Gate Assignment 3

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Download all python codes from

https://github.com/YashasTadikamalla/EE3900/blob/main/GateAssignment3/codes

and latex-tikz codes from

https://github.com/YashasTadikamalla/EE3900/blob/main/GateAssignment3/GateAssignment3.tex

1 Problem (EC-2005 Q25)

A linear system is equivalently represented by two sets of state equations:

$$\dot{X} = AX + BU \text{ and } \dot{W} = CW + DU$$

Eigenvalues of the representations are also computed as $[\lambda]$ and $[\mu]$. Which of the following is true?

1) $[\lambda] = [\mu]$ and $X = W$
2) $[\lambda] = [\mu]$ and $X \neq W$
3) $[\lambda] \neq [\mu]$ and $X = W$
4) $[\lambda] \neq [\mu]$ and $X \neq W$

2 Solution

Definition 2.1 (State Space representation)**.** *It is a mathematical model of a physical system, as a set of input, output and state variables related by first order difference or differential equations. The most general state representation of a linear system with p inputs, q outputs, and n state variables can be written as*

$$\dot{X} = AX + BU \tag{2.0.1}$$

$$Y = CX + DU \tag{2.0.2}$$

where, $X \in R^n$ is the state vector, $Y \in R^q$ is the output vector, $U \in R^p$ is input vector, $A \in R^{n\times n}$ is the state matrix, $B \in R^{n\times p}$ is input matrix, $C \in R^{q\times n}$ is output matrix, $D \in R^{q\times p}$ is feedthrough matrix.

Definition 2.2 (Eigen values of State Space representation)**.** *These are the solutions of the charecteristic equation*

$$\triangle(\lambda) = det(\lambda I - A) = 0 \tag{2.0.3}$$

where A is the state matrix.

Theorem 2.1. *Consider the n-dimensional continuous time linear system*

$$\dot{X} = AX + BU, Y = CX + DU \tag{2.0.4}$$

Let T be an $n \times n$ real non-singular matrix and let $\bar{X} = TX$. Then the state equation

$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}U, Y = \bar{C}\bar{X} + \bar{D}U \tag{2.0.5}$$

where $\bar{A} = TAT^{-1}, \bar{B} = TB, \bar{C} = CT^{-1}, \bar{D} = D$ is said to be equivalent to (2.0.4).

Proof. Given, $\dot{X} = AX + BU$ and $Y = CX + DU$, T is a non-singular matrix such that $\bar{X} = TX$. The same system can be defined using $\bar{X}$ as the state,

$$\dot{\bar{X}} = T\dot{X} = TAX + TBU \tag{2.0.6}$$

$$= TAT^{-1}\bar{X} + TBU \tag{2.0.7}$$

$$Y = CX + DU = CT^{-1}\bar{X} + DU \tag{2.0.8}$$

□

Theorem 2.2. *Equivalent state space representations have same set of eigen values*

Proof. For the representation in (2.0.4), the eigen values $[\lambda]$ are such that

$$Ax = \lambda x \tag{2.0.9}$$

$$\implies (A - \lambda I)x = 0 \tag{2.0.10}$$

$$\implies det(A - \lambda I) = 0 \tag{2.0.11}$$

For the representation in (2.0.5), the eigen values $[\mu]$, are such that

$$\bar{A}x = \mu x \tag{2.0.12}$$

$$\implies (\bar{A} - \mu I)x = 0 \tag{2.0.13}$$

$$\implies (TAT^{-1} - \mu TT^{-1})x = 0 \tag{2.0.14}$$

$$\implies det(T(A - \mu I)T^{-1}) = 0 \tag{2.0.15}$$

$$\implies det(A - \mu I) = 0 \tag{2.0.16}$$

Hence, equivalent state space representations have same set of eigen values. □

Given,

$$\dot{X} = AX + BU \tag{2.0.17}$$

$$\dot{W} = CW + DU \tag{2.0.18}$$

represent the same system. Hence, using (2.1) and (2.2), we can conclude that

$$[\lambda] = [\mu] \text{ and } W = TX$$

where T need not be identity matrix.
Hence, option 2 is the correct answer.
Let us now look at a numerical example to establish the correctness of the obtained result. Consider a SISO LTI system of order 2, represented by the equations

$$\dot{x}_1(t) = -x_1(t) + 1.5x_2(t) + 2u(t) \tag{2.0.19}$$

$$\dot{x}_2(t) = 4x_1(t) + u(t) \tag{2.0.20}$$

$$y(t) = 1.5x_1(t) + 0.625x_2(t) + u(t) \tag{2.0.21}$$

Its state space representation can be given by (2.0.4), where

$$X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, Y = y(t) \tag{2.0.22}$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, U = u(t) \tag{2.0.23}$$

$$A = \begin{bmatrix} -1 & 1.5 \\ 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{2.0.24}$$

$$C = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}, D = 1 \tag{2.0.25}$$

The eigen values for this state representation are

$$det(\lambda I - A) = 0 \tag{2.0.26}$$

$$\begin{vmatrix} \lambda + 1 & -1.5 \\ -4 & \lambda \end{vmatrix} = 0 \tag{2.0.27}$$

$$\lambda^2 + \lambda - 6 = 0 \tag{2.0.28}$$

$$[\lambda] = \{-3, 2\} \tag{2.0.29}$$

Even if we swap the equations, they still should represent the same system. So, consider a different state space representation,

$$W = \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix}, Y = y(t) \tag{2.0.30}$$

$$\dot{W} = \begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix}, U = u(t) \tag{2.0.31}$$

Clearly, $X \neq W$ and $W = TX$, where $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

From (2.1)

$$\bar{A} = \begin{bmatrix} 0 & 4 \\ 1.5 & -1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{2.0.32}$$

$$\bar{C} = \begin{bmatrix} 0.625 & 1.5 \end{bmatrix}, \bar{D} = 1 \tag{2.0.33}$$

Also, the eigen values for this state representation are

$$det(\mu I - A) = 0 \tag{2.0.34}$$

$$\begin{vmatrix} \mu & -4 \\ -1.5 & \mu + 1 \end{vmatrix} = 0 \tag{2.0.35}$$

$$\mu^2 + \mu - 6 = 0 \tag{2.0.36}$$

$$[\mu] = \{-3, 2\} \tag{2.0.37}$$

Hence, both the state space representations are equivalent, and satisfy $[\lambda] = [\mu]$ and $W = TX$, where T need not be identity matrix.