Gate Assignment 3

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Download all python codes from

https://github.com/YashasTadikamalla/EE3900/blob/main/GateAssignment3/codes

and latex-tikz codes from

https://github.com/YashasTadikamalla/EE3900/blob/main/GateAssignment3/GateAssignment3.tex

1 Problem (EC-2005 Q25)

A linear system is equivalently represented by two sets of state equations:

$$\dot{X} = AX + BU \text{ and } \dot{W} = CW + DU$$

Eigenvalues of the representations are also computed as $[\lambda]$ and $[\mu]$. Which of the following is true?

1) $[\lambda] = [\mu]$ and $X = W$
2) $[\lambda] = [\mu]$ and $X \neq W$
3) $[\lambda] \neq [\mu]$ and $X = W$
4) $[\lambda] \neq [\mu]$ and $X \neq W$

2 Solution

Definition 2.1 (State Space representation)**.** *It is a mathematical model of a physical system, as a set of input, output and state variables related by first order difference or differential equations. The most general state-representation of a linear system with p inputs, q outputs, and n state variables can be written as*

$$\dot{X}(t) = A(t)X(t) + B(t)U(t) \tag{2.0.1}$$

$$Y(t) = C(t)X(t) + D(t)U(t) \tag{2.0.2}$$

where, X(.) is the state vector, Y(.) is the output vector, u(.) is input vector, A(.) is the state matrix, B(.) is input matrix, C(.) is output matrix, D(.) is feedthrough matrix.

Definition 2.2 (Eigen values of State Space representation)**.** *Eigen values of a given State Space representation refer to solutions of the charecteristic equation*

$$\triangle(\lambda) = det(\lambda I - A) \tag{2.0.3}$$

Theorem 2.1. *Consider the n-dimensional continuous time linear system*

$$\dot{X} = AX + BU \text{ and } Y = CX + DU \tag{2.0.4}$$

Let T be an $n \times n$ real non-singular matrix and let $\bar{X} = TX$. Then the state equation

$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}U, Y = \bar{C}\bar{X} + \bar{D}U \tag{2.0.5}$$

where $\bar{A} = TAT^{-1}, \bar{B} = TB, \bar{C} = CT^{-1}, \bar{D} = D$ is said to be equivalent to (2.0.4).

Proof. Given, $\dot{X} = AX + BU$ and $Y = CX + DU$, T is a non-singular matrix such that $\bar{X} = TX$. The same system can be defined using $\bar{X}$ as the state,

$$\dot{\bar{X}} = T\dot{X} = TAX + TBU \tag{2.0.6}$$

$$= TAT^{-1}\bar{X} + TBU \tag{2.0.7}$$

$$Y = CX + DU = CT^{-1}\bar{X} + DU \tag{2.0.8}$$

□

Theorem 2.2. *Equivalent state space representations have the same set of eigen values*

Proof.

$$\bar{\triangle}(\lambda) = det(\lambda I - \bar{A}) \tag{2.0.9}$$

$$= det(\lambda TT^{-1} - TAT^{-1}) \tag{2.0.10}$$

$$= det(T(\lambda I - A)T^{-1}) \tag{2.0.11}$$

$$= det(\lambda I - A) = \triangle(\lambda) \tag{2.0.12}$$

□

Given,

$$\dot{X} = AX + BU \tag{2.0.13}$$

$$\dot{W} = CW + DU \tag{2.0.14}$$

represent the same system. Hence, using (2.1) and (2.2), we can conclude that

$$[\lambda] = [\mu] \text{ and } X \neq W$$

Hence, option 2 is the correct answer.

Let us now look at a numerical example to establish the correctness of the obtained result. Consider a

SISO LTI system of order 2, represented by the equations

$$\dot{x}_1(t) = -x_1(t) + 1.5x_2(t) + 2u(t) \quad (2.0.15)$$

$$\dot{x}_2(t) = 4x_1(t) + u(t) \quad (2.0.16)$$

$$y(t) = 1.5x_1(t) + 0.625x_2(t) + u(t) \quad (2.0.17)$$

Its state space representation can be given by (2.0.4), where

$$X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, Y = y(t) \quad (2.0.18)$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, U = u(t) \quad (2.0.19)$$

$$A = \begin{bmatrix} -1 & 1.5 \\ 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (2.0.20)$$

$$C = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}, D = 1 \quad (2.0.21)$$

The eigen values for this state representation are

$$det(\lambda I - A) = 0 \quad (2.0.22)$$

$$\begin{vmatrix} \lambda + 1 & -1.5 \\ -4 & \lambda \end{vmatrix} = 0 \quad (2.0.23)$$

$$\lambda^2 + \lambda - 6 = 0 \quad (2.0.24)$$

$$[\lambda] = \{-3, 2\} \quad (2.0.25)$$

Even if we swap the equations, they still should represent the same system. So, consider a different state space representation,

$$W = \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix}, Y = y(t) \quad (2.0.26)$$

$$\dot{W} = \begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix}, U = u(t) \quad (2.0.27)$$

Clearly, $X \neq W$ and $W = TX$, where $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. From (2.1)

$$\bar{A} = \begin{bmatrix} 0 & 4 \\ 1.5 & -1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (2.0.28)$$

$$\bar{C} = \begin{bmatrix} 0.625 & 1.5 \end{bmatrix}, \bar{D} = 1 \quad (2.0.29)$$

Also, the eigen values for this state representation are

$$det(\mu I - A) = 0 \quad (2.0.30)$$

$$\begin{vmatrix} \mu & -4 \\ -1.5 & \mu + 1 \end{vmatrix} = 0 \quad (2.0.31)$$

$$\mu^2 + \mu - 6 = 0 \quad (2.0.32)$$

$$[\mu] = \{-3, 2\} \quad (2.0.33)$$

Hence, both the state space representations are equivalent, and satisfy $[\lambda] = [\mu]$ and $X \neq W$.