Gate Assignment 3

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Download all python codes from

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https://github.com/YashasTadikamalla/EE3900/blob
    /main/GateAssignment3/codes
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and latex-tikz codes from

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https://github.com/YashasTadikamalla/EE3900/blob
    /main/GateAssignment3/GateAssignment3.tex
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1 Problem (EC-2005 Q25)

A linear system is equivalently represented by two sets of state equations:

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU} \text{ and } \dot{\mathbf{W}} = \mathbf{CW} + \mathbf{DU}$$

Eigenvalues of the representations are also computed as $[\lambda]$ and $[\mu]$. Which of the following is true?

1) $[\lambda] = [\mu]$ and $\mathbf{X} = \mathbf{W}$
2) $[\lambda] = [\mu]$ and $\mathbf{X} \neq \mathbf{W}$
3) $[\lambda] \neq [\mu]$ and $\mathbf{X} = \mathbf{W}$
4) $[\lambda] \neq [\mu]$ and $\mathbf{X} \neq \mathbf{W}$

2 Solution

Definition 2.1 (State Space representation)**.** *It is a mathematical model of a physical system, as a set of input, output and state variables related by first order difference or differential equations. The most general state representation of a linear system with p inputs, q outputs, and n state variables can be written as*

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU} \tag{2.0.1}$$
$$\mathbf{Y} = \mathbf{CX} + \mathbf{DU} \tag{2.0.2}$$

where, $\mathbf{X} \in R^n$ *is the state vector,* $\mathbf{Y} \in R^q$ *is the output vector,* $\mathbf{U} \in R^p$ *is input vector,* $\mathbf{A} \in R^{n\times n}$ *is the state matrix,* $\mathbf{B} \in R^{n\times p}$ *is input matrix,* $\mathbf{C} \in R^{q\times n}$ *is output matrix,* $\mathbf{D} \in R^{q\times p}$ *is feedthrough matrix.*

Definition 2.2 (Eigen values of State Space representation)**.** *These are the solutions of the charecteristic equation*

$$\triangle(\lambda) = det(\lambda\mathbf{I} - \mathbf{A}) = 0 \tag{2.0.3}$$

where A is the state matrix.

Theorem 2.1. *Consider the n-dimensional continuous time linear system*

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}, \mathbf{Y} = \mathbf{CX} + \mathbf{DU} \tag{2.0.4}$$

Let $\mathbf{T}$ *be an* $n \times n$ *real non-singular matrix and let* $\bar{\mathbf{X}} = \mathbf{TX}$*. Then the state equation*

$$\dot{\bar{\mathbf{X}}} = \bar{\mathbf{A}}\bar{\mathbf{X}} + \bar{\mathbf{B}}\mathbf{U}, \mathbf{Y} = \bar{\mathbf{C}}\bar{\mathbf{X}} + \bar{\mathbf{D}}\mathbf{U} \tag{2.0.5}$$

where $\bar{\mathbf{A}} = \mathbf{TAT}^{-1}, \bar{\mathbf{B}} = \mathbf{TB}, \bar{\mathbf{C}} = \mathbf{CT}^{-1}, \bar{\mathbf{D}} = \mathbf{D}$ *is said to be equivalent to* (2.0.4).

Proof. Given, $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}$ and $\mathbf{Y} = \mathbf{CX} + \mathbf{DU}$, T is a non-singular matrix such that $\bar{\mathbf{X}} = \mathbf{TX}$. The same system can be defined using $\bar{\mathbf{X}}$ as the state,

$$\dot{\bar{\mathbf{X}}} = \mathbf{T}\dot{\mathbf{X}} = \mathbf{TAX} + \mathbf{TBU} \tag{2.0.6}$$
$$= \mathbf{TAT}^{-1}\bar{\mathbf{X}} + \mathbf{TBU} \tag{2.0.7}$$
$$\mathbf{Y} = \mathbf{CX} + \mathbf{DU} = \mathbf{CT}^{-1}\bar{\mathbf{X}} + \mathbf{DU} \tag{2.0.8}$$

□

Theorem 2.2. *Equivalent state space representations have same set of eigen values*

Proof. For the representation in (2.0.4), the eigen values $[\lambda]$ are such that

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{2.0.9}$$
$$\Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \tag{2.0.10}$$
$$\Rightarrow det(\mathbf{A} - \lambda\mathbf{I} = 0 \tag{2.0.11}$$

For the representation in (2.0.5), the eigen values $[\mu]$, are such that

$$\bar{\mathbf{A}}\mathbf{x} = \mu\mathbf{x} \tag{2.0.12}$$
$$\Rightarrow (\bar{\mathbf{A}} - \mu\mathbf{I})\mathbf{x} = 0 \tag{2.0.13}$$
$$\Rightarrow (\mathbf{TAT}^{-1} - \mu\mathbf{TT}^{-1})\mathbf{x} = 0 \tag{2.0.14}$$
$$\Rightarrow det(\mathbf{T}(\mathbf{A} - \mu\mathbf{I})\mathbf{T}^{-1}) = 0 \tag{2.0.15}$$
$$\Rightarrow det(\mathbf{A} - \mu\mathbf{I}) = 0 \tag{2.0.16}$$

Hence, equivalent state space representations have same set of eigen values. □

Given,

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU} \tag{2.0.17}$$
$$\dot{\mathbf{W}} = \mathbf{CW} + \mathbf{DU} \tag{2.0.18}$$

represent the same system. Hence, using (2.1) and (2.2), we can conclude that

$$[\lambda] = [\mu] \text{ and } \mathbf{W} = \mathbf{TX}$$

where $\mathbf{T}$ need not be identity matrix.
Hence, option 2 is the correct answer.
Let us now look at a numerical example to establish the correctness of the obtained result. Consider a SISO LTI system of order 2, represented by the equations

$$\dot{x}_1(t) = -x_1(t) + 1.5x_2(t) + 2u(t) \tag{2.0.19}$$
$$\dot{x}_2(t) = 4x_1(t) + u(t) \tag{2.0.20}$$
$$y(t) = 1.5x_1(t) + 0.625x_2(t) + u(t) \tag{2.0.21}$$

Its state space representation can be given by (2.0.4), where

$$\mathbf{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \mathbf{Y} = y(t) \tag{2.0.22}$$
$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, \mathbf{U} = u(t) \tag{2.0.23}$$
$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 4 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{2.0.24}$$
$$\mathbf{C} = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}, \mathbf{D} = 1 \tag{2.0.25}$$

The eigen values for this state representation are

$$det(\lambda\mathbf{I} - \mathbf{A}) = 0 \tag{2.0.26}$$
$$\begin{vmatrix} \lambda + 1 & -1.5 \\ -4 & \lambda \end{vmatrix} = 0 \tag{2.0.27}$$
$$\lambda^2 + \lambda - 6 = 0 \tag{2.0.28}$$
$$[\lambda] = \{-3, 2\} \tag{2.0.29}$$

Even if we swap the equations, they still should represent the same system. So, consider a different state space representation,

$$\mathbf{W} = \begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix}, \mathbf{Y} = y(t) \tag{2.0.30}$$
$$\dot{\mathbf{W}} = \begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix}, \mathbf{U} = u(t) \tag{2.0.31}$$

Clearly, $\mathbf{X} \neq \mathbf{W}$ and $\mathbf{W} = \mathbf{TX}$, where $\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

From (2.1)

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 4 \\ 1.5 & -1 \end{bmatrix}, \bar{\mathbf{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{2.0.32}$$
$$\bar{\mathbf{C}} = \begin{bmatrix} 0.625 & 1.5 \end{bmatrix}, \bar{\mathbf{D}} = 1 \tag{2.0.33}$$

Also, the eigen values for this state representation are

$$det(\mu\mathbf{I} - \mathbf{A}) = 0 \tag{2.0.34}$$
$$\begin{vmatrix} \mu & -4 \\ -1.5 & \mu + 1 \end{vmatrix} = 0 \tag{2.0.35}$$
$$\mu^2 + \mu - 6 = 0 \tag{2.0.36}$$
$$[\mu] = \{-3, 2\} \tag{2.0.37}$$

Hence, both the state space representations are equivalent, and satisfy $[\lambda] = [\mu]$ and $\mathbf{W} = \mathbf{TX}$, where $\mathbf{T}$ need not be identity matrix.