Gate Assignment 3 Presentation

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Question

Problem (EC-2005 Q25)

A linear system is equivalently represented by two sets of state equations:

$$\dot{X} = AX + BU \text{ and } \dot{W} = CW + DU$$

Eigenvalues of the representations are also computed as $[\lambda]$ and $[\mu]$. Which of the following is true?

1. $[\lambda] = [\mu]$ and $X = W$
2. $[\lambda] = [\mu]$ and $X \neq W$
3. $[\lambda] \neq [\mu]$ and $X = W$
4. $[\lambda] \neq [\mu]$ and $X \neq W$

Few prerequisites

Definition (State Space representation)

It is a mathematical model of a physical system, as a set of input, output and state variables related by first order difference or differential equations. The most general state representation of a linear system with p inputs, q outputs, and n state variables can be written as

$$\dot{X} = AX + BU \tag{1}$$

$$Y = CX + DU \tag{2}$$

where, $X \in R^n$ is the state vector, $Y \in R^q$ is the output vector, $U \in R^p$ is input vector, $A \in R^{n\times n}$ is the state matrix, $B \in R^{n\times p}$ is input matrix, $C \in R^{q\times n}$ is output matrix, $D \in R^{q\times p}$ is feedthrough matrix.

Definition (Eigen values of State Space representation)

These are the solutions of the charecteristic equation

$$\triangle(\lambda) = det(\lambda I - A) = 0 \tag{3}$$

where A is the state matrix.

Theorem

Consider the n-dimensional continuous time linear system

$$\dot{X} = AX + BU, Y = CX + DU \tag{4}$$

Let T be $n \times n$ real non-singular matrix and let $\bar{X} = TX$. Then the state equation

$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}U, Y = \bar{C}\bar{X} + \bar{D}U \tag{5}$$

where $\bar{A} = TAT^{-1}, \bar{B} = TB, \bar{C} = CT^{-1}, \bar{D} = D$ is equivalent to (4).

Proof.

Given, $\dot{X} = AX + BU$ and $Y = CX + DU$, T is a non-singular matrix such that $\bar{X} = TX$. The same system can be defined using $\bar{X}$ as the state,

$$\dot{\bar{X}} = T\dot{X} = TAX + TBU \tag{6}$$

$$= TAT^{-1}\bar{X} + TBU \tag{7}$$

$$Y = CX + DU = CT^{-1}\bar{X} + DU \tag{8}$$

□

Theorem

Equivalent state space representations have same set of eigen values

Proof.

For the representation in (4), the eigen values $[\lambda]$ are such that

$$Ax = \lambda x \tag{9}$$

$$\Rightarrow (A - \lambda I)x = 0 \tag{10}$$

$$\Rightarrow det(A - \lambda I) = 0 \tag{11}$$

For the representation in (5), the eigen values $[\mu]$, are such that

$$\bar{A}x = \mu x \tag{12}$$

$$\Rightarrow (\bar{A} - \mu I)x = 0 \tag{13}$$

$$\Rightarrow (TAT^{-1} - \mu TT^{-1})x = 0 \tag{14}$$

$$\Rightarrow det(T(A - \mu I)T^{-1}) = 0 \tag{15}$$

$$\Rightarrow det(A - \mu I) = 0 \tag{16}$$

Hence, equivalent state space representations have same set of eigen values. □

Solution

Given,

$$\dot{X} = AX + BU \tag{17}$$

$$\dot{W} = CW + DU \tag{18}$$

represent the same system. Hence, using (3) and (4), we can conclude that

$$[\lambda] = [\mu] \text{ and } W = TX$$

where T need not be identity matrix.
Hence, option 2 is the correct answer.

Solution Contd.

Let us now look at a numerical example to establish the correctness of the obtained result. Consider a SISO LTI system of order 2, represented by the equations

$$\dot{x}_1(t) = -x_1(t) + 1.5x_2(t) + 2u(t) \tag{19}$$

$$\dot{x}_2(t) = 4x_1(t) + u(t) \tag{20}$$

$$y(t) = 1.5x_1(t) + 0.625x_2(t) + u(t) \tag{21}$$

Solution Contd.

Its state space representation can be given by (4), where

$$X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, Y = y(t) \tag{22}$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, U = u(t) \tag{23}$$

$$A = \begin{bmatrix} -1 & 1.5 \\ 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{24}$$

$$C = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}, D = 1 \tag{25}$$

The eigen values for this state representation are

$$det(\lambda I - A) = 0 \tag{26}$$

$$\begin{vmatrix} \lambda + 1 & -1.5 \\ -4 & \lambda \end{vmatrix} = 0 \tag{27}$$

$$\lambda^2 + \lambda - 6 = 0 \tag{28}$$

$$[\lambda] = \{-3, 2\} \tag{29}$$

Solution Contd.

Even if we swap the equations, they still should represent the same system. So, consider a different state space representation,

$$W = \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix}, Y = y(t) \tag{30}$$

$$\dot{W} = \begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix}, U = u(t) \tag{31}$$

Clearly, $X \neq W$ and $W = TX$, where $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. From (3)

$$\bar{A} = \begin{bmatrix} 0 & 4 \\ 1.5 & -1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{32}$$

$$\bar{C} = \begin{bmatrix} 0.625 & 1.5 \end{bmatrix}, \bar{D} = 1 \tag{33}$$

Solution Contd.

Also, the eigen values for this state representation are

$$det(\mu I - A) = 0 \tag{34}$$

$$\begin{vmatrix} \mu & -4 \\ -1.5 & \mu + 1 \end{vmatrix} = 0 \tag{35}$$

$$\mu^2 + \mu - 6 = 0 \tag{36}$$

$$[\mu] = \{-3, 2\} \tag{37}$$

Hence, both the state space representations are equivalent, and satisfy $[\lambda] = [\mu]$ and $W = TX$, where T need not be identity matrix.