Assignment 4

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Download all python codes from

https://github.com/YashasTadikamalla/EE3900/blob/main/Assignment4/codes

and latex-tikz codes from

https://github.com/YashasTadikamalla/EE3900/blob/main/Assignment4/Assignment4.tex

1 Problem (Linear Forms Q2.33)

If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$$
$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$$

are coplanar, find the value of k.

2 Solution

Given, two coplanar lines,

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \tag{2.0.1}$$
$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5} \tag{2.0.2}$$

For them,

$$\mathbf{m_1} = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \tag{2.0.3}$$
$$\mathbf{A_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{A_2} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} \tag{2.0.4}$$

where $\mathbf{m_1}, \mathbf{m_2}$ are the respective direction vectors and $\mathbf{A_1}, \mathbf{A_2}$ are the respective points through which the lines pass. To find: k.

If two lines are coplanar, they must be either parallel or intersecting. If the lines are parallel, the matrix

$$\begin{pmatrix} -3 & 2k & 2 \\ 3k & 1 & -5 \end{pmatrix} \tag{2.0.5}$$

should have $rank = 1$.

$$\implies \begin{vmatrix} -3 & 2k \\ 3k & 1 \end{vmatrix} = 0 \tag{2.0.6}$$
$$\implies 6k^2 + 3 = 0 \tag{2.0.7}$$

This is not possible. Hence, the lines must be intersecting. In that case,

$$(\mathbf{A_1} - \mathbf{A_2})^T (\mathbf{m_1} \times \mathbf{m_2}) = 0 \tag{2.0.8}$$

$\mathbf{m_1} \times \mathbf{m_2}$ is obtained by row reducing the matrix,

$$\begin{pmatrix} -3 & 2k & 2 \\ 3k & 1 & -5 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2 + kR_1}{1+2k^2}} \begin{pmatrix} -3 & 2k & 2 \\ 0 & 1 & \frac{2k-5}{1+2k^2} \end{pmatrix} \tag{2.0.9}$$
$$\xleftrightarrow{R_1 \leftarrow \frac{R_1 - 2kR_2}{-3}} \begin{pmatrix} 1 & 0 & \frac{10k+2}{-3(1+2k^2)} \\ 0 & 1 & \frac{2k-5}{1+2k^2} \end{pmatrix} \tag{2.0.10}$$

Substituting in (2.0.8),

$$\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}^T \begin{pmatrix} \frac{10k+2}{3(1+2k^2)} \\ \frac{5-2k}{1+2k^2} \\ 1 \end{pmatrix} = 0 \tag{2.0.11}$$
$$\implies -\frac{20k+4}{3+6k^2} + \frac{5-2k}{1+2k^2} = 3 \tag{2.0.12}$$
$$\implies \frac{11-26k}{3+6k^2} = 3 \tag{2.0.13}$$
$$\implies 9k^2 + 13k - 1 = 0 \tag{2.0.14}$$

Solving this, we get

$$k = \frac{-13 \pm \sqrt{205}}{18} \tag{2.0.15}$$

For the above values of k, the lines are intersecting, and hence, coplanar. The lines can be rewritten as

$$\mathbf{x} = \mathbf{A_1} + \lambda_1 \mathbf{m_1} \tag{2.0.16}$$
$$\mathbf{x} = \mathbf{A_2} + \lambda_2 \mathbf{m_1} \tag{2.0.17}$$

Point of intersection can be found by equating them

$$\mathbf{A_1} + \lambda_1 \mathbf{m_1} = \mathbf{A_2} + \lambda_2 \mathbf{m_2} \tag{2.0.18}$$
$$\mathbf{A_1} - \mathbf{A_2} + \lambda_1 \mathbf{m_1} - \lambda_2 \mathbf{m_2} = 0 \tag{2.0.19}$$

Take $\mathbf{A} = \mathbf{A_1} - \mathbf{A_2}, \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \mathbf{m_1} & -\mathbf{m_2} \end{pmatrix}$. Then, (2.0.19) can be rewritten as

$$\mathbf{A} + \mathbf{m}\lambda = 0 \tag{2.0.20}$$

$$\implies \mathbf{m}\lambda = -\mathbf{A} \tag{2.0.21}$$

Substituting in (2.0.21), we get

$$\begin{pmatrix} -3 & -3k \\ 2k & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \tag{2.0.22}$$

The corresponding augmented matrix is

$$\left(\begin{array}{cc|c} -3 & -3k & 2 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \tag{2.0.23}$$

Row reducing the matrix, we get

$$\left(\begin{array}{cc|c} -3 & -3k & 2 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \tag{2.0.24}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cc|c} -3+4k & -3k-2 & 0 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \tag{2.0.25}$$

$$\xleftrightarrow{R_3 \leftarrow \frac{R_3 + 3R_2}{2}} \left(\begin{array}{cc|c} -3+4k & -3k-2 & 0 \\ 2k & -1 & -1 \\ 1+3k & 1 & 0 \end{array}\right) \tag{2.0.26}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + (3k+2)R_3} \left(\begin{array}{cc|c} 9k^2 + 13k - 1 & 0 & 0 \\ 2k & -1 & -1 \\ 1+3k & 1 & 0 \end{array}\right) \tag{2.0.27}$$

Clearly, from (2.0.14), it is

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2k & -1 & -1 \\ 1+3k & 1 & 0 \end{array}\right) \xleftrightarrow{R_3 \leftarrow 2R_3} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2k & -1 & -1 \\ 2+6k & 2 & 0 \end{array}\right) \tag{2.0.28}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 3R_2} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \tag{2.0.29}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_3} \left(\begin{array}{cc|c} 2 & 5 & 3 \\ 2k & -1 & -1 \\ 2 & 5 & 3 \end{array}\right) \tag{2.0.30}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_1} \left(\begin{array}{cc|c} 2 & 5 & 3 \\ 2k & -1 & -1 \\ 0 & 0 & 0 \end{array}\right) \tag{2.0.31}$$

We can observe that, on row reduction, we get a complete row to be 0.

$$\xleftrightarrow{R_1 \leftarrow R_1 + 5R_2} \left(\begin{array}{cc|c} 2+10k & 0 & -2 \\ 2k & -1 & -1 \\ 0 & 0 & 0 \end{array}\right) \tag{2.0.32}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{2+10k}} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ 2k & -1 & -1 \\ 0 & 0 & 0 \end{array}\right) \tag{2.0.33}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ -2k & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \tag{2.0.34}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 2kR_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{-2}{2+10k} \\ 0 & 1 & \frac{6k+2}{2+10k} \\ 0 & 0 & 0 \end{array}\right) \tag{2.0.35}$$

$$\therefore \lambda_1 = \frac{-1}{1+5k}, \lambda_2 = \frac{3k+1}{1+5k} \tag{2.0.36}$$

Substituting λ_1 in (2.0.16) to get the required point of intersection $\mathbf{P}$,

$$\mathbf{P} = \begin{pmatrix} 1 + \frac{3}{1+5k} \\ 2 - \frac{2k}{1+5k} \\ 3 - \frac{2}{1+5k} \end{pmatrix} \tag{2.0.37}$$

$$\implies \mathbf{P} = \begin{pmatrix} \frac{4+5k}{1+5k} \\ \frac{2+8k}{1+5k} \\ \frac{1+15k}{1+5k} \end{pmatrix} \tag{2.0.38}$$

where $k = \dfrac{-13 \pm \sqrt{205}}{18}$.

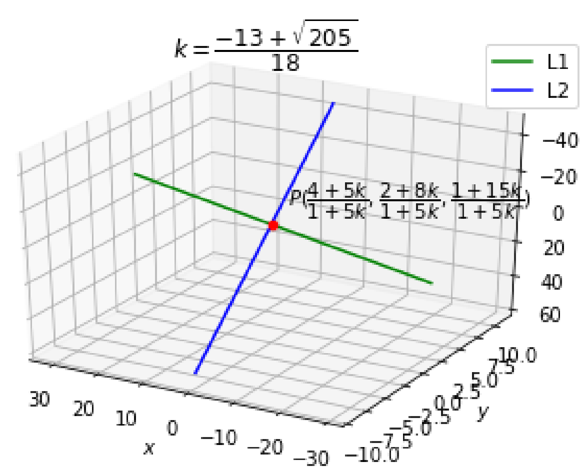

Fig. 0: Intersecting lines - Case1

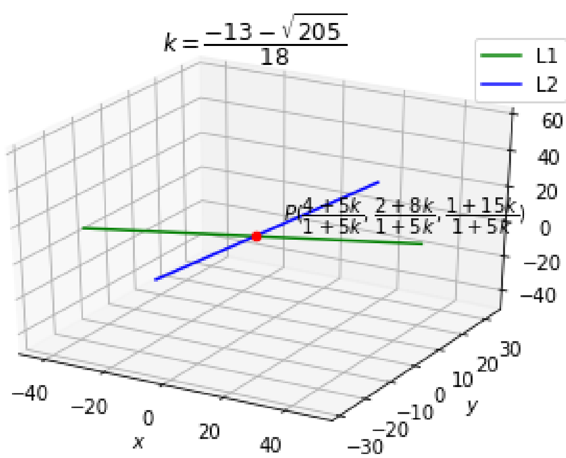

Fig. 0: Intersecting lines - Case2