CSIR-UGC NET-June 2013-Problem(72)

Yashas Tadikamalla

AI20BTECH11027

Convergence in Probability

A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in probability to a random variable X, if

$$\lim_{n\to\infty} \Pr\left(|X_n - X| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{1}$$

Notation : $X_n \xrightarrow{p} X$

Convergence in Distribution

A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in distribution to a random variable X, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{2}$$

for all x at which $F_X(x)$ is continuous.
Notation : $X_n \xrightarrow{d} X$

Random Sampling

A collection of random variables $X_1, X_2, \ldots, X_n$ is said to be a random sample of size n if they are independent and identically distributed, i.e,

1. $X_1, X_2, \ldots, X_n$ are independent random variables
2. They have the same distribution (Let us denote it by $F_X(x)$), i.e,

$$F_X(x) = F_{X_1}(x) = F_{X_2}(x) = \cdots = F_{X_n}(x), \forall x \in \mathbb{R} \tag{3}$$

Order Statistics

Given a random sample $X_1, X_2, \ldots, X_n$, the sequence $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is called the order statistics of it. Here,

$$X_{(1)} = \min\left(X_1, X_2, \ldots, X_n\right) \tag{4}$$

$$X_{(2)} = \text{the } 2^{nd} \text{ smallest of } X_1, X_2, \ldots, X_n \tag{5}$$

$$\vdots \tag{6}$$

$$X_{(n)} = \max\left(X_1, X_2, \ldots, X_n\right) \tag{7}$$

Distribution of the maximum

Let's calculate the CDF, PDF of $X_{(n)}$

$$F_{X_{(n)}}(x) = \Pr\left(X_{(n)} \le x\right) = \Pr\left(X_1 \le x, X_2 \le x, \ldots, X_n \le x\right) \quad (8)$$

$$= \Pr\left(X_1 \le x\right)\Pr\left(X_2 \le x\right)\ldots\Pr\left(X_n \le x\right)\left(\because \text{independence}\right) \quad (9)$$

$$= \left[\Pr\left(X_1 \le x\right)\right]^n \left(\because \text{identical distribution}\right) = \left[F_X(x)\right]^n \quad (10)$$

$$f_{X_{(n)}}(x) = \frac{d}{dx}\left(F_{X_{(n)}}(x)\right) = \frac{d}{dx}\left(\left[F_X(x)\right]^n\right) \quad (11)$$

$$= n\left(\left[F_X(x)\right]^{n-1}\right)\frac{d}{dx}\left(F_X(x)\right) \quad (12)$$

$$= n\left[F_X(x)\right]^{n-1} f_X(x)\left(\because \frac{d}{dx}\left(F_X(x)\right) = f_X(x)\right) \quad (13)$$

Uniform Distribution

A continuous random variable X is said to have a Uniform Distribution over the interval (a,b), shown as $X \sim Uniform(a, b)$, if,

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}, F_X(x) = \begin{cases} x, & a < x < b \\ 1, & x \geq b \\ 0, & otherwise \end{cases} \tag{14}$$

Exponential Distribution

A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$, shown as $X \sim Exponential(\lambda)$, if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}, F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases} \tag{15}$$

Question

CSIR-UGC NET-June 2013-Problem(72)

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables each following a uniform distribution on (0,1). Denote $T_n = \max(X_1, X_2, \ldots, X_n)$. Then, which of the following statements are true?

1. T_n converges to 1 in probability.
2. $n(1 - T_n)$ converges in distribution.
3. $n^2(1 - T_n)$ converges in distribution.
4. $\sqrt{n}(1 - T_n)$ converges to 0 in probability.

Solution

Given, $X_1, X_2, X_3, \ldots$ are independent and identically distributed random variables each following a uniform distribution on (0,1). Let us denote their PDF, CDF by $f_X(x), F_X(x)$ respectively. Then,

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases} \tag{16}$$

$$F_X(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & otherwise \end{cases} \tag{17}$$

Solution Contd.

As $T_n = \max(X_1, X_2, \ldots, X_n) = X_{(n)}$, from (13)

$$f_{T_n}(x) = \begin{cases} nx^{n-1}, & 0 < x < 1 \\ 0, & otherwise \end{cases} \tag{18}$$

Also, from (10)

$$F_{T_n}(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & otherwise \end{cases} \tag{19}$$

Note

Let X be a continuous random variable. If $Y = aX + b$ and $a < 0$, then

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \tag{20}$$

Proof

$$F_Y(y) = \Pr(Y \leq y) = \Pr(aX + b \leq y) \tag{21}$$

$$= \Pr\left(X \geq \left(\frac{y-b}{a}\right)\right) (\because a < 0) \tag{22}$$

$$= 1 - \Pr\left(X \leq \left(\frac{y-b}{a}\right)\right) + \Pr\left(X = \left(\frac{y-b}{a}\right)\right) \tag{23}$$

$$= 1 - F_X\left(\frac{y-b}{a}\right) + 0 = 1 - F_X\left(\frac{y-b}{a}\right) \tag{24}$$

Option 1

Consider the sequence of random variables $X_1, X_2, X_3, \ldots,$ such that $X_n = T_n$. From (1), we need to evaluate $\lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right), \forall \epsilon > 0$

$$\lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = \lim_{n\to\infty} \Pr\left(1 - T_n \geq \epsilon\right) \tag{25}$$

$$= \lim_{n\to\infty} \Pr\left(T_n \leq 1 - \epsilon\right) = \lim_{n\to\infty} F_{T_n}(1-\epsilon) \tag{26}$$

$$F_{T_n}(1-\epsilon) = \begin{cases} (1-\epsilon)^n, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \tag{27}$$

$$\because \lim_{n\to\infty} (1-\epsilon)^n = 0 \text{ for } 0 < \epsilon < 1 \tag{28}$$

$$\therefore \lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{29}$$

$\therefore T_n$ converges to 1 in probability.

Option 2

Consider the sequence of random variables $X_1, X_2, X_3, \ldots,$ such that $X_n = n(1 - T_n)$. From (2), we need to evaluate $\lim_{n\to\infty} F_{n(1-T_n)}(x)$. Substituting $a = -n, b = n$ in (20),

$$F_{n(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n}\right) \tag{30}$$

$$F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} \left(1 - \frac{x}{n}\right)^n, & 0 < x < n \\ 1, & x \le 0 \\ 0, & x \ge n \end{cases} \tag{31}$$

$$\because \lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n = e^{-x} \tag{32}$$

Option 2 Contd.

$$\therefore \lim_{n\to\infty} F_{T_n}\left(1-\frac{x}{n}\right) = \begin{cases} e^{-x}, & x>0 \\ 1, & x\leq 0 \end{cases} \tag{33}$$

$$\because \lim_{n\to\infty} F_{n(1-T_n)}(x) = 1 - \lim_{n\to\infty} F_{T_n}\left(1-\frac{x}{n}\right) \tag{34}$$

$$\therefore \lim_{n\to\infty} F_{n(1-T_n)}(x) = \begin{cases} 1-e^{-x}, & x>0 \\ 0, & x\leq 0 \end{cases} \tag{35}$$

$\because$ The CDF in (35) represents an exponential distribution with $\lambda = 1$
$\therefore n(1-T_n)$ converges in distribution to the random variable
$X \sim Exponential(1)$.

Option 2 Contd.

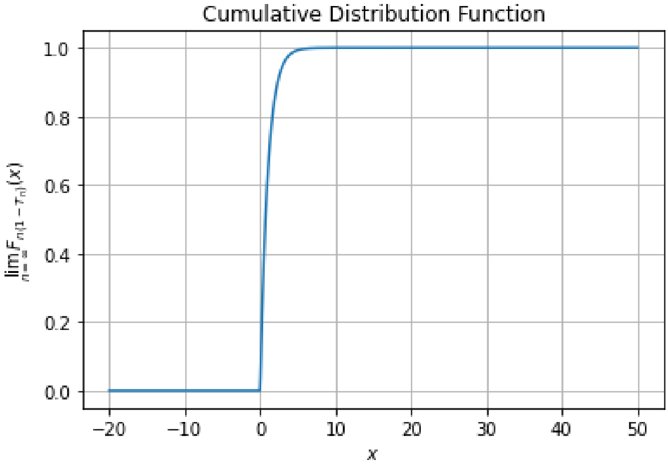

Figure: Plot for the CDF defined in (35)

Option 3

Consider the sequence of random variables $X_1, X_2, X_3, \dots,$ such that $X_n = n^2(1 - T_n)$. From (2), we need to evaluate $\lim_{n\to\infty} F_{n^2(1-T_n)}(x)$. Substituting $a = -n^2, b = n^2$ in (20),

$$F_{n^2(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n^2}\right) \tag{36}$$

$$F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} \left(1 - \frac{x}{n^2}\right)^n, & 0 < x < n^2 \\ 1, & x \leq 0 \\ 0, & x \geq n^2 \end{cases} \tag{37}$$

$$\because \lim_{n\to\infty}\left(1 - \frac{x}{n^2}\right)^n = 1 \tag{38}$$

Option 3 Contd.

$$\therefore \lim_{n\to\infty} F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} 1, & x > 0 \\ 1, & x \leq 0 \end{cases} \tag{39}$$

$$\because \lim_{n\to\infty} F_{n^2(1-T_n)}(x) = 1 - \lim_{n\to\infty} F_{T_n}\left(1 - \frac{x}{n^2}\right) \tag{40}$$

$$\therefore \lim_{n\to\infty} F_{n^2(1-T_n)}(x) = \begin{cases} 0, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{41}$$

$\because$ The CDF in (41) is not valid,
$\therefore$ $n^2(1 - T_n)$ does not converge in distribution.

Option 4

Consider the sequence of random variables $X_1, X_2, X_3, \ldots,$ such that $X_n = \sqrt{n}(1 - T_n)$. From (1), we need to evaluate $\lim_{n\to\infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon\right), \forall \epsilon > 0$

$$\lim_{n\to\infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon\right) = \lim_{n\to\infty} \Pr\left(1 - T_n \geq \frac{\epsilon}{\sqrt{n}}\right) \tag{42}$$

$$= \lim_{n\to\infty} \Pr\left(T_n \leq 1 - \frac{\epsilon}{\sqrt{n}}\right) = \lim_{n\to\infty} F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) \tag{43}$$

$$F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) = \begin{cases} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n, & 0 < \epsilon < \sqrt{n} \\ 0, & \epsilon \geq \sqrt{n} \end{cases} \tag{44}$$

Option 4 Contd.

$$\because \lim_{n\to\infty} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n = 0 \text{ for } 0 < \epsilon < \sqrt{n} \tag{45}$$

$$\therefore \lim_{n\to\infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{46}$$

$\therefore \sqrt{n}(1 - T_n)$ converges to 0 in probability.

Solution Contd.

Therefore, options 1), 2), 4) are correct.