1 Findings

Here is what we have figured out till now:

1. In the Brachistochrone Differential Equation

$$1 + (y')^2 - 2(y_B - y)y'' = 0$$

y_B represents the y coordinate of the starting point, so this time, we work on it using $y_B = 1$.

2. Previously, when we worked using $y_B = 0$, we got polynomial approximation to an inverted cycloid as we constrained it to start from y coordinate 0 and pass through the point (0,1). We will talk more about this in the next section.

3. The solution to the DE

$$1 + (y')^2 - 2(y_B - y)y'' = 0$$

is of the form $(c_1(t - sint) + c_2, y_B - c_1(1 - cost))$, we numerically find the conditions for a Brachistochrone starting from (0,1) and passing through (1,0):

$$c_1 = 0.572917, \quad 0 \le t \le 0.7677\pi, \quad y_B = 1, c_2 = 0$$

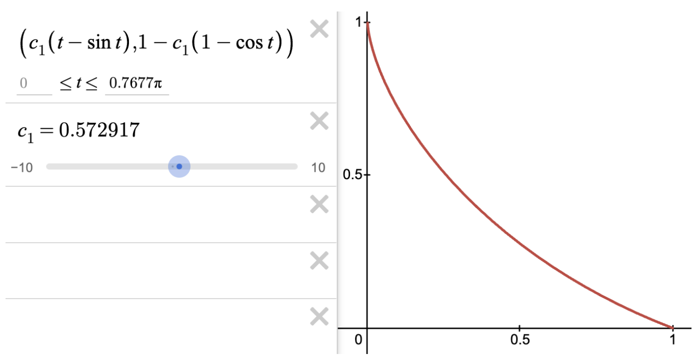

Figure 1: Brachistochrone starting from (0,1) and passing through (1,0).

4. We also find that the slope of the ideal Brachi is almost -0.38188 at (1,0) and tends to -infinity at (0,1).

5. Previously, we solved the DE with $y_B = 0$ and used the conditions y(0) = 1 and y(1) = 0. Now, when we solve the DE using $y_B = 1$, the condition y(0) = 1 (or precisely, $a_0 = 1$) makes all the coefficients imaginary, this arises from the fact that we are using Taylor series of y around 0 and there is a cusp (slope -infinity) at 0.

To sort this out, we are instead finding the polynomial approximation to the Brachi that starts from (-1, 1) and passes through (0,0), and then we shift the polynomial 1 unit to the right. So, if P(x) is our polynomial for the shifted brachi, Q(x) = P(x-1) is the required polynomial. This way, we don't encounter the issue that we are using taylor series around 0 and there is a cusp at 0.

6. so there are two methods that we have come up with:
   In method 1, we solve the DE with the conditions:

   - y(0) = 0
   - y'(0) = -0.38188

   In method 2, we solve the DE with the conditions:

   - y(0) = 0
   - y(-1) = 1

2 What happened previously

We solved the DE using the conditions $y_B = 0$ and y(0) = 1. This resulted in the polynomial approximation of a Brachi that starts from (1,0) and passes through (0,1), so basically this is an inverted Brachi. Note that the cusp problem here did not arise because this Brachi has a finite slope

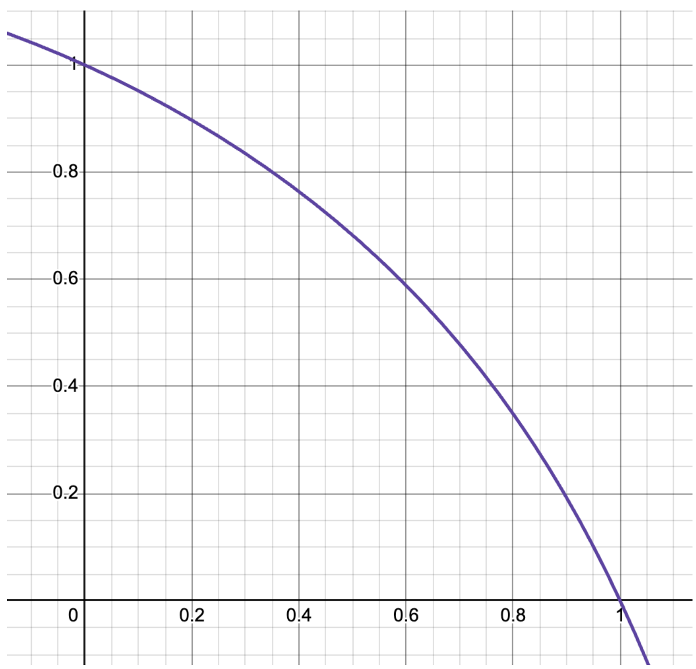

Figure 2: The polynomial approximation we found earlier

at (0,1) and a slope of -infinity at (1,0) so we could approximate it using the Taylor polynomial around 0.
It turns out that this is nothing but the inverted version of our required polynomial. To fix this inverted Brachi Polynomial, we mirrored it about the line x + y = 1 and found that it approximates the real Brachi. So, we got the 7th degree approximation $A_7(x)$ and we plotted the equation

$$x = 1 - A_7(1 - y)$$

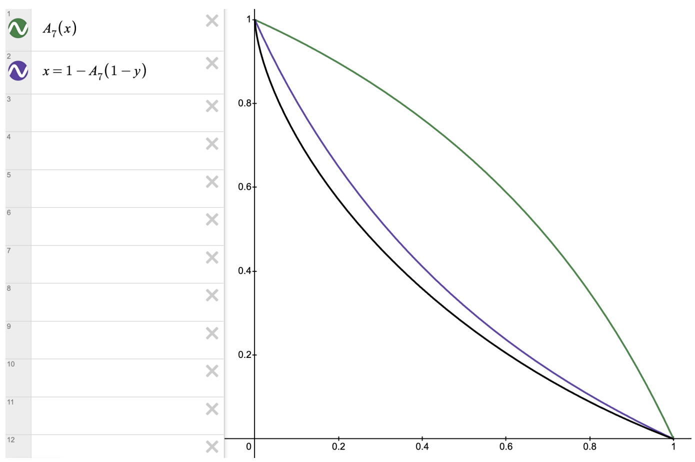

Figure 3: Green: 7th order polynomial that we got earlier; Blue: its mirrored curve around x + y = 1; Black: Ideal Brachi

3 Method 1

In this method, we solve for a shifted Brachi that starts from (-1, 1) and passes through (0,0) and shift it back to get the required polynomial. We do this to avoid the cusp at 0 issue. We use the conditions:

1. $y_B = 1$
2. $y(0) = 0$
3. $y'(0) =$ -0.38188

These conditions directly give $a_0 = 0$ and $a_1 =$ -0.38188. We use our coefficient equations to recursively compute a_i using the coefficients up to a_{i-1}. This method gives us the polynomial that passes through (1,0) and has a slope of -0.38188 at (1,0), but does not pass through (0,1).

Using this method, we have been able to generate up to a 20-degree polynomial so far.

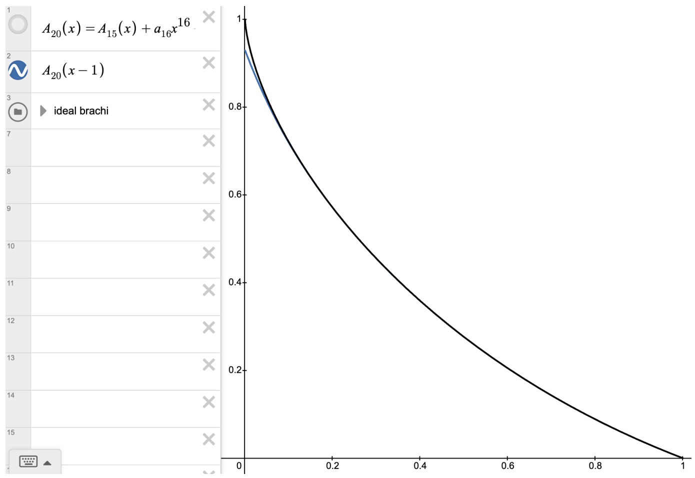

Figure 4: Black: ideal Brachi; Blue: 20-degree polynomial approximation using method 1

4 Method 2

Similar to method 1, we solve for a shifted Brachi and shift the polynomial back to get the required polynomial. In this method, we solve using the conditions:

1. $y_B = 1$
2. $y(0) = 0$
3. $y(-1) = 1$

These conditions give us $a_0 = 0$ and $-a_1 + a_2 - a_3 + a_4 ... = 1$. We proceed by solving the a_i^{th} coefficient equation to get a_i in terms of a_1 only. We put these values in the equation $-a_1 + a_2 - a_3 + a_4 ... = 1$ to get an n^{th} order polynomial in a_1 and solve it numerically to get the value of a_1 and all the other a_is.

This method gives us the polynomial that passes through both (1,0) and (0,1), but has quite a significant slope and value mismatch.

Using this method also, we have been able to generate up to a 20-degree polynomial. Note that as the order is increased, the value of a_1 keeps on decreasing and is expected to converge at -0.38188. For example, for n = 7, a_1 = -0.45043 but for n = 20, a_1 = -0.40476892.

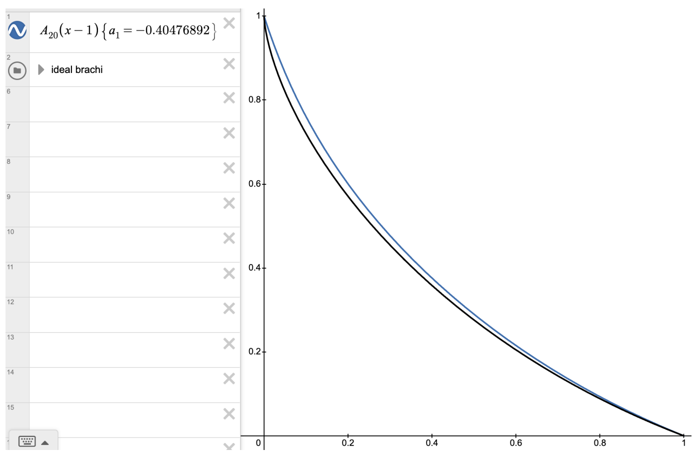

Figure 5: Black: ideal Brachi; Blue: 20-degree polynomial approximation using method 2