# CSIR-UGC NET-Dec 2016-Problem(104)

Yashas Tadikamalla

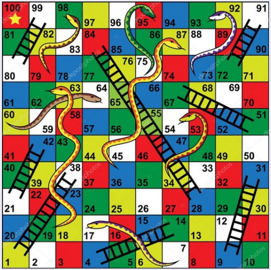
AI20BTECH11027

## Discrete-time Markov chains

Consider a sequence of random variables $\{X_t\}, t = 0, 1, 2, \ldots,$ where $R_{X_i} = S \subset \{0, 1, 2, \ldots\}$, $t$ denotes time. This sequence is said to be a Discrete-time Markov chain if it has the following property,

$$\Pr(X_{t+1} = j | X_t = i, X_{t-1} = i_{t-1}, \ldots, X_0 = i_0) = \Pr(X_{t+1} = j | X_t = i) \quad (1)$$

$\forall j, i, i_{t-1}, \ldots, i_0$. If $S$ is finite, it is said to be a finite Discrete-time Markov chain.



## State transition matrix

The state transition matrix is defined as

$$P = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & \dots & r \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ r \end{array} & \begin{bmatrix} p_{11} & p_{12} & p_{13} & \dots & p_{1r} \\ p_{21} & p_{22} & p_{23} & \dots & p_{2r} \\ p_{31} & p_{32} & p_{33} & \dots & p_{3r} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ p_{r1} & p_{r2} & p_{r3} & \dots & p_{rr} \end{bmatrix} \end{array} \tag{2}$$

where, $p_{ij} = \Pr\left(X_{t+1} = j | X_t = i\right), i, j \in S = \{1, 2, \dots, r\}$.

## n-step State transition matrix

The n-step state transition matrix is defined as

$$P^{(n)} = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & \dots & r \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ r \end{array} & \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} & p_{13}^{(n)} & \dots & p_{1r}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & p_{23}^{(n)} & \dots & p_{2r}^{(n)} \\ p_{31}^{(n)} & p_{32}^{(n)} & p_{33}^{(n)} & \dots & p_{3r}^{(n)} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ p_{r1}^{(n)} & p_{r2}^{(n)} & p_{r3}^{(n)} & \dots & p_{rr}^{(n)} \end{bmatrix} \end{array} \tag{3}$$

where, $p_{ij}^{(n)} = \Pr\left(X_{t+n} = j | X_t = i\right), i, j \in S = \{1, 2, \dots, r\}$.

## Absorbing state

A state is said to be absorbing, if the Markov chain, after attaining that state, remains in it forever. i.e,

$$p_{ii} = 1 \Leftrightarrow \text{State i is absorbing} \tag{4}$$

## Non-absorbing state

A state which is not absorbing, is said to be a non-absorbing state. i.e,

$$p_{ii} \neq 1 \Leftrightarrow \text{State i is non-absorbing} \tag{5}$$

## Standard form of state transition matrix

The standard form of a state transition matrix is

$$P = \begin{array}{cc} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{6}$$

Table: Notations and their meanings

| Notation | Meaning |
| --- | --- |
| $A$ | All absorbing states |
| $N$ | All non-absorbing states |
| $I$ | Identity matrix |
| $O$ | Zero matrix |
| $R, Q$ | Other submatices |

# Question

## CSIR-UGC NET-Dec 2016-Problem(104)

$A$ and $B$ play a game of tossing a fair coin. $A$ starts the game by tossing the coin once and $B$ then tosses the coin twice, followed by $A$ tossing the coin once and $B$ tossing the coin twice and this continues until a head turns up. Whoever gets the first head wins the game. Then,

1. $P(B \text{ Wins}) > P(A \text{ Wins})$
2. $P(B \text{ Wins}) = 2P(A \text{ Wins})$
3. $P(A \text{ Wins}) > P(B \text{ Wins})$
4. $P(A \text{ Wins}) = 1 - P(B \text{ Wins})$

# Solution

Given, a fair coin is tossed till heads turns up.

$$\Pr(H) = p = \frac{1}{2} \tag{7}$$

$$\Pr(T) = q = \frac{1}{2} \tag{8}$$

According to the question, until a player wins, the following set of events occur repetitively,

1. Player $A$'s turn
2. Player $B$'s first turn
3. Player $B$'s second turn

# Solution Contd.

Let's define the problem as a finite Markov chain $\{X_t\}, t = 0, 1, 2, \dots,$ where $R_{X_i} = S = \{1, 2, 3, 4, 5\}$ such that

Table: States and their notations

| Notation | State |
|---|---|
| $S = 1$ | $A$'s turn |
| $S = 2$ | $B$'s first turn |
| $S = 3$ | $B$'s second turn |
| $S = 4$ | $A$ wins |
| $S = 5$ | $B$ wins |

# Solution Contd.

The state transition matrix for the above Markov chain is

$$P = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{9}$$

Clearly, the states $4, 5$ are absorbing, while $1, 2, 3$ are non-absorbing.

# Solution Contd.

Converting (9) into standard form gives

$$P = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix} \end{array} \tag{10}$$

Comparing (10) with (6), we get

$$R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} \tag{11}$$

## Limiting matrices for absorbing Markov chains

If $P$ is the state transition matrix in standard form, i.e,

$$P = \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \tag{12}$$

then $P^{(K)}$ approaches $\bar{P}$ as k increases.

$$\lim_{k \to \infty} P^{(K)} = \bar{P} \tag{13}$$

where,

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix}, F = (I - Q)^{-1} \tag{14}$$

$F$ is called the fundamental matrix of $P$.

# Solution Contd.

Solving (11),(14), we get

$$\bar{P} = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{4}{7} & \frac{3}{7} & 0 & 0 & 0 \\ \frac{1}{7} & \frac{6}{7} & 0 & 0 & 0 \\ \frac{2}{7} & \frac{5}{7} & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{15}$$

A element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability in state $j$, starting from state $i$.

# Solution Contd.

Therefore

1. $\Pr(A \text{ wins}) = \bar{p}_{14} = \frac{4}{7}$
2. $\Pr(B \text{ wins}) = \bar{p}_{15} = \frac{3}{7}$

Clearly,

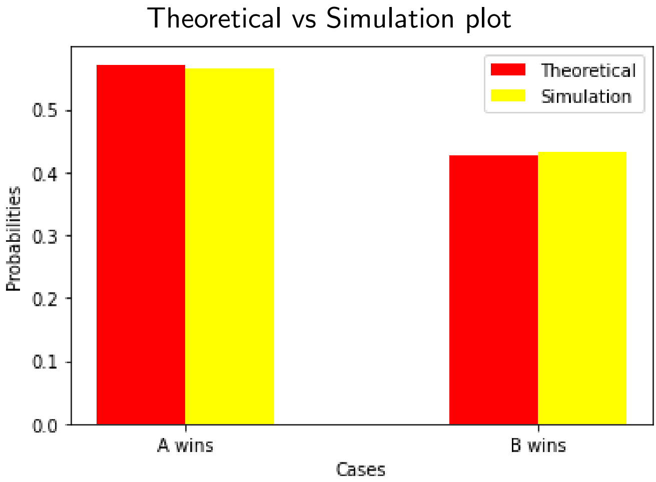
$$\bar{p}_{14} > \bar{p}_{15} \tag{16}$$

and

$$\bar{p}_{14} = 1 - \bar{p}_{15} \tag{17}$$

Therefore, options $3, 4$ are correct.

# Solution Contd.



# Solution Contd.

**Markov chain diagram**