

Assignment 4

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Download all python codes from

<https://github.com/YashasTadikamalla/EE3900/blob/main/Assignment4/codes>

and latex-tikz codes from

<https://github.com/YashasTadikamalla/EE3900/blob/main/Assignment4/Assignment4.tex>

1 PROBLEM (LINEAR FORMS Q2.33)

If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$$

are coplanar, find the value of k .

2 SOLUTION

Given, two coplanar lines,

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \quad (2.0.1)$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5} \quad (2.0.2)$$

For them,

$$\mathbf{m}_1 = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} \quad (2.0.4)$$

where $\mathbf{m}_1, \mathbf{m}_2$ are the respective direction vectors and $\mathbf{A}_1, \mathbf{A}_2$ are the respective points through which the lines pass. To find: k .

If two lines are coplanar, they must be either parallel or intersecting. If the lines are parallel, the matrix

$$\begin{pmatrix} -3 & 2k & 2 \\ 3k & 1 & -5 \end{pmatrix} \quad (2.0.5)$$

should have $\text{rank} = 1$.

$$\Rightarrow \begin{vmatrix} -3 & 2k \\ 3k & 1 \end{vmatrix} = 0 \quad (2.0.6)$$

$$\Rightarrow 6k^2 + 3 = 0 \quad (2.0.7)$$

This is not possible. Hence, the lines must be intersecting. In that case,

$$(\mathbf{A}_1 - \mathbf{A}_2)^T (\mathbf{m}_1 \times \mathbf{m}_2) = 0 \quad (2.0.8)$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix,

$$\begin{pmatrix} -3 & 2k & 2 \\ 3k & 1 & -5 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2 + kR_1}{1+2k^2}} \begin{pmatrix} -3 & 2k & 2 \\ 0 & 1 & \frac{2k-5}{1+2k^2} \end{pmatrix} \quad (2.0.9)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1 - 2kR_2}{-3}} \begin{pmatrix} 1 & 0 & \frac{10k+2}{-3(1+2k^2)} \\ 0 & 1 & \frac{2k-5}{1+2k^2} \end{pmatrix} \quad (2.0.10)$$

Substituting in (2.0.8),

$$\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}^T \begin{pmatrix} \frac{10k+2}{-3(1+2k^2)} \\ \frac{5-2k}{1+2k^2} \\ 1 \end{pmatrix} = 0 \quad (2.0.11)$$

$$\Rightarrow -\frac{20k+4}{3+6k^2} + \frac{5-2k}{1+2k^2} = 3 \quad (2.0.12)$$

$$\Rightarrow \frac{11-26k}{3+6k^2} = 3 \quad (2.0.13)$$

$$\Rightarrow 9k^2 + 13k - 1 = 0 \quad (2.0.14)$$

Solving this, we get

$$k = \frac{-13 \pm \sqrt{205}}{18} \quad (2.0.15)$$

For the above values of k , the lines are intersecting, and hence, coplanar. The lines can be rewritten as

$$\mathbf{x} = \mathbf{A}_1 + \lambda_1 \mathbf{m}_1 \quad (2.0.16)$$

$$\mathbf{x} = \mathbf{A}_2 + \lambda_2 \mathbf{m}_1 \quad (2.0.17)$$

Point of intersection can be found by equating them

$$\mathbf{A}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{A}_2 + \lambda_2 \mathbf{m}_2 \quad (2.0.18)$$

$$\Rightarrow \mathbf{A}_1 - \mathbf{A}_2 + \lambda_1 \mathbf{m}_1 - \lambda_2 \mathbf{m}_2 = 0 \quad (2.0.19)$$

$$\Rightarrow \begin{pmatrix} -3\lambda_1 - 3k\lambda_2 \\ 2k\lambda_1 - \lambda_2 \\ 2\lambda_1 + 5\lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad (2.0.20)$$

We need to solve any two equations to get λ_1, λ_2 . Consider,

$$\begin{pmatrix} 2 & 5 \\ 2k & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad (2.0.21)$$

The corresponding augmented matrix is

$$\left(\begin{array}{cc|c} 2 & 5 & 3 \\ 2k & -1 & -1 \end{array} \right) \quad (2.0.22)$$

Using Gauss Jordan Elimination method,

$$\left(\begin{array}{cc|c} 2 & 5 & 3 \\ 2k & -1 & -1 \end{array} \right) \xrightarrow{R_1 \leftarrow R_1 + 5R_2} \left(\begin{array}{cc|c} 2 + 10k & 0 & -2 \\ 2k & -1 & -1 \end{array} \right) \quad (2.0.23)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{2+10k}} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ 2k & -1 & -1 \end{array} \right) \quad (2.0.24)$$

$$\xrightarrow{R_2 \leftarrow -R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ -2k & 1 & 1 \end{array} \right) \quad (2.0.25)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 2kR_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ 0 & 1 & \frac{6k+2}{2+10k} \end{array} \right) \quad (2.0.26)$$

$$\therefore \lambda_1 = \frac{-1}{1+5k}, \lambda_2 = \frac{3k+1}{1+5k} \quad (2.0.27)$$

Substituting λ_1 in (2.0.16) to get the required point of intersection \mathbf{P} ,

$$\mathbf{P} = \begin{pmatrix} 1 + \frac{3}{1+5k} \\ 2 - \frac{2}{1+5k} \\ 3 - \frac{2}{1+5k} \end{pmatrix} \quad (2.0.28)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} \frac{4+5k}{1+5k} \\ \frac{2+8k}{1+5k} \\ \frac{1+15k}{1+5k} \end{pmatrix} \quad (2.0.29)$$

where $k = \frac{-13 \pm \sqrt{205}}{18}$

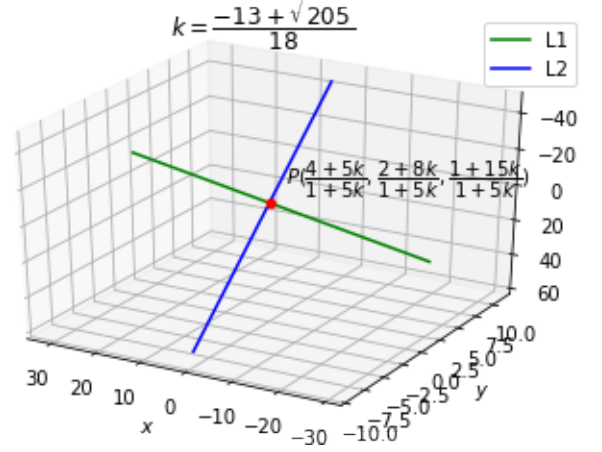


Fig. 0: Intersecting lines - Case1

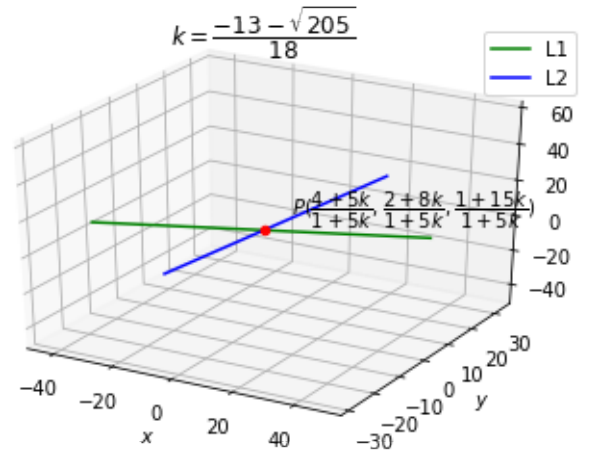


Fig. 0: Intersecting lines - Case2