

AI1103 : Assignment 9

Yashas Tadikamalla - AI20BTECH11027

Download all python codes from

<https://github.com/YashasTadikamalla/AI1103/tree/main/Assignment9/codes>

and latex codes from

<https://github.com/YashasTadikamalla/AI1103/blob/main/Assignment9/Assignment9.tex>

CSIR-UGC NET-JUNE 2013-PROBLEM(72)

Let X_1, X_2, \dots be independent and identically distributed random variables each following a uniform distribution on $(0,1)$. Denote $T_n = \max\{X_1, X_2, \dots, X_n\}$. Then, which of the following statements are true?

- 1) T_n converges to 1 in probability.
- 2) $n(1 - T_n)$ converges in distribution.
- 3) $n^2(1 - T_n)$ converges in distribution.
- 4) $\sqrt{n}(1 - T_n)$ converges to 0 in probability.

CSIR-UGC NET-JUNE 2013-SOLUTION(72)

The PDF, CDF of each X_1, X_2, X_3, \dots is

$$f_{X_i}(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (72.1)$$

$$F_{X_i}(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & \text{otherwise} \end{cases} \quad (72.2)$$

$\forall i \in \mathbb{N}$. Then, as $T_n = \max\{X_1, X_2, \dots, X_n\}$,

$$f_{T_n}(x) = \begin{cases} nx^{n-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (72.3)$$

$$F_{T_n}(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & \text{otherwise} \end{cases} \quad (72.4)$$

NOTE : If $Y = aX + b$, then

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right) \quad (72.5)$$

• OPTION-1:

Convergence in Probability :

A sequence of random variables X_1, X_2, X_3, \dots converges in probability to a random variable X , shown by $X_n \xrightarrow{P} X$, if

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (72.6)$$

To evaluate : $\lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon), \forall \epsilon > 0$

$$\lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon) = \lim_{n \rightarrow \infty} \Pr(1 - T_n \geq \epsilon) \quad (72.7)$$

$$= \lim_{n \rightarrow \infty} \Pr(T_n \leq 1 - \epsilon) = \lim_{n \rightarrow \infty} F_{T_n}(1 - \epsilon) \quad (72.8)$$

$$F_{T_n}(1 - \epsilon) = \begin{cases} (1 - \epsilon)^n, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \quad (72.9)$$

$$\therefore \lim_{n \rightarrow \infty} (1 - \epsilon)^n = 0 \text{ for } 0 < \epsilon < 1 \quad (72.10)$$

$$\therefore \lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (72.11)$$

$\therefore T_n$ converges to 1 in probability.

• OPTION-2:

Convergence in Distribution :

A sequence of random variables X_1, X_2, X_3, \dots converges in distribution to a random variable X , shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \quad (72.12)$$

for all x at which $F_X(x)$ is continuous.

To evaluate : $\lim_{n \rightarrow \infty} F_{n(1-T_n)}(x)$

Substituting $a = -n, b = n$ in (72.5),

$$F_{n(1-T_n)}(x) = F_{T_n}\left(1 - \frac{x}{n}\right) \quad (72.13)$$

$$F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} \left(1 - \frac{x}{n}\right)^n, & 0 < x < n \\ 1, & x \leq 0 \\ 0, & x \geq n \end{cases} \quad (72.14)$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{y}{n}\right)^n = e^{-y} \quad (72.15)$$

$$\therefore \lim_{n \rightarrow \infty} F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} e^{-x}, & 0 < x < n \\ 1, & x \leq 0 \\ 0, & x \geq n \end{cases} \quad (72.16)$$

$\therefore n(1 - T_n)$ converges in distribution to a random variable with CDF in (72.16).

• **OPTION-3:**

Convergence in Distribution :

A sequence of random variables X_1, X_2, X_3, \dots converges in distribution to a random variable X , shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \quad (72.17)$$

for all x at which $F_X(x)$ is continuous.

To evaluate : $\lim_{n \rightarrow \infty} F_{n^2(1-T_n)}(x)$

Substituting $a = -n^2, b = n^2$ in (72.5),

$$F_{n^2(1-T_n)}(x) = F_{T_n}\left(1 - \frac{x}{n^2}\right) \quad (72.18)$$

$$F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} \left(1 - \frac{x}{n^2}\right)^n, & 0 < x < n^2 \\ 1, & x \leq 0 \\ 0, & x \geq n^2 \end{cases} \quad (72.19)$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{y}{n^2}\right)^n \text{ is not defined} \quad (72.20)$$

$\therefore n^2(1 - T_n)$ does not converge in distribution.

• **OPTION-4:**

Convergence in Probability :

A sequence of random variables X_1, X_2, X_3, \dots converges in probability to a random variable X , shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (72.21)$$

To evaluate :

$$\lim_{n \rightarrow \infty} \Pr(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon), \forall \epsilon > 0$$

$$= \lim_{n \rightarrow \infty} \Pr\left(1 - T_n \geq \frac{\epsilon}{\sqrt{n}}\right) \quad (72.22)$$

$$= \lim_{n \rightarrow \infty} \Pr\left(T_n \leq 1 - \frac{\epsilon}{\sqrt{n}}\right) \quad (72.23)$$

$$= \lim_{n \rightarrow \infty} F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) \quad (72.24)$$

$$F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) = \begin{cases} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n, & 0 < \epsilon < \sqrt{n} \\ 0, & \epsilon \geq \sqrt{n} \end{cases} \quad (72.25)$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n = 0 \text{ for } 0 < \epsilon < \sqrt{n} \quad (72.26)$$

$$\therefore \lim_{n \rightarrow \infty} \Pr(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (72.27)$$

$\therefore \sqrt{n}(1 - T_n)$ converges to 0 in probability.

Hence, options 1), 2), 4) are correct.