

Gate Assignment 1

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Download all python codes from

<https://github.com/YashasTadikamalla/EE3900/blob/main/GateAssignment1/codes>

and latex-tikz codes from

<https://github.com/YashasTadikamalla/EE3900/blob/main/GateAssignment1/GateAssignment1.tex>

1 PROBLEM (EC-2013 Q8)

The impulse response of a system is $h(t) = tu(t)$. For an input $u(t-1)$, the output is

- 1) $\frac{t^2}{2}u(t)$
- 2) $\frac{t(t-1)}{2}u(t-1)$
- 3) $\frac{(t-1)^2}{2}u(t-1)$
- 4) $\frac{t^2-1}{2}u(t-1)$

2 SOLUTION

Definition 2.1 (Laplace Transform). *It is an integral transform that converts a function of a real variable t to a function of a complex variable s . The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$.*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (2.0.1)$$

Remark. Laplace transform of $f(t) = t^n, n \geq 1$ is

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0 \quad (2.0.2)$$

Proof. Basis Step: $n = 1$

$$\mathcal{L}\{t\} = \int_0^{\infty} e^{-st} t dt \quad (2.0.3)$$

$$= \left[\frac{te^{-st}}{-s} \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt \quad (2.0.4)$$

$$= 0 + \left[\frac{-1}{s^2} e^{-st} \right]_0^{\infty}, s > 0 \quad (2.0.5)$$

$$= \frac{1}{s^2}, s > 0 \quad (2.0.6)$$

Inductive Step:

$$\mathcal{L}\{t^n\} = \int_0^{\infty} e^{-st} t^n dt \quad (2.0.7)$$

$$= \left[\frac{t^n e^{-st}}{-s} \right]_0^{\infty} + \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} dt \quad (2.0.8)$$

$$= 0 + \frac{n}{s} \mathcal{L}\{t^{n-1}\}, s > 0 \quad (2.0.9)$$

$$= \frac{n}{s} \mathcal{L}\{t^{n-1}\}, s > 0 \quad (2.0.10)$$

To prove that if (2.0.2) holds for $n = k$, it holds for $n = k + 1$. From (2.0.10)

$$\mathcal{L}\{t^{k+1}\} = \frac{k+1}{s} \mathcal{L}\{t^k\} \quad (2.0.11)$$

$$= \frac{(k+1)k!}{s(s^{k+1})} = \frac{(k+1)!}{s^{k+2}}, s > 0 \quad (2.0.12)$$

By mathematical induction, (2.0.2) is true $\forall n \geq 1$ \square

Lemma 2.1. *For any real number c ,*

$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}, s > 0 \quad (2.0.13)$$

Proof.

$$\mathcal{L}\{u(t-c)\} = \int_0^{\infty} e^{-st} u(t-c) dt = \int_c^{\infty} e^{-st} dt \quad (2.0.14)$$

$$= \left[-\frac{e^{-st}}{s} \right]_c^{\infty} = \frac{e^{-cs}}{s}, s > 0 \quad (2.0.15)$$

\square

Definition 2.2 (Inverse Laplace Transform). *It is the transformation of a Laplace transform into a function of time. If $F(s) = \mathcal{L}\{f(t)\}$, then the Inverse laplace transform of $F(s)$ is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.*

Lemma 2.2 (t-shift rule). *For any real number c ,*

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s) \quad (2.0.16)$$

Proof.

$$\mathcal{L}\{u(t-c)f(t-c)\} = \int_0^\infty e^{-st}u(t-c)f(t-c)dt \quad (2.0.17)$$

$$= \int_c^\infty e^{-st}f(t-c)dt \quad (2.0.18)$$

$$= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau \quad (t = \tau + c) \quad (2.0.19)$$

$$= e^{-cs} \int_0^\infty e^{-s\tau}f(\tau)d\tau \quad (2.0.20)$$

$$= e^{-cs}F(s) \quad (2.0.21)$$

□

Corollary 2.2.1.

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c) \quad (2.0.22)$$

Theorem 2.3 (Convolution theorem). *Suppose $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$ exist, then,*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) \quad (2.0.23)$$

Given,

$$h(t) = tu(t) \quad (2.0.24)$$

$$x(t) = u(t-1) \quad (2.0.25)$$

To find: $y(t)$. We know,

$$y(t) = h(t) * x(t) \quad (2.0.26)$$

$$= \mathcal{L}^{-1}\{H(s)X(s)\} \quad (2.0.27)$$

From (2.0.16) and (2.0.2),

$$H(s) = e^0 \mathcal{L}\{t\} = \frac{1}{s^2} \quad (2.0.28)$$

From (2.0.13),

$$X(s) = \frac{e^{-s}}{s} \quad (2.0.29)$$

Substituting in (2.0.27),

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} \quad (2.0.30)$$

Consider

$$p(t) = \frac{t^2}{2} \quad (2.0.31)$$

From (2.0.2)

$$P(s) = \frac{2!}{2s^3} = \frac{1}{s^3} \quad (2.0.32)$$

Further, from (2.0.23), for $c = 1$

$$\mathcal{L}^{-1}\{e^{-s}P(s)\} = u(t-1)p(t-1) \quad (2.0.33)$$

$$= u(t-1)\frac{(t-1)^2}{2} \quad (2.0.34)$$

$$\therefore y(t) = \frac{(t-1)^2}{2}u(t-1) \quad (2.0.35)$$

Option 3 is the correct answer.

$$h(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (2.0.36)$$

$$x(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases} \quad (2.0.37)$$

$$y(t) = \begin{cases} \frac{(t-1)^2}{2}, & t \geq 1 \\ 0, & t < 1 \end{cases} \quad (2.0.38)$$

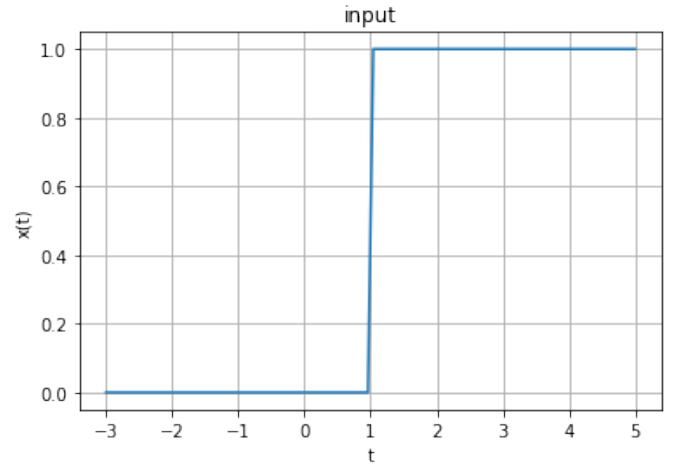


Fig. 4: Plot of $x(t)$

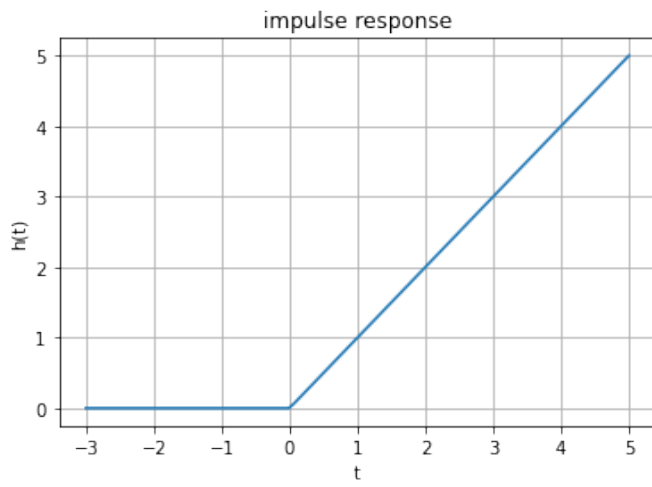


Fig. 4: Plot of $h(t)$

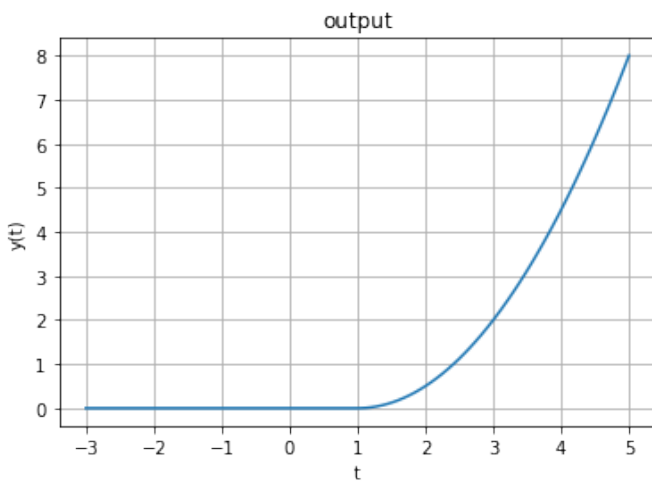


Fig. 4: Plot of $y(t)$