

Assignment 5

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Download all python codes from

<https://github.com/YashasTadikamalla/EE3900/blob/main/Assignment5/codes>

and latex-tikz codes from

<https://github.com/YashasTadikamalla/EE3900/blob/main/Assignment5/Assignment5.tex>

1 PROBLEM (QUADRATIC FORMS Q2.31)

Find the equation of hyperbola with foci $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$ and length of latus rectum 36.

2 SOLUTION

Theorem 2.1. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus \mathbf{F} is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (2.0.1)$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top, \quad (2.0.2)$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \quad (2.0.3)$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (2.0.4)$$

For $|\mathbf{V}| > 0$, the equation represents an ellipse, while for $|\mathbf{V}| < 0$, the equation represents a hyperbola.

Theorem 2.2. The eccentricity of the conic represented by (2.0.1) is given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \quad (2.0.5)$$

Theorem 2.3. For $|\mathbf{V}| \neq 0$, the lengths of the semi-major and semi-minor axes of the conic in (2.0.1) are given by

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}}, \sqrt{\left| \frac{f - \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u}}{\lambda_2} \right|} \quad (2.0.6)$$

Definition 2.1 (Latus rectum). The latus rectum of a conic section is the chord (line segment) that passes

through the focus, is perpendicular to the major axis and has both endpoints on the curve.

Theorem 2.4. For $|\mathbf{V}| \neq 0$, the length of latus rectum (LLR) of the conic in (2.0.1) is given by

$$\frac{2 \left| \frac{f - \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u}}{\lambda_2} \right|}{\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}}} \quad (2.0.7)$$

Given, length of latus rectum is 36 and

$$\mathbf{F} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} \Rightarrow \|\mathbf{F}\| = 12 \quad (2.0.8)$$

Let $\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = \alpha$. From (2.0.6), (2.0.5), (2.0.7)

$$\sqrt{\frac{\alpha}{\lambda_1}} \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = 12 \quad (2.0.9)$$

$$\frac{2 \left(\frac{-\alpha}{\lambda_2} \right)}{\sqrt{\frac{\alpha}{\lambda_1}}} = 36 \quad (2.0.10)$$

Dividing (2.0.9) by (2.0.10) gives

$$\frac{\lambda_1}{\lambda_2} = -3 \quad (2.0.11)$$

$$\Rightarrow e = 2 \quad (2.0.12)$$

$$\Rightarrow \sqrt{\frac{\alpha}{\lambda_1}} = 6 \quad (2.0.13)$$

The directrix of this hyperbola is perpendicular to the y-axis and passes through the point

$$\begin{pmatrix} 0 \\ \sqrt{\frac{\alpha}{e^2 \lambda_1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (2.0.14)$$

Hence, its equation is

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right) = 0 \quad (2.0.15)$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (2.0.16)$$

Comparing it with $\mathbf{n}^\top \mathbf{x} = c$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = 3 \Rightarrow \|\mathbf{n}\| = 1 \quad (2.0.17)$$

Calculating \mathbf{V} , \mathbf{u} and f ,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \quad (2.0.18)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{u} = 3(2^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.20)$$

$$f = 1^2(12^2) - 3^2(2^2) = 108 \quad (2.0.21)$$

Hence, the required equation is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x} + 108 = 0 \quad (2.0.22)$$

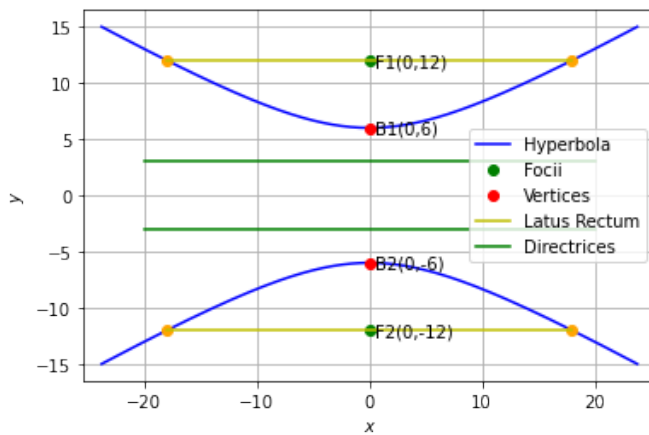


Fig. 0: Hyperbola