AI1103 : Assignment 9

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Download all python codes from

https://github.com/YashasTadikamalla/AI1103/tree/main/Assignment9/codes

and latex codes from

https://github.com/YashasTadikamalla/AI1103/blob/main/Assignment9/Assignment9.tex

CSIR-UGC NET-June 2013-Problem(72)

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables each following a uniform distribution on (0,1). Denote $T_n = max\{X_1, X_2, \ldots, X_n\}$. Then, which of the following statements are true?

1) T_n converges to 1 in probability.
2) $n(1 - T_n)$ converges in distribution.
3) $n^2(1 - T_n)$ converges in distribution.
4) $\sqrt{n}(1 - T_n)$ converges to 0 in probability.

CSIR-UGC NET-June 2013-Solution(72)

The PDF, CDF of each $X_1, X_2, X_3, \ldots$ is

$$f_{X_i}(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases} \tag{72.1}$$

$$F_{X_i}(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & otherwise \end{cases} \tag{72.2}$$

$\forall i \in \mathbb{N}$. Then, as $T_n = max\{X_1, X_2, \ldots, X_n\}$,

$$f_{T_n}(x) = \begin{cases} nx^{n-1}, & 0 < x < 1 \\ 0, & otherwise \end{cases} \tag{72.3}$$

$$F_{T_n}(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & otherwise \end{cases} \tag{72.4}$$

NOTE : If $Y = aX + b$, then

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) \tag{72.5}$$

1) OPTION-1:
   Convergence in Probability :
   A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in probability to a random variable X, shown by $X_n \xrightarrow{p} X$, if

   $$\lim_{n\to\infty} \Pr\left(|X_n - X| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{72.6}$$

   To evaluate : $\lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right), \forall \epsilon > 0$

   $$\lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = \lim_{n\to\infty} \Pr\left(1 - T_n \geq \epsilon\right) \tag{72.7}$$

   $$= \lim_{n\to\infty} \Pr\left(T_n \leq 1 - \epsilon\right) = \lim_{n\to\infty} F_{T_n}(1-\epsilon) \tag{72.8}$$

   $$F_{T_n}(1-\epsilon) = \begin{cases} (1-\epsilon)^n, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \tag{72.9}$$

   $$\because \lim_{n\to\infty} (1-\epsilon)^n = 0 \text{ for } 0 < \epsilon < 1 \tag{72.10}$$

   $$\therefore \lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{72.11}$$

   $\therefore T_n$ converges to 1 in probability.
2) OPTION-2:
   Convergence in Distribution :
   A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{d} X$, if

   $$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{72.12}$$

   for all x at which $F_X(x)$ is continuous.
   To evaluate : $\lim_{n\to\infty} F_{n(1-T_n)}(x)$
   Substituting $a = -n, b = n$ in (72.5),

   $$F_{n(1-T_n)}(x) = F_{T_n}\left(1 - \frac{x}{n}\right) \tag{72.13}$$

$$F_{T_n}\left(1-\frac{x}{n}\right)=\begin{cases}\left(1-\frac{x}{n}\right)^n, & 0<x<n\\ 1, & x\le 0\\ 0, & x\ge n\end{cases} \tag{72.14}$$

$$\because \lim_{n\to\infty}\left(1-\frac{y}{n}\right)^n = e^{-y} \tag{72.15}$$

$$\therefore \lim_{n\to\infty} F_{T_n}\left(1-\frac{x}{n}\right)=\begin{cases}e^{-x}, & 0<x<n\\ 1, & x\le 0\\ 0, & x\ge n\end{cases} \tag{72.16}$$

$\therefore$ $n(1-T_n)$ converges in distribution to a random variable with CDF in (72.16).

3) OPTION-3:
Convergence in Distribution :
A sequence of random variables $X_1, X_2, X_3, \dots$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{72.17}$$

for all x at which $F_X(x)$ is continuous.
To evaluate : $\lim_{n\to\infty} F_{n^2(1-T_n)}(x)$
Substituting $a=-n^2, b=n^2$ in (72.5),

$$F_{n^2(1-T_n)}(x) = F_{T_n}\left(1-\frac{x}{n^2}\right) \tag{72.18}$$

$$F_{T_n}\left(1-\frac{x}{n^2}\right)=\begin{cases}\left(1-\frac{x}{n^2}\right)^n, & 0<x<n^2\\ 1, & x\le 0\\ 0, & x\ge n^2\end{cases} \tag{72.19}$$

$$\because \lim_{n\to\infty}\left(1-\frac{y}{n^2}\right)^n \text{ is not defined} \tag{72.20}$$

$\therefore n^2(1-T_n)$ does not converge in distribution.

4) OPTION-4:
Convergence in Probability :
A sequence of random variables $X_1, X_2, X_3, \dots$ converges in probability to a random variable X, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n\to\infty} \Pr\left(|X_n - X| \ge \epsilon\right) = 0, \forall \epsilon > 0 \tag{72.21}$$

To evaluate :
$\lim_{n\to\infty} \Pr\left(|\sqrt{n}(1-T_n)-0| \ge \epsilon\right), \forall\epsilon>0$

$$= \lim_{n\to\infty} \Pr\left(1-T_n \ge \frac{\epsilon}{\sqrt{n}}\right) \tag{72.22}$$

$$= \lim_{n\to\infty} \Pr\left(T_n \le 1-\frac{\epsilon}{\sqrt{n}}\right) \tag{72.23}$$

$$= \lim_{n\to\infty} F_{T_n}\left(1-\frac{\epsilon}{\sqrt{n}}\right) \tag{72.24}$$

$$F_{T_n}\left(1-\frac{\epsilon}{\sqrt{n}}\right)=\begin{cases}\left(1-\frac{\epsilon}{\sqrt{n}}\right)^n, & 0<\epsilon<\sqrt{n}\\ 0, & \epsilon\ge\sqrt{n}\end{cases} \tag{72.25}$$

$$\because \lim_{n\to\infty}\left(1-\frac{\epsilon}{\sqrt{n}}\right)^n = 0 \text{ for } 0<\epsilon<\sqrt{n} \tag{72.26}$$

$$\therefore \lim_{n\to\infty} \Pr\left(|\sqrt{n}(1-T_n)-0| \ge \epsilon\right) = 0, \forall\epsilon>0 \tag{72.27}$$

$\therefore$ $\sqrt{n}(1-T_n)$ converges to 0 in probability.

$\boxed{\text{Hence, options 1), 2), 4) are correct.}}$