $$\phi_i = \frac{e^{\eta_i}}{\sum_{j=1}^{k} e^{\eta_j}}$$

$$p(y=i/\phi) = \phi_i = \frac{e^{\eta_i}}{\sum_{j=1}^{k} e^{\eta_j}}$$

$$p(y/\phi) = \prod_{i=1}^{k} \phi_i^{T(y)_i}$$

$$l(\theta) = \sum_{i=1}^{k} T(y)_i \phi_i$$