Gate Assignment 1 Presentation

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Question

Problem (EC-2013 Q8)

The impulse response of a system is $h(t) = tu(t)$. For an input $u(t-1)$, the output is

1. $\dfrac{t^2}{2}u(t)$
2. $\dfrac{t(t-1)}{2}u(t-1)$
3. $\dfrac{(t-1)^2}{2}u(t-1)$
4. $\dfrac{t^2-1}{2}u(t-1)$

Few prerequisites

Definition (Laplace Transform)

It is an integral transform that converts a function of a real variable t to a function of a complex variable s. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \tag{1}$$

Remark

Laplace transform of $f(t) = t^n, n \geq 1$ *is*

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, Re(s) > 0 \tag{2}$$

Proof.

Basis Step: $n = 1$

$$\mathcal{L}\left\{t\right\} = \int_0^{\infty} e^{-st} t dt = \left[\frac{te^{-st}}{-s}\right]_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st} dt \tag{3}$$

$$= 0 + \left[\frac{-1}{s^2} e^{-st}\right]_0^{\infty}, Re(s) > 0 = \frac{1}{s^2}, Re(s) > 0 \tag{4}$$

Inductive Step:

$$\mathcal{L}\left\{t^n\right\} = \int_0^{\infty} e^{-st} t^n dt = \left[\frac{t^n e^{-st}}{-s}\right]_0^{\infty} + \frac{n}{s}\int_0^{\infty} t^{n-1} e^{-st} dt \tag{5}$$

$$= 0 + \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, Re(s) > 0 = \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, Re(s) > 0 \tag{6}$$

Proof.

To prove that if (2) holds for $n = k$, it holds for $n = k + 1$. From (6)

$$\mathcal{L}\left\{t^{k+1}\right\} = \frac{k+1}{s}\mathcal{L}\left\{t^{k}\right\} \tag{7}$$

$$= \frac{(k+1)k!}{s(s^{k+1})} = \frac{(k+1)!}{s^{k+2}}, Re(s) > 0 \tag{8}$$

By mathematical induction, (2) is true $\forall n \geq 1$ $\square$

Lemma

For any real number c,

$$\mathcal{L}\left\{u(t-c)\right\} = \frac{e^{-cs}}{s}, Re(s) > 0 \tag{9}$$

Proof.

$$\mathcal{L}\left\{u(t-c)\right\} = \int_0^\infty e^{-st} u(t-c) dt = \int_c^\infty e^{-st} dt \tag{10}$$

$$= \left[-\frac{e^{-st}}{s}\right]_c^\infty = \frac{e^{-cs}}{s}, Re(s) > 0 \tag{11}$$

□

Definition (Inverse Laplace Transform)

Its the transformation of a Laplace transform into a function of time. If $F(s) = \mathcal{L}\left\{f(t)\right\}$, then Inverse laplace transform of $F(s)$ is $\mathcal{L}^{-1}\left\{F(s)\right\} = f(t)$.

Lemma (t-shift rule)

For any real number c,

$$\mathcal{L}\left\{u(t-c)f(t-c)\right\} = e^{-cs}F(s) \tag{12}$$

Proof.

$$\mathcal{L}\left\{u(t-c)f(t-c)\right\} = \int_0^\infty e^{-st}u(t-c)f(t-c)dt \tag{13}$$

$$= \int_c^\infty e^{-st}f(t-c)dt \tag{14}$$

$$= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau \,(t=\tau+c) \tag{15}$$

$$= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)d\tau = e^{-cs}F(s) \tag{16}$$

□

Corollary

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u(t-c)f(t-c) \tag{17}$$

Theorem (Convolution theorem)

Suppose $F(s) = \mathcal{L}\{f(t)\}, G(s) = \mathcal{L}\{g(t)\}$ *exist, then,*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) \tag{18}$$

Solution

Given,

$$h(t) = tu(t) \tag{19}$$
$$x(t) = u(t-1) \tag{20}$$

To find: $y(t)$. We know,

$$y(t) = h(t) * x(t) \tag{21}$$
$$= \mathcal{L}^{-1}\left\{H(s)X(s)\right\} \tag{22}$$

From (12) and (2),

$$H(s) = e^{0}\mathcal{L}\left\{t\right\} = \frac{1}{s^2} \tag{23}$$

From (9),

$$X(s) = \frac{e^{-s}}{s} \tag{24}$$

Solution Contd.

Substituting in (22),

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} \tag{25}$$

Consider

$$p(t) = \frac{t^2}{2} \tag{26}$$

From (2)

$$P(s) = \frac{2!}{2s^3} = \frac{1}{s^3} \tag{27}$$

Further, from (18), for $c = 1$

$$\mathcal{L}^{-1}\left\{e^{-s}P(s)\right\} = u(t-1)p(t-1) = u(t-1)\frac{(t-1)^2}{2} \tag{28}$$

$$\therefore y(t) = \frac{(t-1)^2}{2}u(t-1) \tag{29}$$

Option 3 is the correct answer.

Solution Contd.

$$h(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{30}$$

$$x(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{31}$$

$$y(t) = \begin{cases} \dfrac{(t-1)^2}{2}, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{32}$$

Solution Contd.

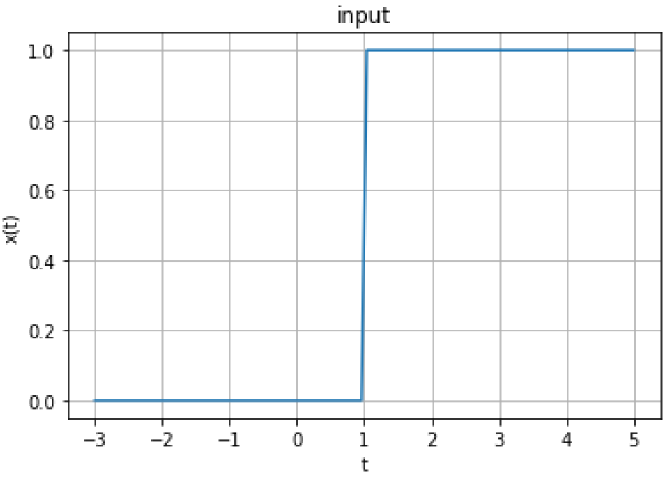

Figure: Plot of $x(t)$

Solution Contd.

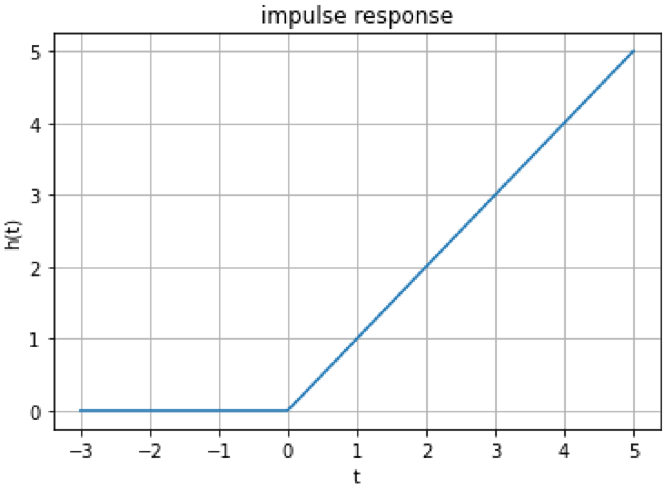

Figure: Plot of $h(t)$

Solution Contd.

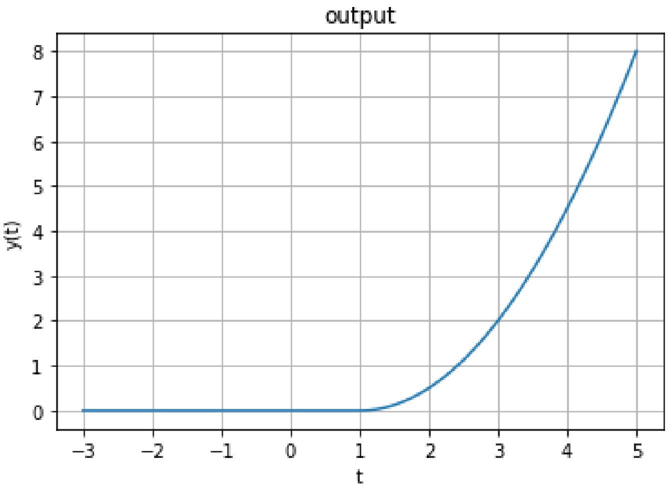

Figure: Plot of $y(t)$