Assignment 4 Presentation

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Question

Problem (Linear Forms Q2.33)

If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$$

are coplanar, find the value of k.

Solution

Given, two coplanar lines,

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \tag{1}$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5} \tag{2}$$

For them,

$$\mathrm{m}_1 = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathrm{m}_2 = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \tag{3}$$

$$\mathrm{A}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathrm{A}_2 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} \tag{4}$$

where $\mathrm{m}_1, \mathrm{m}_2$ are the respective direction vectors and $\mathrm{A}_1, \mathrm{A}_2$ are the respective points through which the lines pass.

Solution Contd.

As the lines are given to be coplanar, they are either parallel or intersecting. Let's check if our lines are parallel. Consider the matrix

$$\begin{pmatrix} -3 & 2k & 2 \\ 3k & 1 & -5 \end{pmatrix} \tag{5}$$

If the lines are parallel, the *rank* of the above matrix will be 1.

$$\Rightarrow \begin{vmatrix} -3 & 2k \\ 3k & 1 \end{vmatrix} = 0 \tag{6}$$

$$\Rightarrow 6k^2 + 3 = 0 \tag{7}$$

This is not possible. Hence, the lines must cannot be parallel. That means the lines must be intersecting.

Solution Contd.

That means the lines must be intersecting. In that case,

$$\left(\mathbf{A}_1 - \mathbf{A}_2\right)^T \left(\mathbf{m}_1 \times \mathbf{m}_2\right) = 0 \tag{8}$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix,

$$\begin{pmatrix} -3 & 2k & 2 \\ 3k & 1 & -5 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2 + kR_1}{1+2k^2}} \begin{pmatrix} -3 & 2k & 2 \\ 0 & 1 & \frac{2k-5}{1+2k^2} \end{pmatrix} \tag{9}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1 - 2kR_2}{-3}} \begin{pmatrix} 1 & 0 & \frac{10k+2}{-3(1+2k^2)} \\ 0 & 1 & \frac{2k-5}{1+2k^2} \end{pmatrix} \tag{10}$$

Solution Contd.

Substituting in (8),

$$\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}^T \begin{pmatrix} \frac{10k+2}{3(1+2k^2)} \\ \frac{5-2k}{1+2k^2} \\ 1 \end{pmatrix} = 0 \tag{11}$$

$$\Rightarrow -\frac{20k+4}{3+6k^2} + \frac{5-2k}{1+2k^2} = 3 \tag{12}$$

$$\Rightarrow \frac{11-26k}{3+6k^2} = 3 \tag{13}$$

$$\Rightarrow 9k^2 + 13k - 1 = 0 \tag{14}$$

Solving this, we get

$$k = \frac{-13 \pm \sqrt{205}}{18} \tag{15}$$

For the above values of k, the lines are intersecting, and hence, coplanar.

Solution Contd.

The lines can be rewritten as

$$x = A_1 + \lambda_1 m_1 \tag{16}$$

$$x = A_2 + \lambda_2 m_1 \tag{17}$$

The point of intersection can be found by equating them

$$A_1 + \lambda_1 m_1 = A_2 + \lambda_2 m_2 \tag{18}$$

$$A_1 - A_2 + \lambda_1 m_1 - \lambda_2 m_2 = 0 \tag{19}$$

Take $A = A_1 - A_2, \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, m = \begin{pmatrix} m_1 & -m_2 \end{pmatrix}$. Then, (19) can be rewritten as

$$A + m\lambda = 0 \tag{20}$$

$$\Rightarrow m\lambda = -A \tag{21}$$

Solution Contd.

Substituting in (21), we get

$$\begin{pmatrix} -3 & -3k \\ 2k & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \tag{22}$$

The corresponding augmented matrix is

$$\left(\begin{array}{cc|c} -3 & -3k & 2 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \tag{23}$$

Row reducing the matrix, we get

$$\left(\begin{array}{cc|c} -3 & -3k & 2 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cc|c} -3+4k & -3k-2 & 0 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \tag{24}$$

$$\xleftrightarrow{R_3 \leftarrow \frac{R_3+3R_2}{2}} \left(\begin{array}{cc|c} -3+4k & -3k-2 & 0 \\ 2k & -1 & -1 \\ 1+3k & 1 & 0 \end{array}\right) \tag{25}$$

Solution Contd.

$$\xleftrightarrow{R_1 \leftarrow R_1+(3k+2)R_3} \left(\begin{array}{cc|c} 9k^2+13k-1 & 0 & 0 \\ 2k & -1 & -1 \\ 1+3k & 1 & 0 \end{array}\right) \tag{26}$$

Clearly, from (14), it is

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2k & -1 & -1 \\ 1+3k & 1 & 0 \end{array}\right) \xleftrightarrow{R_3 \leftarrow 2R_3} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2k & -1 & -1 \\ 2+6k & 2 & 0 \end{array}\right) \tag{27}$$

$$\xleftrightarrow{R_3 \leftarrow R_3-3R_2} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \xleftrightarrow{R_1 \leftarrow R_1+R_3} \left(\begin{array}{cc|c} 2 & 5 & 3 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \tag{28}$$

$$\xleftrightarrow{R_3 \leftarrow R_3+R_1} \left(\begin{array}{cc|c} 2 & 5 & 3 \\ 2k & -1 & -1 \\ 0 & 0 & 0 \end{array}\right) \tag{29}$$

Solution Contd.

We can observe that, on row reduction, we get a complete row to be 0.

$$\xleftrightarrow{R_1 \leftarrow R_1 + 5R_2} \left(\begin{array}{cc|c} 2+10k & 0 & -2 \\ 2k & -1 & -1 \\ 0 & 0 & 0 \end{array}\right) \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2+10k}} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ 2k & -1 & -1 \\ 0 & 0 & 0 \end{array}\right) \tag{30}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ -2k & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 + 2kR_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ 0 & 1 & \frac{6k+2}{2+10k} \\ 0 & 0 & 0 \end{array}\right) \tag{31}$$

$$\therefore \lambda_1 = \frac{-1}{1+5k}, \lambda_2 = \frac{3k+1}{1+5k} \tag{32}$$

Solution Contd.

Substituting λ_1 in (16) to get the required point of intersection P,

$$\mathrm{P} = \begin{pmatrix} 1 + \frac{3}{1+5k} \\ 2 - \frac{2k}{1+5k} \\ 3 - \frac{2}{1+5k} \end{pmatrix} \Rightarrow \mathrm{P} = \begin{pmatrix} \frac{4+5k}{1+5k} \\ \\ \frac{2+8k}{1+5k} \\ \\ \frac{1+15k}{1+5k} \end{pmatrix} \tag{33}$$

where $k = \dfrac{-13 \pm \sqrt{205}}{18}$.

Solution Contd.

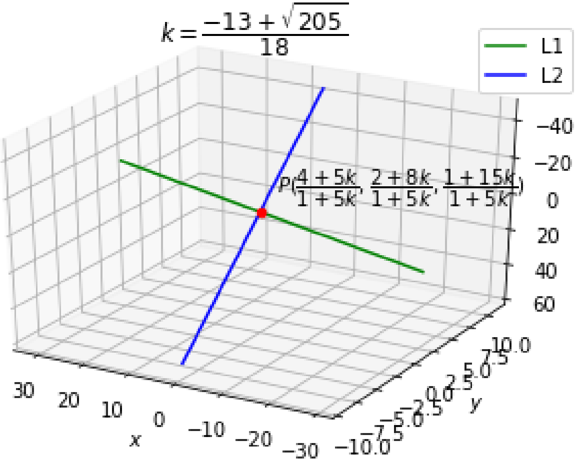

Figure: Intersecting lines - Case1

Solution Contd.

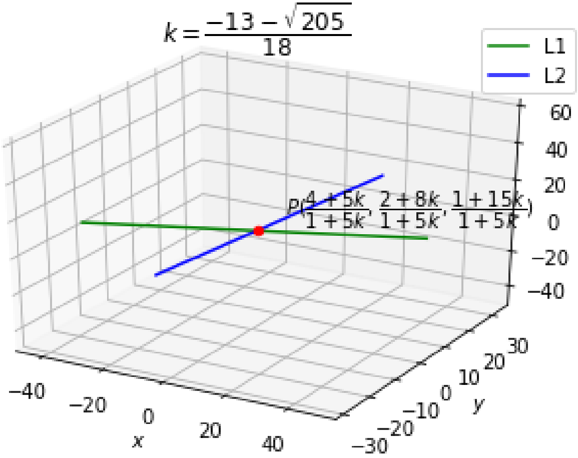

Figure: Intersecting lines - Case2