AI1103 : Assignment 9

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Download all python codes from

https://github.com/YashasTadikamalla/AI1103/tree/main/Assignment9/codes

and latex codes from

https://github.com/YashasTadikamalla/AI1103/blob/main/Assignment9/Assignment9.tex

CSIR-UGC NET-June 2013-Problem(72)

Let $X_1, X_2, \dots$ be independent and identically distributed random variables each following a uniform distribution on (0,1). Denote $T_n = max\{X_1, X_2, \dots, X_n\}$. Then, which of the following statements are true?

1) T_n converges to 1 in probability.
2) $n(1 - T_n)$ converges in distribution.
3) $n^2(1 - T_n)$ converges in distribution.
4) $\sqrt{n}(1 - T_n)$ converges to 0 in probability.

CSIR-UGC NET-June 2013-Solution(72)

Random Sampling :
A collection of random variables $X_1, X_2, \dots, X_n$ is said to be a random sample of size n if they are independent and identically distributed, i.e,

1) $X_1, X_2, \dots, X_n$ are independent random variables
2) They have the same distribution (Let us denote it by $F_X(x)$), i.e,

$$F_X(x) = F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_n}(x), \forall x \in \mathbb{R} \tag{0.0.1}$$

Order Statistics :
Given a random sample $X_1, X_2, \dots, X_n$, the sequence $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ is called the order statistics of it. Here,

$$X_{(1)} = \min(X_1, X_2, \dots, X_n) \tag{0.0.2}$$
$$X_{(2)} = \text{the } 2^{nd} \text{ smallest of } X_1, X_2, \dots, X_n \tag{0.0.3}$$
$$\vdots \tag{0.0.4}$$
$$X_{(n)} = \max(X_1, X_2, \dots, X_n) \tag{0.0.5}$$

Distribution of the maximum :
Let's calculate the CDF, PDF of $X_{(n)}$

$$F_{X_{(n)}}(x) = \Pr(X_{(n)} \le x) \tag{0.0.6}$$
$$= \Pr(X_1 \le x, X_2 \le x, \dots, X_n \le x) \tag{0.0.7}$$
$$= \Pr(X_1 \le x)\Pr(X_2 \le x)\dots\Pr(X_n \le x) \tag{0.0.8}$$
$$= [\Pr(X_1 \le x)]^n \ (\because \text{identical distribution}) \tag{0.0.9}$$
$$= [F_X(x)]^n \tag{0.0.10}$$

$$f_{X_{(n)}}(x) = \frac{d}{dx}\left(F_{X_{(n)}}(x)\right) = \frac{d}{dx}\left([F_X(x)]^n\right) \tag{0.0.11}$$
$$= n\left([F_X(x)]^{n-1}\right)\frac{d}{dx}(F_X(x)) \tag{0.0.12}$$
$$= n[F_X(x)]^{n-1} f_X(x)\left(\because \frac{d}{dx}(F_X(x)) = f_X(x)\right) \tag{0.0.13}$$

The PDF, CDF of each $X_1, X_2, X_3, \dots$ is

$$f_{X_i}(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases}, F_{X_i}(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & x \ge 1 \\ 0, & otherwise \end{cases} \tag{0.0.14}$$

$\forall i \in \mathbb{N}$. Then, as $T_n = max\{X_1, X_2, \dots, X_n\} = X_{(n)}$,

$$f_{T_n}(x) = \begin{cases} nx^{n-1}, & 0 < x < 1 \\ 0, & otherwise \end{cases} \tag{0.0.15}$$

$$F_{T_n}(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & x \ge 1 \\ 0, & otherwise \end{cases} \tag{0.0.16}$$

NOTE : If $Y = aX + b$ and $a < 0$, then

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \tag{0.0.17}$$

1) OPTION-1:
Convergence in Probability :
A sequence of random variables $X_1, X_2, X_3, \dots$

converges in probability to a random variable X, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n\to\infty} \Pr\left(|X_n - X| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{0.0.18}$$

To evaluate : $\lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right), \forall \epsilon > 0$

$$\lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = \lim_{n\to\infty} \Pr\left(1 - T_n \geq \epsilon\right) \tag{0.0.19}$$

$$= \lim_{n\to\infty} \Pr\left(T_n \leq 1 - \epsilon\right) = \lim_{n\to\infty} F_{T_n}(1-\epsilon) \tag{0.0.20}$$

$$F_{T_n}(1-\epsilon) = \begin{cases} (1-\epsilon)^n, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \tag{0.0.21}$$

$$\because \lim_{n\to\infty} (1-\epsilon)^n = 0 \text{ for } 0 < \epsilon < 1 \tag{0.0.22}$$

$$\therefore \lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{0.0.23}$$

$\therefore T_n$ converges to 1 in probability.

2) OPTION-2:
Convergence in Distribution :
A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{0.0.24}$$

for all x at which $F_X(x)$ is continuous.
To evaluate : $\lim_{n\to\infty} F_{n(1-T_n)}(x)$
Substituting $a = -n, b = n$ in (0.0.17),

$$F_{n(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n}\right) \tag{0.0.25}$$

$$F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} \left(1 - \frac{x}{n}\right)^n, & 0 < x < n \\ 1, & x \leq 0 \\ 0, & x \geq n \end{cases} \tag{0.0.26}$$

$$\because \lim_{n\to\infty} \left(1 - \frac{y}{n}\right)^n = e^{-y} \tag{0.0.27}$$

$$\therefore \lim_{n\to\infty} F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} e^{-x}, & x > 0 \\ 1, & x \leq 0 \end{cases} \tag{0.0.28}$$

$$\because \lim_{n\to\infty} F_{n(1-T_n)}(x) = 1 - \lim_{n\to\infty} F_{T_n}\left(1 - \frac{x}{n}\right) \tag{0.0.29}$$

$$\therefore \lim_{n\to\infty} F_{n(1-T_n)}(x) = \begin{cases} 1 - e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{0.0.30}$$

$\therefore$ $n(1 - T_n)$ converges in distribution to the random variable $X \sim Exponential(1)$.

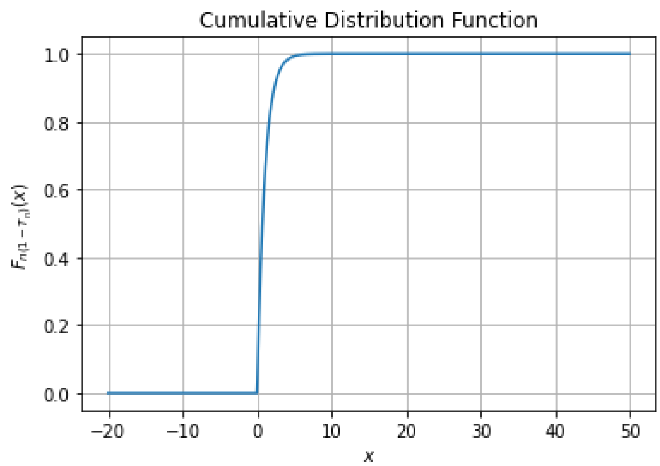

Fig. 1: CDF

3) OPTION-3:
Convergence in Distribution :
A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{0.0.31}$$

for all x at which $F_X(x)$ is continuous.
To evaluate : $\lim_{n\to\infty} F_{n^2(1-T_n)}(x)$
Substituting $a = -n^2, b = n^2$ in (0.0.17),

$$F_{n^2(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n^2}\right) \tag{0.0.32}$$

$$F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} \left(1 - \frac{x}{n^2}\right)^n, & 0 < x < n^2 \\ 1, & x \leq 0 \\ 0, & x \geq n^2 \end{cases} \tag{0.0.33}$$

$$\because \lim_{n\to\infty} \left(1 - \frac{x}{n^2}\right)^n = 1 \tag{0.0.34}$$

$$\therefore \lim_{n\to\infty} F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} 1, & x > 0 \\ 1, & x \leq 0 \end{cases} \tag{0.0.35}$$

$$\because \lim_{n\to\infty} F_{n^2(1-T_n)}(x) = 1 - \lim_{n\to\infty} F_{T_n}\left(1 - \frac{x}{n^2}\right) \tag{0.0.36}$$

$$\therefore \lim_{n\to\infty} F_{n^2(1-T_n)}(x) = \begin{cases} 0, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{0.0.37}$$

$\because$ The CDF in (0.0.37) is not valid,
$\therefore n^2(1 - T_n)$ does not converge in distribution.

4) OPTION-4:
Convergence in Probability :
A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in probability to a random variable X, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n\to\infty} \Pr\left(|X_n - X| \ge \epsilon\right) = 0, \forall \epsilon > 0 \tag{0.0.38}$$

To evaluate :
$\lim_{n\to\infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \ge \epsilon\right), \forall \epsilon > 0$

$$= \lim_{n\to\infty} \Pr\left(1 - T_n \ge \frac{\epsilon}{\sqrt{n}}\right) \tag{0.0.39}$$

$$= \lim_{n\to\infty} \Pr\left(T_n \le 1 - \frac{\epsilon}{\sqrt{n}}\right) \tag{0.0.40}$$

$$= \lim_{n\to\infty} F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) \tag{0.0.41}$$

$$F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) = \begin{cases} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n, & 0 < \epsilon < \sqrt{n} \\ 0, & \epsilon \ge \sqrt{n} \end{cases} \tag{0.0.42}$$

$$\because \lim_{n\to\infty} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n = 0 \text{ for } 0 < \epsilon < \sqrt{n} \tag{0.0.43}$$

$$\therefore \lim_{n\to\infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \ge \epsilon\right) = 0, \forall \epsilon > 0 \tag{0.0.44}$$

$\therefore \sqrt{n}(1 - T_n)$ converges to 0 in probability.

$\boxed{\text{Hence, options 1), 2), 4) are correct.}}$