Quiz 2

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Download all python codes from

```
https://github.com/YashasTadikamalla/EE3900/blob
/main/Quiz2/codes
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and latex-tikz codes from

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https://github.com/YashasTadikamalla/EE3900/blob
/main/Quiz2/Quiz2.tex
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1 Problem 3.25

Consider a right-sided sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} = \frac{z^2}{(z-a)(z-b)}$$

In section 3.3 we considered the determination of $x[n]$ by carrying out a partial fraction expansion, with $X(z)$ considered as a ratio of polynomials in z^{-1}. Carry out a partial fraction expansion of $X(z)$, considered as a ratio of polynomials in z, and determine $x[n]$ from this expansion.

2 Solution

Definition 2.1 (Z-transform)**.** *The z-transform of a discrete time signal $x[n]$ is defined as*

$$X(Z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{2.0.1}$$

Remark. The inverse z-transform of $X(Z)$ is $x[n]$

Definition 2.2 (Poles and zeroes of Z-transform)**.** *The values of z for which $X(z)$ is zero are called zeroes, while the values of z for which $X(z)$ is infinite, are called poles.*

Definition 2.3 (ROC)**.** *The region of convergence (ROC) is the set of values of z for which z-transform converges.*

Remark. For a right-sided sequence, the ROC extends outward from the outermost finite pole of $X(z)$ to $z = \infty$

Lemma 2.1. *z-tranform of $x[n] = \delta[n]$ is $X(z) = 1, ROC : \forall z$*

Proof.

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} \tag{2.0.2}$$

$$X(z) = z^0 = 1 \tag{2.0.3}$$

As it is defined $\forall z, ROC : \forall z$ □

Corollary 2.1.1. *Inverse z-transform of $X(z) = 1, ROC : \forall z$ is $x[n] = \delta[n]$*

Lemma 2.2. *z-transform of $x[n] = a^{n-k}u[n-k]$ is $X(z) = \frac{z^{-k}}{1-az^{-1}}, ROC : |z| > |a|$*

Proof.

$$x[n] = a^{n-k}u[n-k] \tag{2.0.4}$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} a^{n-k}u[n-k]z^{-n} \tag{2.0.5}$$

$$= \sum_{n=k}^{\infty} a^{n-k}z^{-n} = a^{-k}\sum_{n=k}^{\infty} a^n z^{-n} \tag{2.0.6}$$

If $|az^{-1}| < 1$,

$$X(z) = a^{-k}\frac{a^k z^k}{1-az^{-1}}, ROC : |z| > |a| \tag{2.0.7}$$

$$= \frac{z^{-k}}{1-az^{-1}}, ROC : |z| > |a| \tag{2.0.8}$$

□

Corollary 2.2.1. *Inverse z-transform of $X(z) = \frac{z^{1-k}}{z-a}, ROC : |z| > |a|$ is $x[n] = a^{n-k}u[n-k]$*

Given, a right-sided sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} \tag{2.0.9}$$

$$= \frac{z^2}{(z-a)(z-b)} \tag{2.0.10}$$

Clearly, $X(n)$ has two zeroes at $z = 0$ and $z = a$,$z = b$ are the poles. Without loss of generality, let $z = a$

be the outermost pole. Then, the ROC is $|z| > |a|$.
Adding and subtracting 1 in RHS

$$X(z) = 1 + \frac{z^2}{(z-a)(z-b)} - 1 \tag{2.0.11}$$
$$= 1 + \frac{z(a+b) - ab}{(z-a)(z-b)} \tag{2.0.12}$$

Multiplying and dividing with $(a-b)$

$$X(z) = 1 + \frac{a^2(z-b) - b^2(z-a)}{(z-a)(z-b)(a-b)} \tag{2.0.13}$$
$$= 1 + \frac{1}{a-b}\left(\frac{a^2}{z-a} - \frac{b^2}{z-b}\right) \tag{2.0.14}$$
$$= 1 + \frac{a^2}{a-b}\left(\frac{1}{z-a}\right) - \frac{b^2}{a-b}\left(\frac{1}{z-b}\right) \tag{2.0.15}$$

Using (2.1.1),(2.2.1) for $(k = 1)$,

$$x[n] = \delta[n] + \frac{a^2}{a-b}a^{n-1}u[n-1] - \frac{b^2}{a-b}b^{n-1}u[n-1] \tag{2.0.16}$$

$$\therefore x[n] = \delta[n] + \frac{a^{n+1}}{a-b}u[n-1] - \frac{b^{n+1}}{a-b}u[n-1] \tag{2.0.17}$$

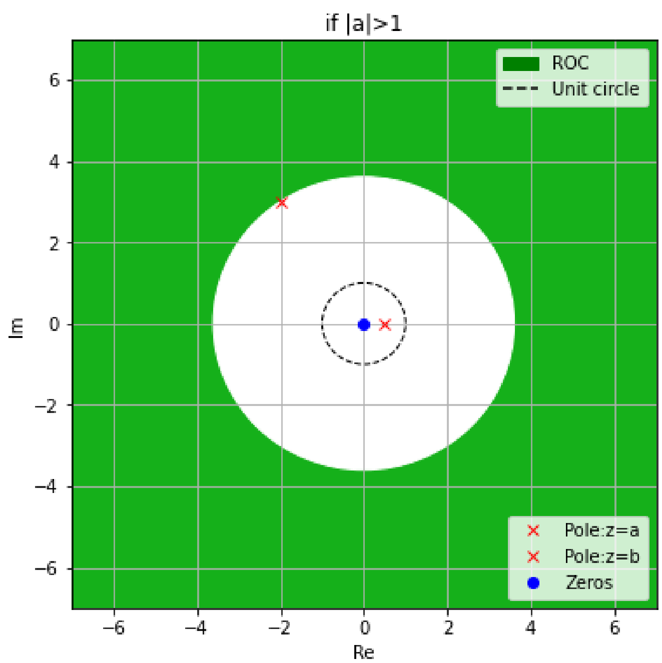

Fig. 0: A possible pole-zero plot with *ROC*

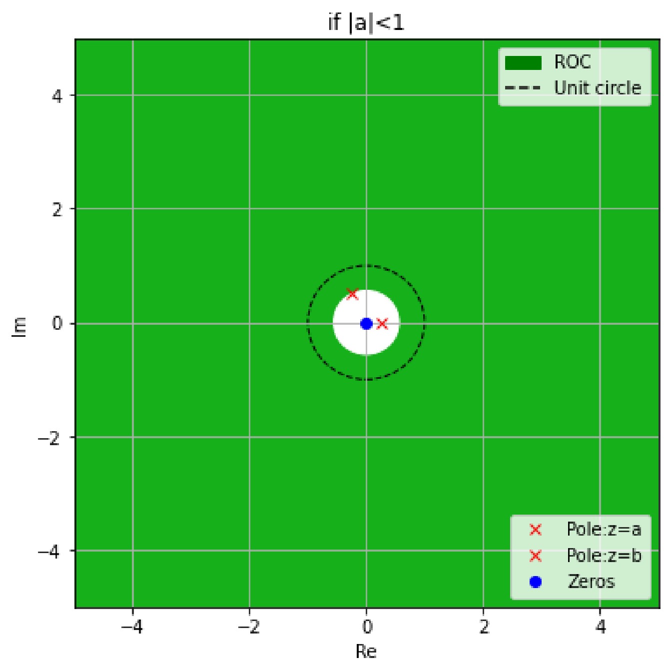

Fig. 0: A possible pole-zero plot with *ROC*