AI1103 : Assignment 9

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CSIR-UGC NET-June 2013-Problem(72)

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables each following a uniform distribution on (0,1). Denote $T_n = max\{X_1, X_2, \ldots, X_n\}$. Then, which of the following statements are true?

1) T_n converges to 1 in probability.
2) $n(1 - T_n)$ converges in distribution.
3) $n^2(1 - T_n)$ converges in distribution.
4) $\sqrt{n}(1 - T_n)$ converges to 0 in probability.

CSIR-UGC NET-June 2013-Solution(72)

- Convergence in Distribution :
  A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \qquad (72.1)$$

  for all x at which $F_X(x)$ is continuous.

- Convergence in Probability :
  A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in probability to a random variable X, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n \to \infty} \Pr\left(|X_n - X| \geq \epsilon\right) = 0, \forall \epsilon > 0 \qquad (72.2)$$

The PDF, CDF of each $X_1, X_2, X_3, \ldots$ is

$$f_{X_i}(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases} \qquad (72.3)$$

$$F_{X_i}(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & otherwise \end{cases} \qquad (72.4)$$

$\forall i \in \mathbb{N}$. Then, as $T_n = max\{X_1, X_2, \ldots, X_n\}$,

$$f_{T_n}(x) = \begin{cases} nx^{n-1}, & 0 < x < 1 \\ 0, & otherwise \end{cases} \qquad (72.5)$$

$$F_{T_n}(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & otherwise \end{cases} \qquad (72.6)$$

NOTE : If $Y = aX + b$, then

$$F_Y(y) = F_X\left(\frac{y - b}{a}\right) \qquad (72.7)$$

OPTION-1

To evaluate : $\lim_{n \to \infty} \Pr\left(|T_n - 1| \geq \epsilon\right), \forall \epsilon > 0$

$$\lim_{n \to \infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = \lim_{n \to \infty} \Pr\left(1 - T_n \geq \epsilon\right) \quad (72.8)$$

$$= \lim_{n \to \infty} \Pr\left(T_n \leq 1 - \epsilon\right) = \lim_{n \to \infty} F_{T_n}(1 - \epsilon) \qquad (72.9)$$

$$F_{T_n}(1 - \epsilon) = \begin{cases} (1 - \epsilon)^n, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \qquad (72.10)$$

$$\because \lim_{n \to \infty} (1 - \epsilon)^n = 0 \text{ for } 0 < \epsilon < 1 \qquad (72.11)$$

$$\therefore \lim_{n \to \infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = 0, \forall \epsilon > 0 \qquad (72.12)$$

Therefore T_n converges to 1 in probability.

OPTION-2

To evaluate : $\lim_{n \to \infty} F_{n(1-T_n)}(x)$
Substituting $a = -n, b = n$ in (72.7),

$$F_{n(1-T_n)}(x) = F_{T_n}\left(1 - \frac{x}{n}\right) \qquad (72.13)$$

$$F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} \left(1 - \frac{x}{n}\right)^n, & 0 < x < n \\ 1, & x \le 0 \\ 0, & x \ge n \end{cases} \quad (72.14)$$

$$\because \lim_{n \to \infty} \left(1 - \frac{y}{n}\right)^n = e^{-y} \quad (72.15)$$

$$\therefore \lim_{n \to \infty} F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} e^{-x}, & 0 < x < n \\ 1, & x \le 0 \\ 0, & x \ge n \end{cases} \quad (72.16)$$

Therefore $n(1 - T_n)$ converges in distribution to a random variable with CDF in (72.16).

OPTION-3

To evaluate : $\lim_{n \to \infty} F_{n^2(1-T_n)}(x)$

Substituting $a = -n^2, b = n^2$ in (72.7),

$$F_{n^2(1-T_n)}(x) = F_{T_n}\left(1 - \frac{x}{n^2}\right) \quad (72.17)$$

$$F_{T_n}\left(1 - \frac{x}{n^2}\right) = \begin{cases} \left(1 - \frac{x}{n^2}\right)^n, & 0 < x < n^2 \\ 1, & x \le 0 \\ 0, & x \ge n^2 \end{cases} \quad (72.18)$$

$$\because \lim_{n \to \infty} \left(1 - \frac{y}{n^2}\right)^n \text{ is not defined} \quad (72.19)$$

Therefore $n^2(1 - T_n)$ does converge in distribution.

OPTION-4

To evaluate : $\lim_{n \to \infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \ge \epsilon\right), \forall \epsilon > 0$

$$= \lim_{n \to \infty} \Pr\left(1 - T_n \ge \frac{\epsilon}{\sqrt{n}}\right) = \lim_{n \to \infty} \Pr\left(T_n \le 1 - \frac{\epsilon}{\sqrt{n}}\right) \quad (72.20)$$

$$= \lim_{n \to \infty} F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) \quad (72.21)$$

$$F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) = \begin{cases} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n, & 0 < \epsilon < \sqrt{n} \\ 0, & \epsilon \ge \sqrt{n} \end{cases} \quad (72.22)$$

$$\because \lim_{n \to \infty} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n = 0 \text{ for } 0 < \epsilon < \sqrt{n} \quad (72.23)$$

$$\therefore \lim_{n \to \infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \ge \epsilon\right) = 0, \forall \epsilon > 0 \quad (72.24)$$

Therefore $\sqrt{n}(1 - T_n)$ converges to 0 in probability. Hence, options 1), 2), 4) are correct.