

AI1103 : Assignment 8

Yashas Tadikamalla - AI20BTECH11027

Download all python codes from

<https://github.com/YashasTadikamalla/AI1103/tree/main/Assignment8/codes>

and latex codes from

<https://github.com/YashasTadikamalla/AI1103/blob/main/Assignment8/Assignment8.tex>

CSIR-UGC NET-DEC 2016-PROBLEM(104)

A and B play a game of tossing a fair coin. A starts the game by tossing the coin once and B then tosses the coin twice, followed by A tossing the coin once and B tossing the coin twice and this continues until a head turns up. Whoever gets the first head wins the game. Then,

- 1) $P(B \text{ Wins}) > P(A \text{ Wins})$
- 2) $P(B \text{ Wins}) = 2P(A \text{ Wins})$
- 3) $P(A \text{ Wins}) > P(B \text{ Wins})$
- 4) $P(A \text{ Wins}) = 1 - P(B \text{ Wins})$

CSIR-UGC NET-DEC 2016-SOLUTION(104)

Given, a fair coin is tossed till heads turns up.

$$p = \frac{1}{2}, q = \frac{1}{2} \quad (104.1)$$

Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3, 4, 5\}$, such that

TABLE 1: States and their notations

Notation	State
$S = 1$	A 's turn
$S = 2$	B 's first turn
$S = 3$	B 's second turn
$S = 4$	A wins
$S = 5$	B wins

The state transition matrix for the Markov chain is

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \quad (104.2)$$

Clearly, the states 1, 2, 3 are transient, while 4, 5 are absorbing. The standard form of a state transition matrix is

$$P = \begin{matrix} & \begin{matrix} A & N \end{matrix} \\ \begin{matrix} A \\ N \end{matrix} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{matrix} \quad (104.3)$$

where,

TABLE 2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other submatrices

Converting (104.2) to standard form, we get

$$P = \begin{matrix} & \begin{matrix} 4 & 5 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix} \end{matrix} \quad (104.4)$$

From (104.4),

$$R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} \quad (104.5)$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \quad (104.6)$$

where,

$$F = (I - Q)^{-1} \quad (104.7)$$

is called the fundamental matrix of P .

On solving, we get

$$\bar{P} = \begin{matrix} & \begin{matrix} 4 & 5 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5714 & 0.4285 & 0 & 0 & 0 \\ 0.1428 & 0.8571 & 0 & 0 & 0 \\ 0.2857 & 0.7142 & 0 & 0 & 0 \end{bmatrix} \end{matrix} \quad (104.8)$$

A element \bar{p}_{ij} of \bar{P} denotes the absorption probability in state j , starting from state i . Then,

$$1) Pr(A \text{ wins}) = \bar{p}_{14} \approx 0.5714$$

$$2) Pr(B \text{ wins}) = \bar{p}_{15} \approx 0.4285$$

$$\therefore p_{14} > p_{15} \quad (104.9)$$

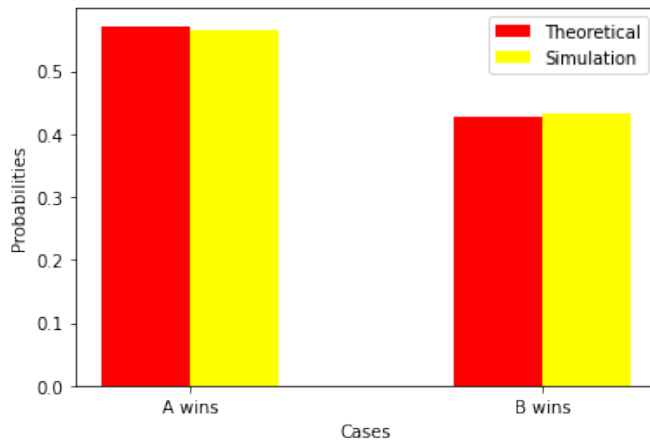
Also, in \bar{P} , all the terms in every row should sum to 1.

$$\Rightarrow p_{14} + p_{15} + 0 + 0 + 0 = 1 \quad (104.10)$$

$$\therefore p_{14} = 1 - p_{15} \quad (104.11)$$

Therefore, options 3), 4) are correct.

Theoretical vs Simulation plot



Markov chain diagram

