

Gate Assignment 3

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Download all python codes from

<https://github.com/YashasTadikamalla/EE3900/blob/main/GateAssignment3/codes>

and latex-tikz codes from

<https://github.com/YashasTadikamalla/EE3900/blob/main/GateAssignment3/GateAssignment3.tex>

1 PROBLEM (EC-2005 Q25)

A linear system is equivalently represented by two sets of state equations:

$$\dot{X} = AX + BU \text{ and } \dot{W} = CW + DU$$

Eigenvalues of the representations are also computed as $[\lambda]$ and $[\mu]$. Which of the following is true?

- 1) $[\lambda] = [\mu]$ and $X = W$
- 2) $[\lambda] = [\mu]$ and $X \neq W$
- 3) $[\lambda] \neq [\mu]$ and $X = W$
- 4) $[\lambda] \neq [\mu]$ and $X \neq W$

2 SOLUTION

Definition 2.1 (State Space representation). *It is a mathematical model of a physical system, as a set of input, output and state variables related by first order difference or differential equations. The most general state representation of a linear system with p inputs, q outputs, and n state variables can be written as*

$$\dot{X} = AX + BU \quad (2.0.1)$$

$$Y = CX + DU \quad (2.0.2)$$

where, $X \in R^n$ is the state vector, $Y \in R^q$ is the output vector, $U \in R^p$ is input vector, $A \in R^{n \times n}$ is the state matrix, $B \in R^{n \times p}$ is input matrix, $C \in R^{q \times n}$ is output matrix, $D \in R^{q \times p}$ is feedthrough matrix.

Definition 2.2 (Eigen values of State Space representation). *These are the solutions of the characteristic equation*

$$\Delta(\lambda) = \det(\lambda I - A) = 0 \quad (2.0.3)$$

where A is the state matrix.

Theorem 2.1. *Consider the n -dimensional continuous time linear system*

$$\dot{X} = AX + BU \text{ and } Y = CX + DU \quad (2.0.4)$$

Let T be an $n \times n$ real non-singular matrix and let $\bar{X} = TX$. Then the state equation

$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}U, Y = \bar{C}\bar{X} + \bar{D}U \quad (2.0.5)$$

where $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$ is said to be equivalent to (2.0.4).

Proof. Given, $\dot{X} = AX + BU$ and $Y = CX + DU$, T is a non-singular matrix such that $\bar{X} = TX$. The same system can be defined using \bar{X} as the state,

$$\dot{\bar{X}} = T\dot{X} = TAX + TBU \quad (2.0.6)$$

$$= TAT^{-1}\bar{X} + TBU \quad (2.0.7)$$

$$Y = CX + DU = CT^{-1}\bar{X} + DU \quad (2.0.8)$$

□

Theorem 2.2. *Equivalent state space representations have the same set of eigen values*

Proof.

$$\bar{\Delta}(\lambda) = \det(\lambda I - \bar{A}) \quad (2.0.9)$$

$$= \det(\lambda TT^{-1} - TAT^{-1}) \quad (2.0.10)$$

$$= \det(T(\lambda I - A)T^{-1}) \quad (2.0.11)$$

$$= \det(\lambda I - A) = \Delta(\lambda) \quad (2.0.12)$$

□

Given,

$$\dot{X} = AX + BU \quad (2.0.13)$$

$$\dot{W} = CW + DU \quad (2.0.14)$$

represent the same system. Hence, using (2.1) and (2.2), we can conclude that

$$[\lambda] = [\mu] \text{ and } X \neq W$$

Hence, option 2 is the correct answer.

Let us now look at a numerical example to establish the correctness of the obtained result. Consider a SISO LTI system of order 2, represented by the equations

$$\dot{x}_1(t) = -x_1(t) + 1.5x_2(t) + 2u(t) \quad (2.0.15)$$

$$\dot{x}_2(t) = 4x_1(t) + u(t) \quad (2.0.16)$$

$$y(t) = 1.5x_1(t) + 0.625x_2(t) + u(t) \quad (2.0.17)$$

Its state space representation can be given by (2.0.4), where

$$X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, Y = y(t) \quad (2.0.18)$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, U = u(t) \quad (2.0.19)$$

$$A = \begin{bmatrix} -1 & 1.5 \\ 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (2.0.20)$$

$$C = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}, D = 1 \quad (2.0.21)$$

The eigen values for this state representation are

$$\det(\lambda I - A) = 0 \quad (2.0.22)$$

$$\begin{vmatrix} \lambda + 1 & -1.5 \\ -4 & \lambda \end{vmatrix} = 0 \quad (2.0.23)$$

$$\lambda^2 + \lambda - 6 = 0 \quad (2.0.24)$$

$$[\lambda] = \{-3, 2\} \quad (2.0.25)$$

Even if we swap the equations, they still should represent the same system. So, consider a different state space representation,

$$W = \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix}, Y = y(t) \quad (2.0.26)$$

$$\dot{W} = \begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix}, U = u(t) \quad (2.0.27)$$

Clearly, $X \neq W$ and $W = TX$, where $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

From (2.1)

$$\bar{A} = \begin{bmatrix} 0 & 4 \\ 1.5 & -1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (2.0.28)$$

$$\bar{C} = \begin{bmatrix} 0.625 & 1.5 \end{bmatrix}, \bar{D} = 1 \quad (2.0.29)$$

Also, the eigen values for this state representation

are

$$\det(\mu I - A) = 0 \quad (2.0.30)$$

$$\begin{vmatrix} \mu & -4 \\ -1.5 & \mu + 1 \end{vmatrix} = 0 \quad (2.0.31)$$

$$\mu^2 + \mu - 6 = 0 \quad (2.0.32)$$

$$[\mu] = \{-3, 2\} \quad (2.0.33)$$

Hence, both the state space representations are equivalent, and satisfy $[\lambda] = [\mu]$ and $X \neq W$.