

# CSIR-UGC NET-June 2013-Problem(72)

Yashas Tadikamalla

AI20BTECH11027

## Convergence in Probability

A sequence of random variables  $X_1, X_2, X_3, \dots$  converges in probability to a random variable  $X$ , if

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (1)$$

Notation :  $X_n \xrightarrow{p} X$

## Convergence in Distribution

A sequence of random variables  $X_1, X_2, X_3, \dots$  converges in distribution to a random variable  $X$ , if

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \quad (2)$$

for all  $x$  at which  $F_X(x)$  is continuous.

Notation :  $X_n \xrightarrow{d} X$

## Random Sampling

A collection of random variables  $X_1, X_2, \dots, X_n$  is said to be a random sample of size  $n$  if they are independent and identically distributed, i.e.,

- 1  $X_1, X_2, \dots, X_n$  are independent random variables
- 2 They have the same distribution (Let us denote it by  $F_X(x)$ ), i.e.,

$$F_X(x) = F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_n}(x), \forall x \in \mathbb{R} \quad (3)$$

## Order Statistics

Given a random sample  $X_1, X_2, \dots, X_n$ , the sequence  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  is called the order statistics of it. Here,

$$X_{(1)} = \min(X_1, X_2, \dots, X_n) \quad (4)$$

$$X_{(2)} = \text{the } 2^{\text{nd}} \text{ smallest of } X_1, X_2, \dots, X_n \quad (5)$$

$$\vdots \quad (6)$$

$$X_{(n)} = \max(X_1, X_2, \dots, X_n) \quad (7)$$

## Distribution of the maximum

Let's calculate the CDF, PDF of  $X_{(n)}$

$$F_{X_{(n)}}(x) = \Pr(X_{(n)} \leq x) = \Pr(X_1 \leq x, X_2 \leq x, \dots, X_n \leq x) \quad (8)$$

$$= \Pr(X_1 \leq x) \Pr(X_2 \leq x) \dots \Pr(X_n \leq x) (\because \text{independence}) \quad (9)$$

$$= [\Pr(X_1 \leq x)]^n (\because \text{identical distribution}) = [F_X(x)]^n \quad (10)$$

$$f_{X_{(n)}}(x) = \frac{d}{dx} (F_{X_{(n)}}(x)) = \frac{d}{dx} ([F_X(x)]^n) \quad (11)$$

$$= n ([F_X(x)]^{n-1}) \frac{d}{dx} (F_X(x)) \quad (12)$$

$$= n [F_X(x)]^{n-1} f_X(x) \left( \because \frac{d}{dx} (F_X(x)) = f_X(x) \right) \quad (13)$$

## Uniform Distribution

A continuous random variable  $X$  is said to have a Uniform Distribution over the interval  $(a,b)$ , shown as  $X \sim \text{Uniform}(a, b)$ , if,

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}, F_X(x) = \begin{cases} x, & a < x < b \\ 1, & x \geq b \\ 0, & \text{otherwise} \end{cases} \quad (14)$$

## Exponential Distribution

A continuous random variable  $X$  is said to have an exponential distribution with parameter  $\lambda > 0$ , shown as  $X \sim \text{Exponential}(\lambda)$ , if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}, F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

# Question

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Let  $X_1, X_2, \dots$  be independent and identically distributed random variables each following a uniform distribution on  $(0,1)$ . Denote  $T_n = \max(X_1, X_2, \dots, X_n)$ . Then, which of the following statements are true?

- ①  $T_n$  converges to 1 in probability.
- ②  $n(1 - T_n)$  converges in distribution.
- ③  $n^2(1 - T_n)$  converges in distribution.
- ④  $\sqrt{n}(1 - T_n)$  converges to 0 in probability.

## Solution

Given,  $X_1, X_2, X_3, \dots$  are independent and identically distributed random variables each following a uniform distribution on  $(0,1)$ . Let us denote their PDF, CDF by  $f_X(x), F_X(x)$  respectively. Then,

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

$$F_X(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

## Solution Contd.

As  $T_n = \max(X_1, X_2, \dots, X_n) = X_{(n)}$ , from (13)

$$f_{T_n}(x) = \begin{cases} nx^{n-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (18)$$

Also, from (10)

$$F_{T_n}(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & \text{otherwise} \end{cases} \quad (19)$$



## Note

Let  $X$  be a continuous random variable. If  $Y = aX + b$  and  $a < 0$ , then

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \quad (20)$$

## Proof

$$F_Y(y) = \Pr(Y \leq y) = \Pr(aX + b \leq y) \quad (21)$$

$$= \Pr\left(X \geq \left(\frac{y-b}{a}\right)\right) (\because a < 0) \quad (22)$$

$$= 1 - \Pr\left(X \leq \left(\frac{y-b}{a}\right)\right) + \Pr\left(X = \left(\frac{y-b}{a}\right)\right) \quad (23)$$

$$= 1 - F_X\left(\frac{y-b}{a}\right) + 0 = 1 - F_X\left(\frac{y-b}{a}\right) \quad (24)$$

## Option 1

Consider the sequence of random variables  $X_1, X_2, X_3, \dots$ , such that  $X_n = T_n$ . From (1), we need to evaluate  $\lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon), \forall \epsilon > 0$

$$\lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon) = \lim_{n \rightarrow \infty} \Pr(1 - T_n \geq \epsilon) \quad (25)$$

$$= \lim_{n \rightarrow \infty} \Pr(T_n \leq 1 - \epsilon) = \lim_{n \rightarrow \infty} F_{T_n}(1 - \epsilon) \quad (26)$$

$$F_{T_n}(1 - \epsilon) = \begin{cases} (1 - \epsilon)^n, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \quad (27)$$

$$\therefore \lim_{n \rightarrow \infty} (1 - \epsilon)^n = 0 \text{ for } 0 < \epsilon < 1 \quad (28)$$

$$\therefore \lim_{n \rightarrow \infty} \Pr(|T_n - 1| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (29)$$

$\therefore T_n$  converges to 1 in probability.

## Option 2

Consider the sequence of random variables  $X_1, X_2, X_3, \dots$ , such that  $X_n = n(1 - T_n)$ . From (2), we need to evaluate  $\lim_{n \rightarrow \infty} F_{n(1-T_n)}(x)$ .

Substituting  $a = -n, b = n$  in (20),

$$F_{n(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n}\right) \quad (30)$$

$$F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} \left(1 - \frac{x}{n}\right)^n, & 0 < x < n \\ 1, & x \leq 0 \\ 0, & x \geq n \end{cases} \quad (31)$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{x}{n}\right)^n = e^{-x} \quad (32)$$

## Option 2 Contd.

$$\therefore \lim_{n \rightarrow \infty} F_{T_n} \left( 1 - \frac{x}{n} \right) = \begin{cases} e^{-x}, & x > 0 \\ 1, & x \leq 0 \end{cases} \quad (33)$$

$$\therefore \lim_{n \rightarrow \infty} F_{n(1-T_n)}(x) = 1 - \lim_{n \rightarrow \infty} F_{T_n} \left( 1 - \frac{x}{n} \right) \quad (34)$$

$$\therefore \lim_{n \rightarrow \infty} F_{n(1-T_n)}(x) = \begin{cases} 1 - e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (35)$$

$\therefore$  The CDF in (35) represents an exponential distribution with  $\lambda = 1$   
 $\therefore n(1 - T_n)$  converges in distribution to the random variable  
 $X \sim \text{Exponential}(1)$ .

## Option 2 Contd.

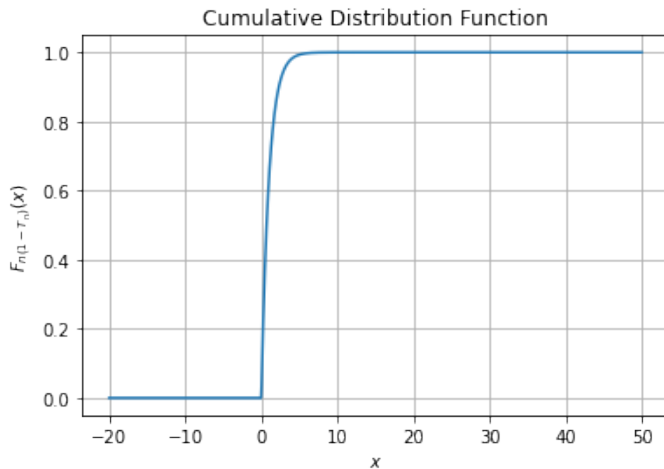


Figure: Plot for the CDF defined in (35)

## Option 3

Consider the sequence of random variables  $X_1, X_2, X_3, \dots$ , such that  $X_n = n^2(1 - T_n)$ . From (2), we need to evaluate  $\lim_{n \rightarrow \infty} F_{n^2(1-T_n)}(x)$ .

Substituting  $a = -n^2, b = n^2$  in (20),

$$F_{n^2(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n^2}\right) \quad (36)$$

$$F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} \left(1 - \frac{x}{n^2}\right)^n, & 0 < x < n^2 \\ 1, & x \leq 0 \\ 0, & x \geq n^2 \end{cases} \quad (37)$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{x}{n^2}\right)^n = 1 \quad (38)$$

## Option 3 Contd.

$$\therefore \lim_{n \rightarrow \infty} F_{T_n} \left( 1 - \frac{x}{n^2} \right) = \begin{cases} 1, & x > 0 \\ 1, & x \leq 0 \end{cases} \quad (39)$$

$$\therefore \lim_{n \rightarrow \infty} F_{n^2(1-T_n)}(x) = 1 - \lim_{n \rightarrow \infty} F_{T_n} \left( 1 - \frac{x}{n^2} \right) \quad (40)$$

$$\therefore \lim_{n \rightarrow \infty} F_{n^2(1-T_n)}(x) = \begin{cases} 0, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (41)$$

$\therefore$  The CDF in (41) is not valid,  
 $\therefore n^2(1 - T_n)$  does not converge in distribution.

## Option 4

Consider the sequence of random variables  $X_1, X_2, X_3, \dots$ , such that  $X_n = \sqrt{n}(1 - T_n)$ . From (1), we need to evaluate

$$\lim_{n \rightarrow \infty} \Pr(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon), \forall \epsilon > 0$$

$$\lim_{n \rightarrow \infty} \Pr(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon) = \lim_{n \rightarrow \infty} \Pr\left(1 - T_n \geq \frac{\epsilon}{\sqrt{n}}\right) \quad (42)$$

$$= \lim_{n \rightarrow \infty} \Pr\left(T_n \leq 1 - \frac{\epsilon}{\sqrt{n}}\right) = \lim_{n \rightarrow \infty} F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) \quad (43)$$

$$F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) = \begin{cases} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n, & 0 < \epsilon < \sqrt{n} \\ 0, & \epsilon \geq \sqrt{n} \end{cases} \quad (44)$$



## Option 4 Contd.

$$\therefore \lim_{n \rightarrow \infty} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n = 0 \text{ for } 0 < \epsilon < \sqrt{n} \quad (45)$$

$$\therefore \lim_{n \rightarrow \infty} \Pr(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon) = 0, \forall \epsilon > 0 \quad (46)$$

$\therefore \sqrt{n}(1 - T_n)$  converges to 0 in probability.

## Solution Contd.

Therefore, options 1), 2), 4) are correct.