Boundary value ODE Using Interval Arithmetic

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Problem Statement

- This project aims to solve a Boundary Value problem which involves non-linear differential equations subject to endpoint constraints.
- We aim to solve the problem using techniques like Interval Arithmetic which ensures rigorous bounds.
- Specifically, we are solving the Brachistochrone Problem. For this problem, traditional analytical solutions already exist, but we are trying to solve it using Taylor Series and Interval Arithmetic methods.

Introduction to Brachistochrone Problem

- The Brachistochrone problem is a classical problem in calculus of variations
- Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time. The term derives from the Greek (brachistos) "the shortest" and (chronos) "time, delay."

Snell's Law Solution

Which path Light would take?

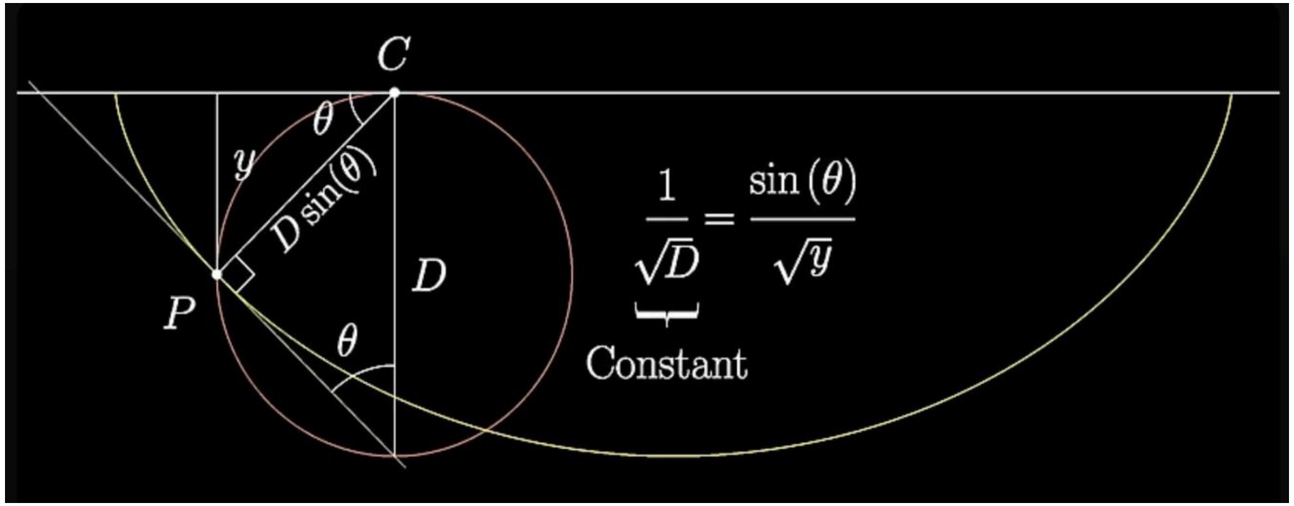
- Use the empirical fact that Light takes the fastest path
- The problem is modelled as if Light is travelling through vertical column with different velocity = √ y, and the path light will take in such a scenario to travel between 2 point will be the shortest path
- Then using Snell's law the angle θ made by the path with the vertical will satisfy

$\frac{\sin\theta}{v} =$ constant

$\frac{\sin\theta}{\sqrt{y}} =$ constant

- This path is just the expression

of the path of a Cycloid



Existing Analytical Solution

- The Time to traverse a segment of the curve is given by:

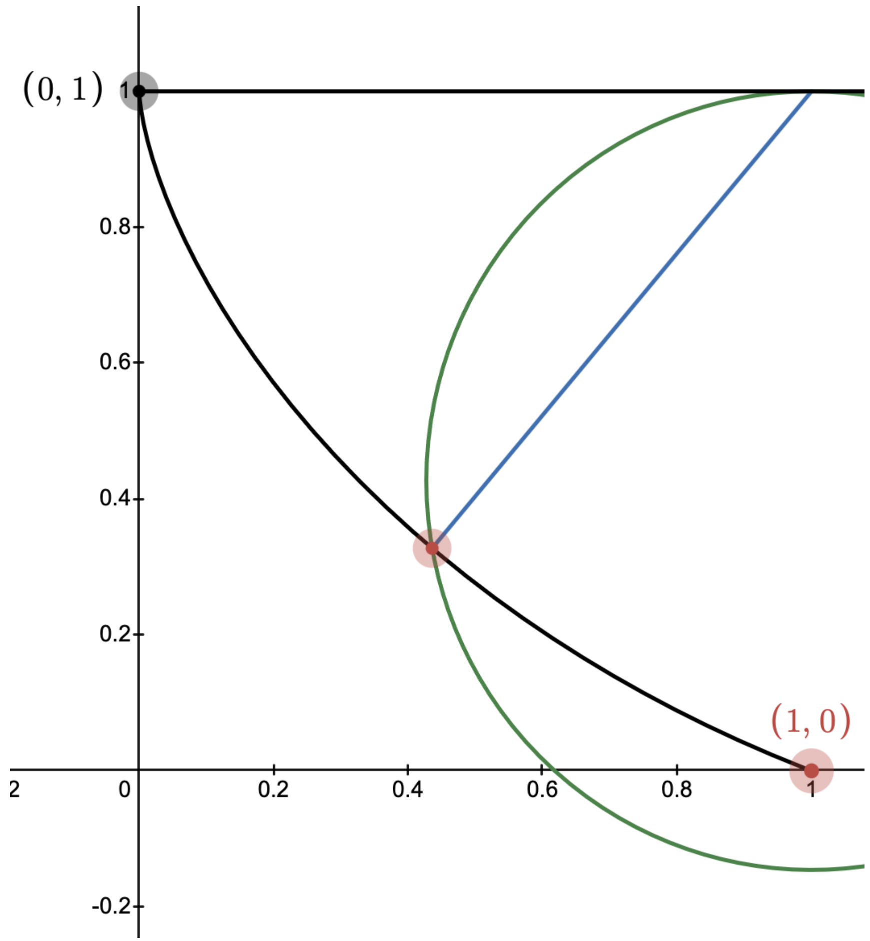
$$T[y, y'] = \int_A^B \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}} dx.$$

- This equation may be simplified to:

$$1 + (y')^2 - 2(y_B - y)y'' = 0$$

- The solution curve of the equation given as parametric equations of an upside down cycloid are:

$$x(\phi) = c_1(\phi - \sin\phi) + c_2,$$
$$y(\phi) = y_B - c_1(1 - cos\phi)$$



Motivation

- Computers work in discrete steps and use floating-point numbers, which can cause rounding errors. These small errors can build up over time, so we use *interval arithmetic* to track possible ranges of values and keep the results accurate.
- When we use Taylor series, we get polynomial equations. These can be solved accurately using interval version of Newton-Raphson method, which gives guaranteed bounds on the answer.
- Intervals appear naturally in math. For example, in the Mean Value

$$f(\hat{x}) = f(\tilde{x}) + (\hat{x} - \tilde{x})\, f'(\varepsilon) \qquad \hat{x} \le \varepsilon \le \tilde{x}$$

with ε in real numb is unknown. But the relation $f(\hat{x}) \in f(\tilde{x}) + (\hat{x} - \tilde{x})\, f'(X)$ with $X := [\hat{x}, \tilde{x}]$ can be used numerically with the aid of interval analysis.

Motivation

- Computers work in discrete steps and use floating-point numbers, which can cause rounding errors. These small errors can build up over time, so we use *interval arithmetic* to track possible ranges of values and keep the results accurate.
- Intervals appear naturally in math. For example, in the Mean Value Theorem:

$$f(\hat{x}) = f(\tilde{x}) + (\hat{x} - \tilde{x})\, f'(\varepsilon) \quad \text{with} \quad \hat{x} \leq \varepsilon \leq \tilde{x}$$

cannot be treated numerically in real numbers because ε is unknown

Overview of Our Solution

(Using Taylor Series & Interval Arithmetic)

$$1 + (y')^2 - 2(y_B - y)y'' = 0$$

- The Taylor series expansion is used to approximate the solution to the nonlinear boundary value ODE, providing a systematic way to construct accurate power series representations of the solution over intervals
- The series is substituted into our differential equation to obtain simultaneous equations by coefficient matching
- Interval Newton Method to be used for finding rigorous bounds of the coefficients of the Power series, and the curve represented as an Interval valued function to ensure the solution is enclosed
- By applying interval arithmetic within the boundary value ODE framework, the solution aims to precisely capture uncertainties and verify solution correctness

Taylor Series & Interval Arithmetic Solution

Solving for Initial case

- The Differential Equation: $1+(y')^2-2(y_B-y)y''=0$
- The Conditions: $y_B=1$ (i) $y(0)=1$ (ii) $y(1)=0$
- Taylor Series : $y=a_0+a_1x+a_2x^2+a_3x^3+\ ...$

$$y'=a_1+2a_2x+3a_3x^2+\ ...$$

$$y''=2a_2+6a_3x+\ ...$$

- Expression obtained after substitution

$$1+\left(a_1+2a_2x+3x^2a_3\right)^2-2\left(2a_2+6a_3x\right)\left(1-a_0-a_1x-x^2a_2-x^3a_3\right)=0$$

Taylor Series & Interval Arithmetic Solution

Equations obtained on matching coefficients:

$$1 - 4\,(1 - a_0)a_2 + (a_1)^2 = 0$$

$$-12\,(1 - a_0)a_3 + 8a_1a_2 = 0$$

$$\frac{1}{2}\left(36a_1a_3 + 16(a_2)^2\right) = 0$$

$$28a_2a_3 = 0$$

Particularly for n = 3, we have 4 variables. We already have 2 conditions from the boundary values, so we take the first two equations of the above 4 equations obtained.

Problem with this case

So, the 4 equations that we solve are: $a_0 = 1$

$$\sum_{n=1}^{4} a_i = 0$$

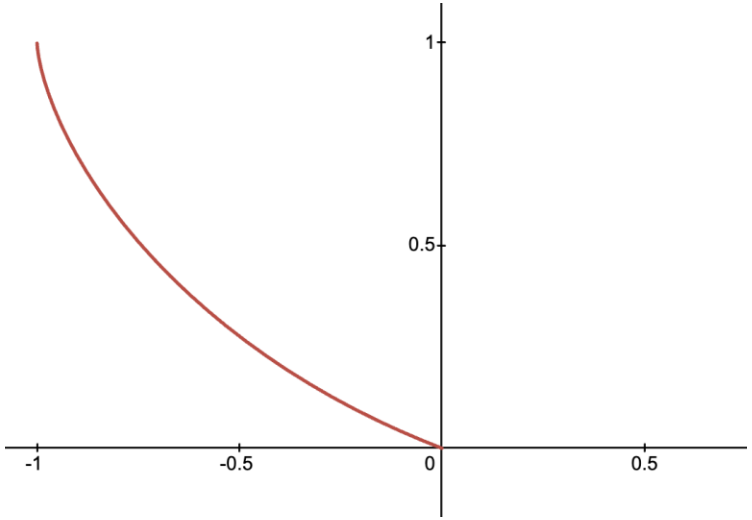
$$1 - 4\left(1 - a_0\right)a_2 + a_1^2 = 0$$

$$-12\left(1 - a_0\right)a_3 + 8a_1a_2 = 0$$

When we try to plug in $a_0 = 1$ we arrive at $a_1^2 + 1 = 0$

Updated Case: Shifted Brachistochrone

Instead, solve for a shifted Brachistochrone which starts from (-1,1) and passes through (0,0)



So, our conditions now become:

1. $y_B = 1$
2. $y(0) = 0$
3. $y(-1) = 1$

Solution Method

We have 2 equations already

$$y(0) = 0$$

$$y(-1) = 1$$

We take the first n-1 equations of the equations obtained

So, our conditions now become

1. $y_B = 1$
2. $y(0) = 0$
3. $y(-1) = 1$

Example: Solving for second order

The polynomial: $A_2(x) = a_0 + a_1 x + a_2 x^2$

The conditions: $a_0 = 0$ $\quad a_0 - a_1 + a_2 = 1$ $\quad 1 - 4(1 - a_0)a_2 + a_1^2 = 0$

After solving these, we get: $a_1 = -0.64575$ $\quad a_2 = 0.35425$

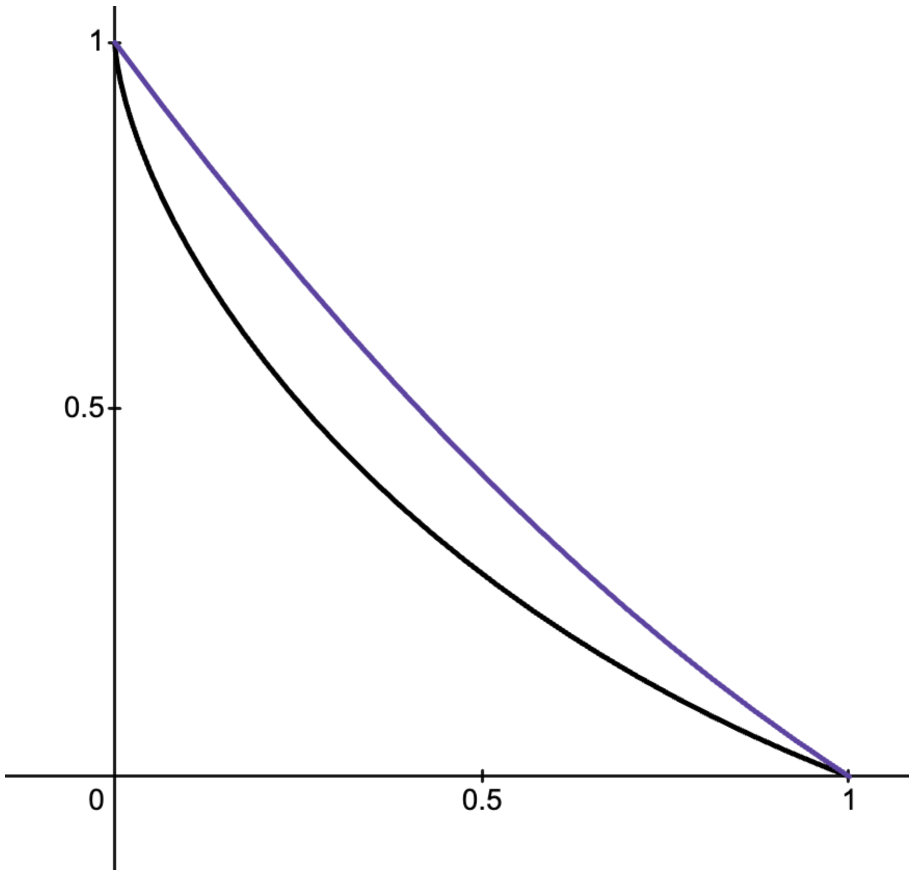
Since we shifted the Brachistochrone, to shift it back, the required Polynomial should be

$$y = A_2(x - 1)$$

Plot of second order polynomial

This is the Plot of

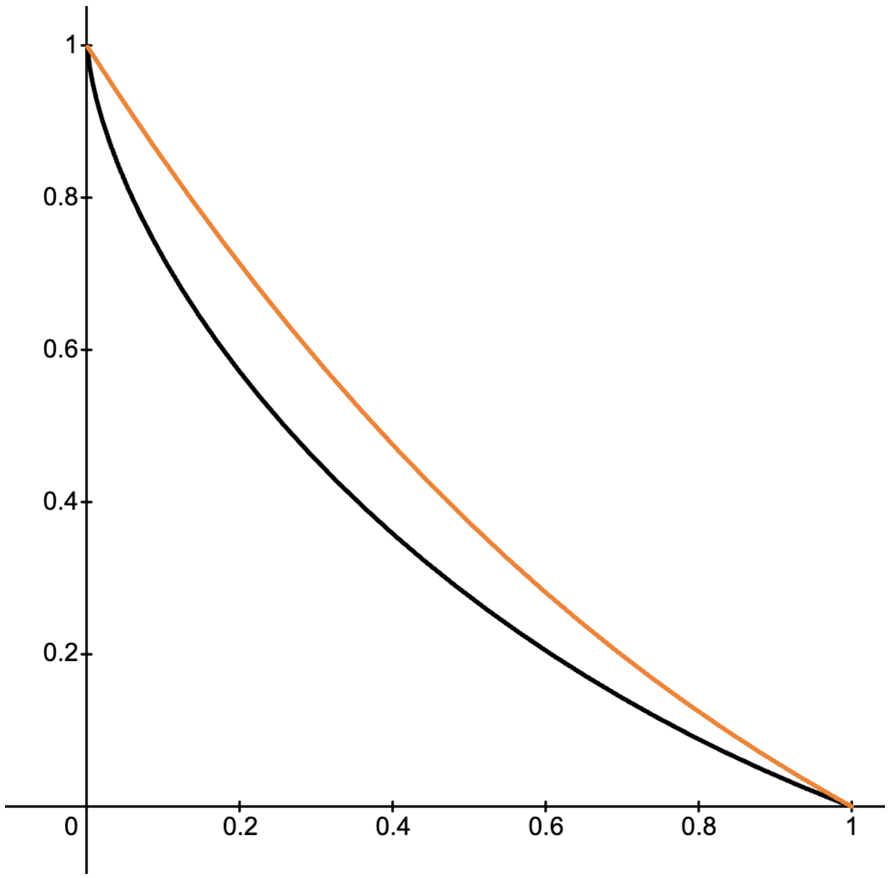
$$y = A_2(x-1)$$



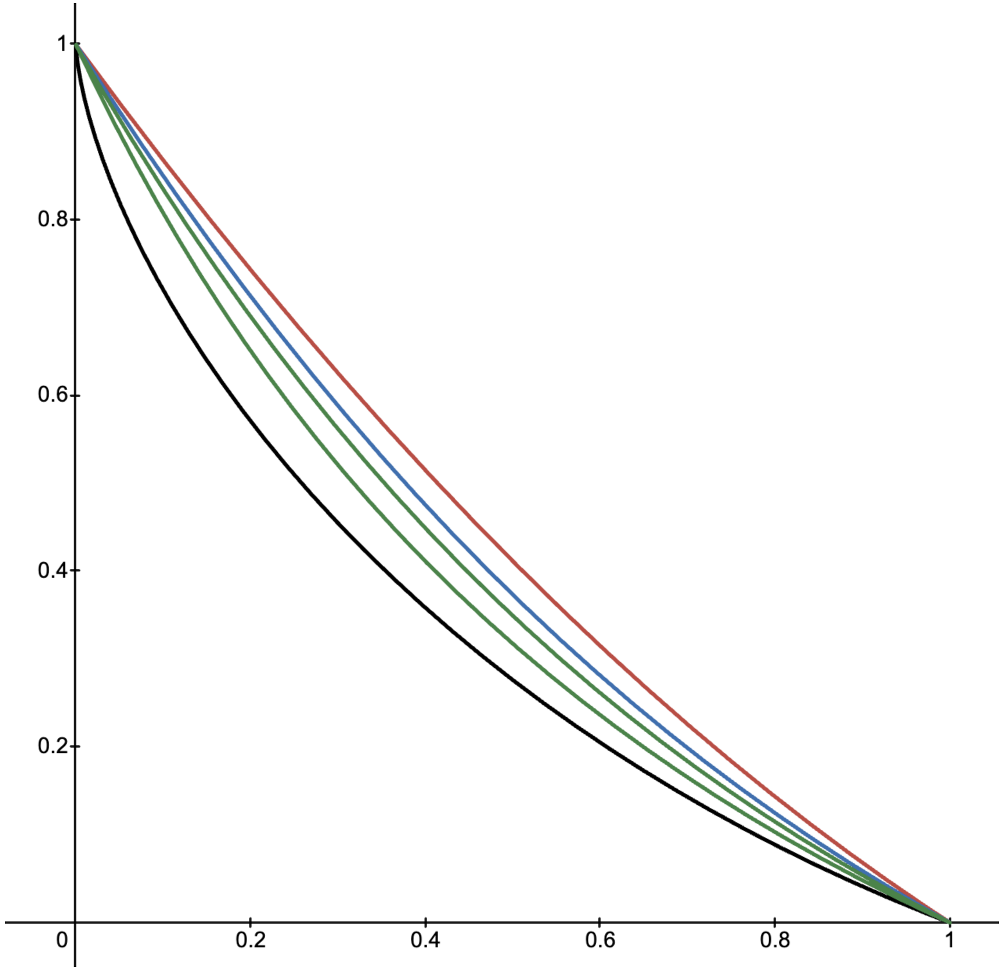
Black: Original Brachistochrone
Purple: Second order approximation

Third order polynomial

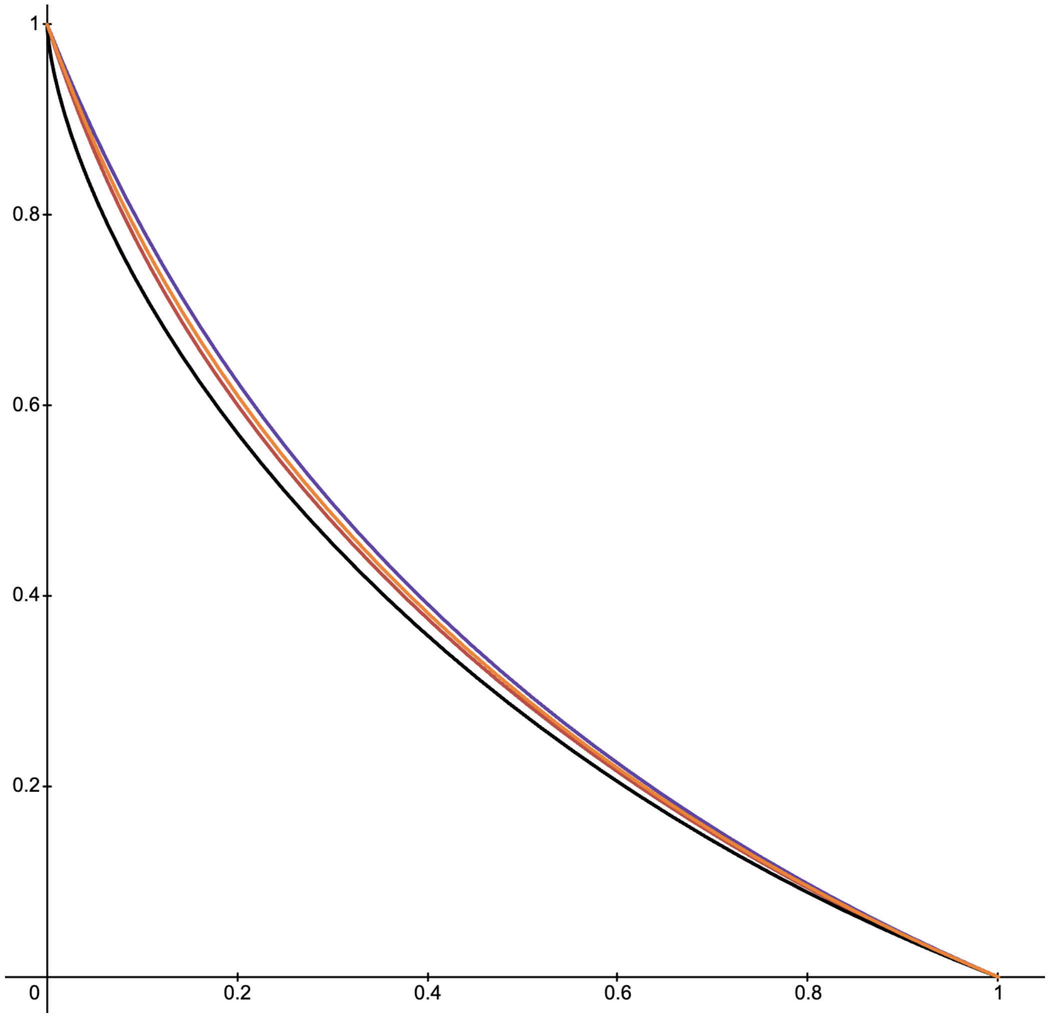
Similarly, this is the Plot of a third order approximation:



Plots of 2nd, 3rd, 4th and 7th orders



Plots of 11th, 15th and 20th orders



Future Work

- Implement Interval Newton Raphson Algorithm to find rigorous bounds for the coefficients of Power Series and plot the Interval Valued curve solution of Brachistochrone Problem
- Use our knowledge of the solution for Brachistochrone problem to implement an algorithm to solve general Boundary value ODE problems

Thank You