Reduction to integral equation

Let $p(x) = y'(x)$. From the ODE,

$$1 + p^2 - 2(1-y)p' = 0 \implies p' = \frac{1+p^2}{2(1-y)}.$$

$$p\frac{dp}{dy} = \frac{1+p^2}{2(1-y)}$$

$$\implies \frac{dp}{dy} = \frac{1+p^2}{2(1-y)p}$$

$$\implies \int_{p_0}^{p(y)} \frac{2s}{1+s^2}\, ds = \int_0^y \frac{dt}{1-t}.$$

$$\ln \frac{1+p(y)^2}{1+p_0^2} = -\ln(1-y) \implies 1 + p(y)^2 = \frac{1+p_0^2}{1-y}$$

$$p(y) = \pm\sqrt{\frac{p_0^2 + y}{1-y}}$$

(The negative root is taken becoz $y'(0) < 0$.)

$$\frac{dx}{dy} = \frac{1}{p(y)} = -\sqrt{\frac{1-y}{p_0^2+y}}$$

$$x(1) - x(0) = \int_0^1 \frac{dx}{dy}\, dy = -\int_0^1 \sqrt{\frac{1-y}{p_0^2+y}}\, dy$$

$$I(p_0) := \int_0^1 \sqrt{\frac{1-y}{p_0^2+y}}\, dy = 1$$