Gate Assignment 3 Presentation

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Question

Problem (EC-2005 Q25)

A linear system is equivalently represented by two sets of state equations:

$$\dot{\mathrm{X}} = \mathrm{AX} + \mathrm{BU} \text{ and } \dot{\mathrm{W}} = \mathrm{CW} + \mathrm{DU}$$

Eigenvalues of the representations are also computed as $[\lambda]$ and $[\mu]$. Which of the following is true?

1. $[\lambda] = [\mu]$ and $\mathrm{X} = \mathrm{W}$
2. $[\lambda] = [\mu]$ and $\mathrm{X} \neq \mathrm{W}$
3. $[\lambda] \neq [\mu]$ and $\mathrm{X} = \mathrm{W}$
4. $[\lambda] \neq [\mu]$ and $\mathrm{X} \neq \mathrm{W}$

Few prerequisites

Definition (State Space representation)

It is a mathematical model of a physical system, as a set of input, output and state variables related by first order difference or differential equations. The most general state representation of a linear system with p inputs, q outputs, and n state variables can be written as

$$\dot{X} = AX + BU \tag{1}$$

$$Y = CX + DU \tag{2}$$

where, $X \in R^n$ is the state vector, $Y \in R^q$ is the output vector, $U \in R^p$ is input vector, $A \in R^{n\times n}$ is the state matrix, $B \in R^{n\times p}$ is input matrix, $C \in R^{q\times n}$ is output matrix, $D \in R^{q\times p}$ is feedthrough matrix.

Definition (Eigen values of State Space representation)

These are the solutions of the charecteristic equation

$$\triangle(\lambda) = det(\lambda \mathrm{I} - \mathrm{A}) = 0 \tag{3}$$

where A is the state matrix.

Theorem

Consider the n-dimensional continuous time linear system

$$\dot{\mathrm{X}} = \mathrm{AX} + \mathrm{BU}, \mathrm{Y} = \mathrm{CX} + \mathrm{DU} \tag{4}$$

Let T *be an* $n \times n$ *real non-singular matrix and let* $\bar{\mathrm{X}} = \mathrm{TX}$*. Then the state equation*

$$\dot{\bar{\mathrm{X}}} = \bar{\mathrm{A}}\bar{\mathrm{X}} + \bar{\mathrm{B}}\mathrm{U}, \mathrm{Y} = \bar{\mathrm{C}}\bar{\mathrm{X}} + \bar{\mathrm{D}}\mathrm{U} \tag{5}$$

where $\bar{\mathrm{A}} = \mathrm{TAT}^{-1}, \bar{\mathrm{B}} = \mathrm{TB}, \bar{\mathrm{C}} = \mathrm{CT}^{-1}, \bar{\mathrm{D}} = \mathrm{D}$ *is said to be equivalent to* (4)*.*

Proof.

Given, $\dot{X} = AX + BU$ and $Y = CX + DU$, T is a non-singular matrix such that $\bar{X} = TX$. The same system can be defined using $\bar{X}$ as the state,

$$\dot{\bar{X}} = T\dot{X} = TAX + TBU \tag{6}$$

$$= TAT^{-1}\bar{X} + TBU \tag{7}$$

$$Y = CX + DU = CT^{-1}\bar{X} + DU \tag{8}$$

□

Theorem

Equivalent state space representations have same set of eigen values

Proof.

For the representation in (4), the eigen values $[\lambda]$ are such that

$$\mathsf{Ax} = \lambda \mathsf{x} \tag{9}$$
$$\Rightarrow (\mathsf{A} - \lambda \mathsf{I})\mathsf{x} = 0 \tag{10}$$
$$\Rightarrow det(\mathsf{A} - \lambda \mathsf{I} = 0 \tag{11}$$

For the representation in (5), the eigen values $[\mu]$, are such that

$$\bar{\mathsf{A}}\mathsf{x} = \mu \mathsf{x} \tag{12}$$
$$\Rightarrow (\bar{\mathsf{A}} - \mu \mathsf{I})\mathsf{x} = 0 \tag{13}$$
$$\Rightarrow (\mathsf{TAT}^{-1} - \mu \mathsf{TT}^{-1})\mathsf{x} = 0 \tag{14}$$
$$\Rightarrow det(\mathsf{T}(\mathsf{A} - \mu \mathsf{I})\mathsf{T}^{-1}) = 0 \tag{15}$$
$$\Rightarrow det(\mathsf{A} - \mu \mathsf{I}) = 0 \tag{16}$$

Hence, equivalent state space representations have same set of eigen values. □

Solution

Given,

$$\dot{X} = AX + BU \tag{17}$$

$$\dot{W} = CW + DU \tag{18}$$

represent the same system. Hence, using (3) and (4), we can conclude that

$$[\lambda] = [\mu] \text{ and } W = TX$$

where T need not be identity matrix.
Hence, option 2 is the correct answer.

Solution Contd.

Let us now look at a numerical example to establish the correctness of the obtained result. Consider a SISO LTI system of order 2, represented by the equations

$$\dot{x}_1(t) = -x_1(t) + 1.5x_2(t) + 2u(t) \tag{19}$$

$$\dot{x}_2(t) = 4x_1(t) + u(t) \tag{20}$$

$$y(t) = 1.5x_1(t) + 0.625x_2(t) + u(t) \tag{21}$$

Solution Contd.

Its state space representation can be given by (4), where

$$\mathrm{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \mathrm{Y} = y(t) \tag{22}$$

$$\dot{\mathrm{X}} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, \mathrm{U} = u(t) \tag{23}$$

$$\mathrm{A} = \begin{bmatrix} -1 & 1.5 \\ 4 & 0 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{24}$$

$$\mathrm{C} = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}, \mathrm{D} = 1 \tag{25}$$

The eigen values for this state representation are

$$det(\lambda \mathrm{I} - \mathrm{A}) = 0 \tag{26}$$

$$\begin{vmatrix} \lambda + 1 & -1.5 \\ -4 & \lambda \end{vmatrix} = 0 \tag{27}$$

$$\lambda^2 + \lambda - 6 = 0 \tag{28}$$

$$[\lambda] = \{-3, 2\} \tag{29}$$

Solution Contd.

Even if we swap the equations, they still should represent the same system. So, consider a different state space representation,

$$\mathrm{W} = \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix}, \mathrm{Y} = y(t) \tag{30}$$

$$\dot{\mathrm{W}} = \begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix}, \mathrm{U} = u(t) \tag{31}$$

Clearly, $\mathrm{X} \neq \mathrm{W}$ and $\mathrm{W} = \mathrm{TX}$, where $\mathrm{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. From (3)

$$\bar{\mathrm{A}} = \begin{bmatrix} 0 & 4 \\ 1.5 & -1 \end{bmatrix}, \bar{\mathrm{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{32}$$

$$\bar{\mathrm{C}} = \begin{bmatrix} 0.625 & 1.5 \end{bmatrix}, \bar{\mathrm{D}} = 1 \tag{33}$$

Solution Contd.

Also, the eigen values for this state representation are

$$det(\mu \mathrm{I} - \mathrm{A}) = 0 \tag{34}$$

$$\begin{vmatrix} \mu & -4 \\ -1.5 & \mu + 1 \end{vmatrix} = 0 \tag{35}$$

$$\mu^2 + \mu - 6 = 0 \tag{36}$$

$$[\mu] = \{-3, 2\} \tag{37}$$

Hence, both the state space representations are equivalent, and satisfy $[\lambda] = [\mu]$ and $\mathrm{W} = \mathrm{TX}$, where T need not be identity matrix.