AI1103 : Challenging Problem 7

Yashas Tadikamalla - AI20BTECH11027

Download latex codes from

https://github.com/YashasTadikamalla/AI1103/blob/main/ChallengingProblem7/ChallengingProblem7.tex

Problem - IES/ISS exam 2015, statistics p1, Q.3(c)

Two points are chosen on a line of unit length. Find the probability that each of the 3 line segments will have length greater than $\frac{1}{4}$?

Solution - IES/ISS exam 2015, statistics p1, Q.3(c)

Let the two points chosen, be represented by X,Y. Let the random variables X, Y denote the distances of the points X,Y from one end. X, Y are i.i.d random variables with a uniform distribution in [0,1].

$$F_X(x) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x < 0 \\ 1, & otherwise \end{cases} \tag{0.0.1}$$

$$F_Y(y) = \begin{cases} y, & 0 \le y \le 1 \\ 0, & y < 0 \\ 1, & otherwise \end{cases} \tag{0.0.2}$$

Without loss of generality, let $Y = max\{X, Y\}$.
To find : $\Pr\left(|X - 0| > \frac{1}{4}, |Y - X| > \frac{1}{4}, |1 - Y| > \frac{1}{4}\right)$
Let us write down the simplified equations,

$$X > \frac{1}{4} \tag{0.0.3}$$

$$Y - X > \frac{1}{4} \tag{0.0.4}$$

$$Y < \frac{3}{4} \tag{0.0.5}$$

Adding (0.0.3),(0.0.4),

$$Y > \frac{1}{2} \tag{0.0.6}$$

Subtracting (0.0.5) from (0.0.4),

$$X < \frac{1}{2} \tag{0.0.7}$$

Therefore,

$$\frac{1}{4} < X < \frac{1}{2} \tag{0.0.8}$$

$$\frac{1}{2} < Y < \frac{3}{4} \tag{0.0.9}$$

Hence, the required probability is

$$\begin{aligned} &\Pr\left(|X - 0| > \frac{1}{4}, |Y - X| > \frac{1}{4}, |1 - Y| > \frac{1}{4}\right) \\ &= \Pr\left(\frac{1}{4} < X < \frac{1}{2}, \frac{1}{2} < Y < \frac{3}{4}\right) \\ &= \Pr\left(\frac{1}{4} < X < \frac{1}{2}\right)\Pr\left(\frac{1}{2} < Y < \frac{3}{4}\right) \end{aligned} \tag{0.0.10}$$

as X, Y are independent.

$$\begin{aligned} \Pr\left(\frac{1}{4} < X < \frac{1}{2}\right) &= F_X\left(\frac{1}{2}\right) - F_X\left(\frac{1}{4}\right) - \Pr\left(X = \frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{4} - 0 = \frac{1}{4} \end{aligned} \tag{0.0.11}$$

$$\begin{aligned} \Pr\left(\frac{1}{2} < Y < \frac{3}{4}\right) &= F_Y\left(\frac{3}{4}\right) - F_X\left(\frac{1}{2}\right) - \Pr\left(Y = \frac{3}{4}\right) \\ &= \frac{3}{4} - \frac{1}{2} - 0 = \frac{1}{4} \end{aligned} \tag{0.0.12}$$

Therefore, the required probability is

$$\begin{aligned} &\Pr\left(\frac{1}{4} < X < \frac{1}{2}\right)\Pr\left(\frac{1}{2} < Y < \frac{3}{4}\right) \\ &= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16} \end{aligned} \tag{0.0.13}$$