AI1103 : Assignment 8

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Download all python codes from

```
https://github.com/YashasTadikamalla/AI1103/tree/
    main/Assignment8/codes
```

and latex codes from

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https://github.com/YashasTadikamalla/AI1103/blob/
    main/Assignment8/Assignment8.tex
```

CSIR-UGC NET-Dec 2016-Problem(104)

A and B play a game of tossing a fair coin. A starts the game by tossing the coin once and B then tosses the coin twice, followed by A tossing the coin once and B tossing the coin twice and this continues until a head turns up. Whoever gets the first head wins the game. Then,

1) $P(B \text{ Wins}) > P(A \text{ Wins})$
2) $P(B \text{ Wins}) = 2P(A \text{ Wins})$
3) $P(A \text{ Wins}) > P(B \text{ Wins})$
4) $P(A \text{ Wins}) = 1 - P(B \text{ Wins})$

CSIR-UGC NET-Dec 2016-Solution(104)

Given, a fair coin is tossed till heads turns up.

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{104.1}$$

Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3, 4, 5\}$, such that

TABLE 1: States and their notations

Notation	State
$S = 1$	A's turn
$S = 2$	B's first turn
$S = 3$	B's second turn
$S = 4$	A wins
$S = 4$	B wins

The state transition matrix for the Markov chain is

$$P = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{104.2}$$

Clearly, the states $1, 2, 3$ are transient, while $4, 5$ are absorbing. The standard form of a state transition matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{104.3}$$

where,

TABLE 2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other submatices

Converting (104.2) to standard form, we get

$$P = \begin{array}{c c} & \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix} \end{array} \tag{104.4}$$

From (104.4),

$$R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} \tag{104.5}$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \tag{104.6}$$

where,

$$F = (I - Q)^{-1} \tag{104.7}$$

is called the fundamental matrix of P.
On solving, we get

$$\bar{P} = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5714 & 0.4285 & 0 & 0 & 0 \\ 0.1428 & 0.8571 & 0 & 0 & 0 \\ 0.2857 & 0.7142 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{104.8}$$

A element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability in state j, starting from state i. Then,

1) $Pr(A \text{ wins}) = \bar{p}_{14} \approx 0.5714$
2) $Pr(B \text{ wins}) = \bar{p}_{15} \approx 0.4285$

$$\therefore \bar{p}_{14} > \bar{p}_{15} \tag{104.9}$$

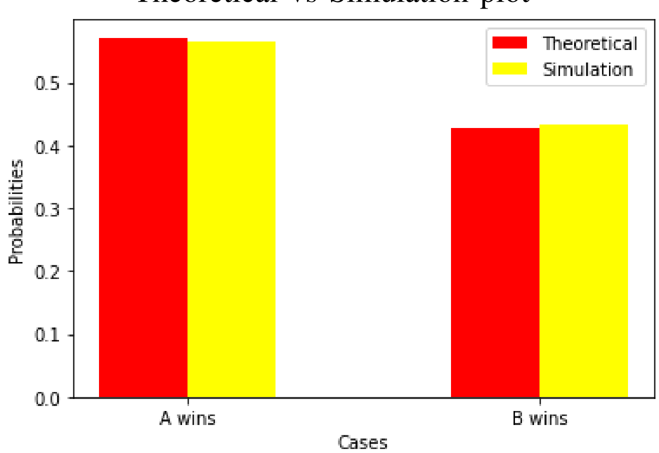
Also, in $\bar{P}$, all the terms in every row should sum to 1.

$$\Rightarrow \bar{p}_{14} + \bar{p}_{15} + 0 + 0 + 0 = 1 \tag{104.10}$$

$$\therefore \bar{p}_{14} = 1 - \bar{p}_{15} \tag{104.11}$$

Therefore, options 3), 4) are correct.

Theoretical vs Simulation plot



Markov chain diagram