

AI1103 : Assignment 9

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Download all python codes from

<https://github.com/YashasTadikamalla/AI1103/tree/main/Assignment9/codes>

and latex codes from

<https://github.com/YashasTadikamalla/AI1103/blob/main/Assignment9/Assignment9.tex>

CSIR-UGC NET-JUNE 2013-PROBLEM(72)

Let X_1, X_2, \dots be independent and identically distributed random variables each following a uniform distribution on $(0,1)$. Denote $T_n = \max\{X_1, X_2, \dots, X_n\}$. Then, which of the following statements are true?

- 1) T_n converges to 1 in probability.
- 2) $n(1 - T_n)$ converges in distribution.
- 3) $n^2(1 - T_n)$ converges in distribution.
- 4) $\sqrt{n}(1 - T_n)$ converges to 0 in probability.

CSIR-UGC NET-JUNE 2013-SOLUTION(72)

Random Sampling :

A collection of random variables X_1, X_2, \dots, X_n is said to be a random sample of size n if they are independent and identically distributed, i.e,

- 1) X_1, X_2, \dots, X_n are independent random variables
- 2) They have the same distribution (Let us denote it by $F_X(x)$), i.e,

$$F_X(x) = F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_n}(x), \forall x \in \mathbb{R} \quad (0.0.1)$$

Order Statistics :

Given a random sample X_1, X_2, \dots, X_n , the sequence $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ is called the order statistics of it. Here,

$$X_{(1)} = \min(X_1, X_2, \dots, X_n) \quad (0.0.2)$$

$$X_{(2)} = \text{the } 2^{\text{nd}} \text{ smallest of } X_1, X_2, \dots, X_n \quad (0.0.3)$$

$$\vdots \quad (0.0.4)$$

$$X_{(n)} = \max(X_1, X_2, \dots, X_n) \quad (0.0.5)$$

Distribution of the maximum :

Let's calculate the CDF, PDF of $X_{(n)}$

$$F_{X_{(n)}}(x) = \Pr(X_{(n)} \leq x) \quad (0.0.6)$$

$$= \Pr(X_1 \leq x, X_2 \leq x, \dots, X_n \leq x) \quad (0.0.7)$$

$$= \Pr(X_1 \leq x) \Pr(X_2 \leq x) \dots \Pr(X_n \leq x) \quad (0.0.8)$$

$$= [\Pr(X_1 \leq x)]^n (\because \text{identical distribution}) \quad (0.0.9)$$

$$= [F_X(x)]^n \quad (0.0.10)$$

$$f_{X_{(n)}}(x) = \frac{d}{dx} (F_{X_{(n)}}(x)) = \frac{d}{dx} ([F_X(x)]^n) \quad (0.0.11)$$

$$= n [F_X(x)]^{n-1} \frac{d}{dx} (F_X(x)) \quad (0.0.12)$$

$$= n [F_X(x)]^{n-1} f_X(x) \left(\because \frac{d}{dx} (F_X(x)) = f_X(x) \right) \quad (0.0.13)$$

The PDF, CDF of each X_1, X_2, X_3, \dots is

$$f_{X_i}(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}, F_{X_i}(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & \text{otherwise} \end{cases} \quad (0.0.14)$$

$\forall i \in \mathbb{N}$. Then, as $T_n = \max\{X_1, X_2, \dots, X_n\} = X_{(n)}$,

$$f_{T_n}(x) = \begin{cases} nx^{n-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (0.0.15)$$

$$F_{T_n}(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & \text{otherwise} \end{cases} \quad (0.0.16)$$

NOTE : If $Y = aX + b$ and $a < 0$, then

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \quad (0.0.17)$$

1) OPTION-1:

Convergence in Probability :

A sequence of random variables X_1, X_2, X_3, \dots

converges in probability to a random variable X , shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (0.0.18)$$

To evaluate : $\lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon), \forall \epsilon > 0$

$$\lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon) = \lim_{n \rightarrow \infty} \Pr(1 - T_n \geq \epsilon) \quad (0.0.19)$$

$$= \lim_{n \rightarrow \infty} \Pr(T_n \leq 1 - \epsilon) = \lim_{n \rightarrow \infty} F_{T_n}(1 - \epsilon) \quad (0.0.20)$$

$$F_{T_n}(1 - \epsilon) = \begin{cases} (1 - \epsilon)^n, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \quad (0.0.21)$$

$$\therefore \lim_{n \rightarrow \infty} (1 - \epsilon)^n = 0 \text{ for } 0 < \epsilon < 1 \quad (0.0.22)$$

$$\therefore \lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (0.0.23)$$

$\therefore T_n$ converges to 1 in probability.

2) OPTION-2:

Convergence in Distribution :

A sequence of random variables X_1, X_2, X_3, \dots converges in distribution to a random variable X , shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \quad (0.0.24)$$

for all x at which $F_X(x)$ is continuous.

To evaluate : $\lim_{n \rightarrow \infty} F_{n(1-T_n)}(x)$

Substituting $a = -n, b = n$ in (0.0.17),

$$F_{n(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n}\right) \quad (0.0.25)$$

$$F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} \left(1 - \frac{x}{n}\right)^n, & 0 < x < n \\ 1, & x \leq 0 \\ 0, & x \geq n \end{cases} \quad (0.0.26)$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{y}{n}\right)^n = e^{-y} \quad (0.0.27)$$

$$\therefore \lim_{n \rightarrow \infty} F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} e^{-x}, & x > 0 \\ 1, & x \leq 0 \end{cases} \quad (0.0.28)$$

$$\therefore \lim_{n \rightarrow \infty} F_{n(1-T_n)}(x) = 1 - \lim_{n \rightarrow \infty} F_{T_n}\left(1 - \frac{x}{n}\right) \quad (0.0.29)$$

$$\therefore \lim_{n \rightarrow \infty} F_{n(1-T_n)}(x) = \begin{cases} 1 - e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (0.0.30)$$

$\therefore n(1 - T_n)$ converges in distribution to the random variable $X \sim \text{Exponential}(1)$.

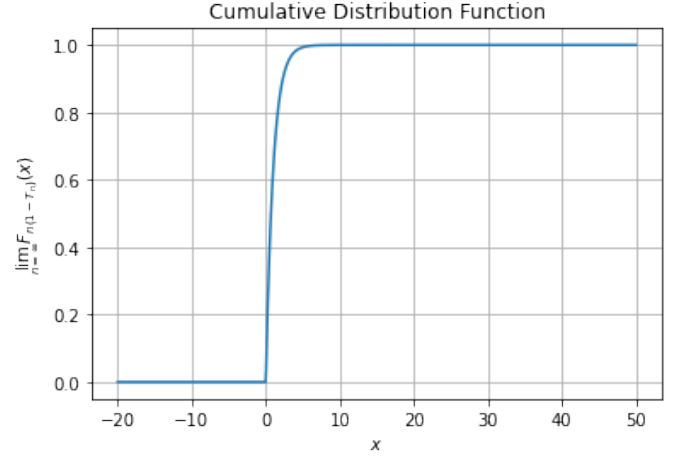


Fig. 1: CDF

3) OPTION-3:

Convergence in Distribution :

A sequence of random variables X_1, X_2, X_3, \dots converges in distribution to a random variable X , shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \quad (0.0.31)$$

for all x at which $F_X(x)$ is continuous.

To evaluate : $\lim_{n \rightarrow \infty} F_{n^2(1-T_n)}(x)$

Substituting $a = -n^2, b = n^2$ in (0.0.17),

$$F_{n^2(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n^2}\right) \quad (0.0.32)$$

$$F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} \left(1 - \frac{x}{n^2}\right)^n, & 0 < x < n^2 \\ 1, & x \leq 0 \\ 0, & x \geq n^2 \end{cases} \quad (0.0.33)$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{x}{n^2}\right)^n = 1 \quad (0.0.34)$$

$$\therefore \lim_{n \rightarrow \infty} F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} 1, & x > 0 \\ 1, & x \leq 0 \end{cases} \quad (0.0.35)$$

$$\therefore \lim_{n \rightarrow \infty} F_{n^2(1-T_n)}(x) = 1 - \lim_{n \rightarrow \infty} F_{T_n} \left(1 - \frac{x}{n^2} \right) \quad (0.0.36)$$

$$\therefore \lim_{n \rightarrow \infty} F_{n^2(1-T_n)}(x) = \begin{cases} 0, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (0.0.37)$$

\therefore The CDF in (0.0.37) is not valid,

$\therefore n^2(1 - T_n)$ does not converge in distribution.

4) OPTION-4:

Convergence in Probability :

A sequence of random variables X_1, X_2, X_3, \dots converges in probability to a random variable X , shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (0.0.38)$$

To evaluate :

$$\lim_{n \rightarrow \infty} \Pr(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon), \forall \epsilon > 0$$

$$= \lim_{n \rightarrow \infty} \Pr\left(1 - T_n \geq \frac{\epsilon}{\sqrt{n}}\right) \quad (0.0.39)$$

$$= \lim_{n \rightarrow \infty} \Pr\left(T_n \leq 1 - \frac{\epsilon}{\sqrt{n}}\right) \quad (0.0.40)$$

$$= \lim_{n \rightarrow \infty} F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) \quad (0.0.41)$$

$$F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) = \begin{cases} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n, & 0 < \epsilon < \sqrt{n} \\ 0, & \epsilon \geq \sqrt{n} \end{cases} \quad (0.0.42)$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n = 0 \text{ for } 0 < \epsilon < \sqrt{n} \quad (0.0.43)$$

$$\therefore \lim_{n \rightarrow \infty} \Pr(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (0.0.44)$$

$\therefore \sqrt{n}(1 - T_n)$ converges to 0 in probability.

Hence, options 1), 2), 4) are correct.