Assignment 5

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Download all python codes from

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https://github.com/YashasTadikamalla/EE3900/blob
    /main/Assignment5/codes
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and latex-tikz codes from

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https://github.com/YashasTadikamalla/EE3900/blob
    /main/Assignment5/Assignment5.tex
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1 Problem (Quadratic Forms Q2.31)

Find the equation of hyperbola with foci $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$ and length of latus rectum 36.

2 Solution

Theorem 2.1. *The equation of a conic with directrix* $\mathbf{n}^\top \mathbf{x} = c$*, eccentricity* e *and focus* $\mathbf{F}$ *is given by*

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{2.0.2}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{2.0.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{2.0.4}$$

For $|\mathbf{V}| > 0$*, the equation represents an ellipse, while for* $|\mathbf{V}| < 0$*, the equation represents a hyperbola.*

Theorem 2.2. *For* $|\mathbf{V}| \neq 0$ *the equations of minor and major axes of the conic in* (2.0.1) *are given by*

$$\mathbf{p_i}^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{2.0.5}$$

Theorem 2.3. *The eccentricity of the conic in* (2.0.1) *is given by*

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{2.0.6}$$

Definition 2.1 (Latus rectum)**.** *The latus rectum of a conic section is the chord (line segment) that passes through the focus, is perpendicular to the major axis and has both endpoints on the curve.*

Theorem 2.4. *The equation latus rectum of the conic in* (2.0.1) *is given by*

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{2.0.7}$$

Theorem 2.5. *For* $|\mathbf{V}| \neq 0$*, the lengths of semi-major and semi-minor axes of the conic in* (2.0.1) *are*

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}}, \sqrt{\left|\frac{f - \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}\right|} \tag{2.0.8}$$

Theorem 2.6. *For* $|\mathbf{V}| \neq 0$*, the length of latus rectum (LLR) of the conic in* (2.0.1) *is given by*

$$LLR = \frac{2\left|\frac{f - \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}\right|}{\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}}} \tag{2.0.9}$$

Proof. Using (2.0.8), we can write

$$\mathbf{F} = \mathbf{c} \pm \left(\sqrt{\frac{(\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}} \right) \mathbf{p_1} \tag{2.0.10}$$

Also, we know, for $|\mathbf{V}| \neq 0$,

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p_1} \tag{2.0.11}$$

$$\implies \mathbf{p_1}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{2.0.12}$$

is the equation of latus rectum. Solving this with (2.0.1) and simplifying, we get

$$\mathbf{H} = \mathbf{F} \pm \sqrt{\lambda_1 \left(\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2^2} \right)} \mathbf{p_2} \tag{2.0.13}$$

as the points of intersection. Hence, the length of line segment is

$$LLR = \left\| 2 \sqrt{\lambda_1 \left(\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2^2} \right)} \mathbf{p_2} \right\| = \frac{2\left|\frac{f - \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}\right|}{\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}}} \tag{2.0.14}$$

$\square$

Given, length of latus rectum is 36 and focii are $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$. Let us consider $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$ for solving the problem.

$$\mathbf{F} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} \implies \|\mathbf{F}\| = 12 \tag{2.0.15}$$

Let $\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = \alpha$. From (2.0.8),(2.0.6),(2.0.9)

$$\sqrt{\frac{\alpha}{\lambda_1}} \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = 12 \tag{2.0.16}$$

$$\frac{2\left(\frac{-\alpha}{\lambda_2}\right)}{\sqrt{\frac{\alpha}{\lambda_1}}} = 36 \tag{2.0.17}$$

Dividing (2.0.16) by (2.0.17) gives

$$\frac{\lambda_1}{\lambda_2} = -3 \tag{2.0.18}$$

$$\implies e = 2 \tag{2.0.19}$$

$$\implies \sqrt{\frac{\alpha}{\lambda_1}} = 6 \tag{2.0.20}$$

The associated directrix is perpendicular to the y-axis and passes through the point

$$\begin{pmatrix} 0 \\ \sqrt{\frac{\alpha}{e^2 \lambda_1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.0.21}$$

Hence, its equation is

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right) = 0 \tag{2.0.22}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{2.0.23}$$

Comparing it with $\mathbf{n}^\top \mathbf{x} = c$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = 3 \implies \|\mathbf{n}\| = 1 \tag{2.0.24}$$

Calculating $\mathbf{V}, \mathbf{u}$ and f,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{2.0.25}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \tag{2.0.26}$$

$$\mathbf{u} = 3(2^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.27}$$

$$f = 1^2(12^2) - 3^2(2^2) = 108 \tag{2.0.28}$$

Hence, the required equation is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x} + 108 = 0 \tag{2.0.29}$$

Also, from (2.0.7), the equations of latus rectum is

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 12 \end{pmatrix} \right) = 0 \tag{2.0.30}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 12 \tag{2.0.31}$$

Similarly, the equations of directrix and latus rectum associated with $\begin{pmatrix} 0 \\ -12 \end{pmatrix}$ are given by

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -3 \tag{2.0.32}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \tag{2.0.33}$$

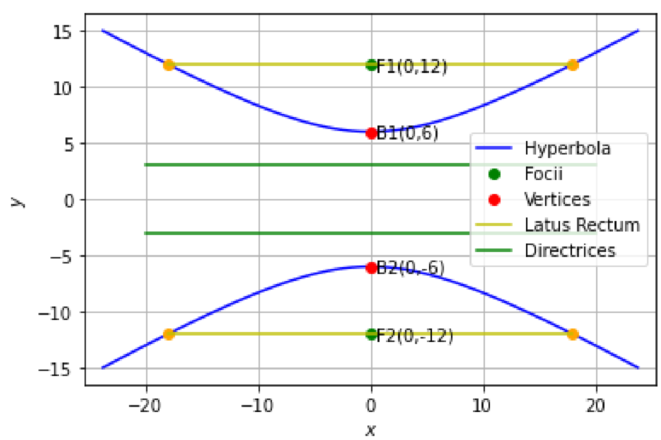

Fig. 0: Hyperbola