Gate Assignment 1

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Download all python codes from

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https://github.com/YashasTadikamalla/EE3900/blob
    /main/GateAssignment1/codes
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and latex-tikz codes from

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https://github.com/YashasTadikamalla/EE3900/blob
    /main/GateAssignment1/GateAssignment1.tex
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1 Problem (EC-2013 Q8)

The impulse response of a system is $h(t) = tu(t)$. For an input $u(t-1)$, the output is

1) $\frac{t^2}{2}u(t)$
2) $\frac{t(t-1)}{2}u(t-1)$
3) $\frac{(t-1)^2}{2}u(t-1)$
4) $\frac{t^2-1}{2}u(t-1)$

2 Solution

Definition 2.1 (Laplace Transform)**.** *It is an integral transform that converts a function of a real variable t to a function of a complex variable s. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$.*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \tag{2.0.1}$$

Remark. Laplace transform of $f(t) = t^n, n \geq 1$ is

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0 \tag{2.0.2}$$

Proof. Basis Step: $n = 1$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt \tag{2.0.3}$$

$$= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \tag{2.0.4}$$

$$= 0 + \left[\frac{-1}{s^2}e^{-st}\right]_0^\infty, s > 0 \tag{2.0.5}$$

$$= \frac{1}{s^2}, s > 0 \tag{2.0.6}$$

Inductive Step:

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}t^n dt \tag{2.0.7}$$

$$= \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \tag{2.0.8}$$

$$= 0 + \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, s > 0 \tag{2.0.9}$$

$$= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, s > 0 \tag{2.0.10}$$

To prove that if (2.0.2) holds for $n = k$, it holds for $n = k+1$. From (2.0.10)

$$\mathcal{L}\left\{t^{k+1}\right\} = \frac{k+1}{s}\mathcal{L}\left\{t^k\right\} \tag{2.0.11}$$

$$= \frac{(k+1)k!}{s(s^{k+1})} = \frac{(k+1)!}{s^{k+2}}, s > 0 \tag{2.0.12}$$

By mathematical induction, (2.0.2) is true $\forall n \geq 1$ □

Lemma 2.1. *For any real number c,*

$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}, s > 0 \tag{2.0.13}$$

Proof.

$$\mathcal{L}\{u(t-c)\} = \int_0^\infty e^{-st}u(t-c)dt = \int_c^\infty e^{-st}dt \tag{2.0.14}$$

$$= \left[-\frac{e^{-st}}{s}\right]_c^\infty = \frac{e^{-cs}}{s}, s > 0 \tag{2.0.15}$$

□

Definition 2.2 (Inverse Laplace Transform)**.** *It is the transformation of a Laplace transform into a function of time. If $F(s) = \mathcal{L}\{f(t)\}$, then the Inverse laplace transform of $F(s)$ is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.*

Lemma 2.2 (t-shift rule)**.** *For any real number c,*

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s) \tag{2.0.16}$$

Proof.

$$\mathcal{L}\{u(t-c)f(t-c)\} = \int_0^\infty e^{-st}u(t-c)f(t-c)dt \tag{2.0.17}$$

$$= \int_c^\infty e^{-st}f(t-c)dt \tag{2.0.18}$$

$$= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau\,(t=\tau+c) \tag{2.0.19}$$

$$= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)d\tau \tag{2.0.20}$$

$$= e^{-cs}F(s) \tag{2.0.21}$$

□

Corollary 2.2.1.

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c) \tag{2.0.22}$$

Theorem 2.3 (Convolution theorem)**.** *Suppose $F(s) = \mathcal{L}\{f(t)\}, G(s) = \mathcal{L}\{g(t)\}$ exist, then,*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) \tag{2.0.23}$$

Given,

$$h(t) = tu(t) \tag{2.0.24}$$

$$x(t) = u(t-1) \tag{2.0.25}$$

i.e,

$$h(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2.0.26}$$

$$x(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{2.0.27}$$

To find: $y(t)$. We know,

$$y(t) = h(t) * x(t) \tag{2.0.28}$$

$$= \mathcal{L}^{-1}\{H(s)X(s)\} \tag{2.0.29}$$

From (2.0.16) and (2.0.2),

$$H(s) = e^0\mathcal{L}\{t\} = \frac{1}{s^2} \tag{2.0.30}$$

From (2.0.13),

$$X(s) = \frac{e^{-s}}{s} \tag{2.0.31}$$

Substituting in (2.0.29),

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} \tag{2.0.32}$$

Consider

$$p(t) = \frac{t^2}{2} \tag{2.0.33}$$

From (2.0.2)

$$P(s) = \frac{2!}{2s^3} = \frac{1}{s^3} \tag{2.0.34}$$

Further, from (2.0.23), for $c = 1$

$$\mathcal{L}^{-1}\{e^{-s}P(s)\} = u(t-1)p(t-1) \tag{2.0.35}$$

$$= u(t-1)\frac{(t-1)^2}{2} \tag{2.0.36}$$

$$\therefore y(t) = \frac{(t-1)^2}{2}u(t-1) \tag{2.0.37}$$

Option 3 is the correct answer.

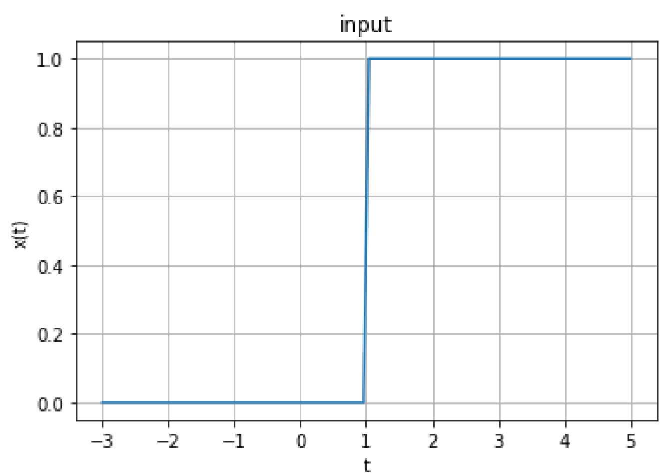

Fig. 4: Plot of $x(t)$

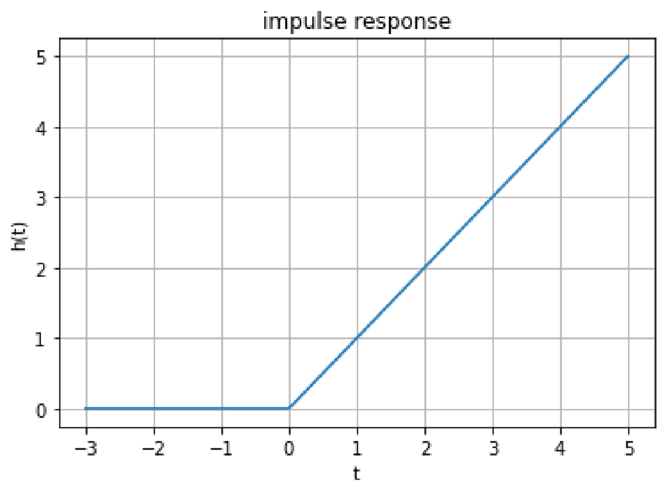

Fig. 4: Plot of $h(t)$

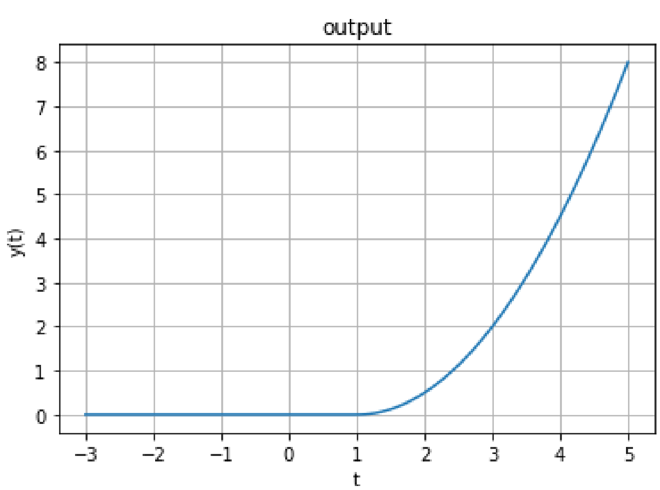

Fig. 4: Plot of $y(t)$