INDIAN INSTITUTE OF INFORMATION TECHNOLOGY

Probabilistic Fermat's Primality test and Deterministic Primes in P (AKS algorithm)

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Basics on Prime number

- A **prime number** (or a prime) is a natural number greater than 1 that is not a product of two smaller natural numbers.
- A natural number greater than 1 that is not prime is called a **composite number.**
- For example,
  - 5 is prime because the only ways of writing it as a product, 1 × 5 or 5 × 1, involve 5 itself.
  - 4 is composite because it is a product (2 × 2) in which both numbers are smaller than 4.
- The first 25 prime numbers (all the prime numbers less than 100) are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
- Primes are used in several routines in information technology,
  - such as public-key cryptography, which relies on the difficulty of factoring large numbers into their prime factors.

https://en.wikipedia.org/wiki/Prime_number
https://en.wikipedia.org/wiki/Prime_(disambiguation)

Other Primality Tests ① Lucas Lehmer - works for Mersenne numbers where $M_p = 2^p - 1$

② Pepin's Test - can be applied to Fermat numbers where $F_n = 2^{2^n} + 1$

Probabilistic Fermat Primality Test

Fermat's Little Theorem - for a prime number 'p' and any number 'a', 'p' divides '$a^p - a$' or p leaves a remainder of 'a' on dividing a^p. This can be expressed in terms of modulo expression as $a^p \equiv a \mod p$ As p is prime i.e. $\gcd(a, p) \equiv 1$, we can express this eqn as

$$a^{p-1} \bmod p \equiv 1.$$

* If a' gives $a^{p-1} \bmod p \equiv 1$ for a prime number, it is called the fermat witness.

* If a' gives $a^{c-1} \bmod c \equiv 1$ for a composite number, it is called a fermat liar & c is called a pseudo prime (eg c = 511, a' = 8)

Fermat primality test

- aks Fermat's little theorem
- Fermat's little theorem is the basis for the Fermat primality test and is one of the fundamental results of elementary number theory.
- The theorem is named after Pierre de Fermat, who stated it in 1640. It is called the "little theorem" to distinguish it from Fermat's Last Theorem.
- We can pick random integers ***a*** not divisible by ***p*** and see whether the equality holds. If the equality does not hold for value of ***a***, then ***p*** is composite. This congruence is unlikely to hold for random ***a*** if ***p*** is composite.

$$1 < a < p - 1$$

- If the congruency $\cong$ does hold for one or more values of ***a***, then we say that ***p*** is **probably prime**.

$$a^p \equiv a \pmod{p}$$

https://en.wikipedia.org/wiki/Fermat%27s_little_theorem

Fermat primality test

- For large p, the exhaustive search is hard for primality test, thus use Fermat primality test
- If p is a prime number, then for any integer a, the number $a^p - a$ is an integer multiple of p. In the notation of modular arithmetic, this is expressed as

$$a^p \equiv a \pmod{p}$$

- To test whether p is prime,
  - if p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$
  - this means, if a is not divisible by p, $a^{p-1} - 1$ is an integer multiple of p
- For example,
  - if $a = 2$ and $p = 7$, then $2^6 = 64$, and $64 - 1 = 63 = 7 \times 9$, thus it is multiple of 7.
  - if $a = 2$ and $p = 7$, then $2^7 = 128$, and $128 - 2 = 126 = 7 \times 18$ is multiple of 7.

https://en.wikipedia.org/wiki/Fermat%27s_little_theorem

Fermat primality test

- Any ***a*** such that when ***n*** is composite is known as a Fermat liar. In this case ***n*** is called Fermat pseudoprime to base ***a***.
- Suppose we wish to determine whether n = 221 is prime. Randomly pick **1 < a < 220**, say a = 38.

$$a^{n-1} = 38^{220} \equiv 1 \pmod{221}$$

- Either 221 is prime, or 38 is a Fermat liar, so we take another ***a***, say 24:

$$a^{n-1} = 24^{220} \equiv 81 \not\equiv 1 \pmod{221}$$

- So 221 is composite and 38 was indeed Fermat liar. Furthermore, 24 is Fermat witness for the compositeness of 221.

https://en.wikipedia.org/wiki/Fermat%27s_little_theorem

Fermat primality test to AKS algorithm

- Many primality tests are known that work only for numbers with certain properties.
- For example,
  - the Lucas–Lehmer test works only for Mersenne numbers (a Mersenne prime is prime number that is one less than a power of two), the prime numbers of form $M_p = 2^p - 1$ for some prime p
  - while Pépin's test can be applied to Fermat numbers $F_n = 2^{2^n} + 1$
- AKS algorithm is a generalization to polynomials of Fermat's little theorem.
- The AKS algorithm can be used to verify the primality of any given number.

https://en.wikipedia.org/wiki/AKS_primality_test

PRIMES is in P
(A hope for NP problems in P)

PRIMES is in P

- In 2002, it was shown that the problem of determining if a number is prime is in P.
- AKS (Agrawal–Kayal–Saxena) primality test,
  - famous research of IIT Kanpur, and
  - authors received the 2006 Gödel Prize and the 2006 Fulkerson Prize
- AKS primality test:
  - "an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite"

Agrawal, Manindra, Neeraj Kayal, and Nitin Saxena. "PRIMES is in P." *Annals of mathematics* (2004): 781-793.
https://en.wikipedia.org/wiki/AKS_primality_test

PRIMES is in P

- The key idea is to find the coefficient of x^i in $((x+a)^n - (x^n + a))$
  - if all coefficients are multiple of n, then n is prime
  - else composite number
- We work out it for $a = -1$.
  - What are the coefficient of x^i in $((x-1)^n - (x^n - 1))$?

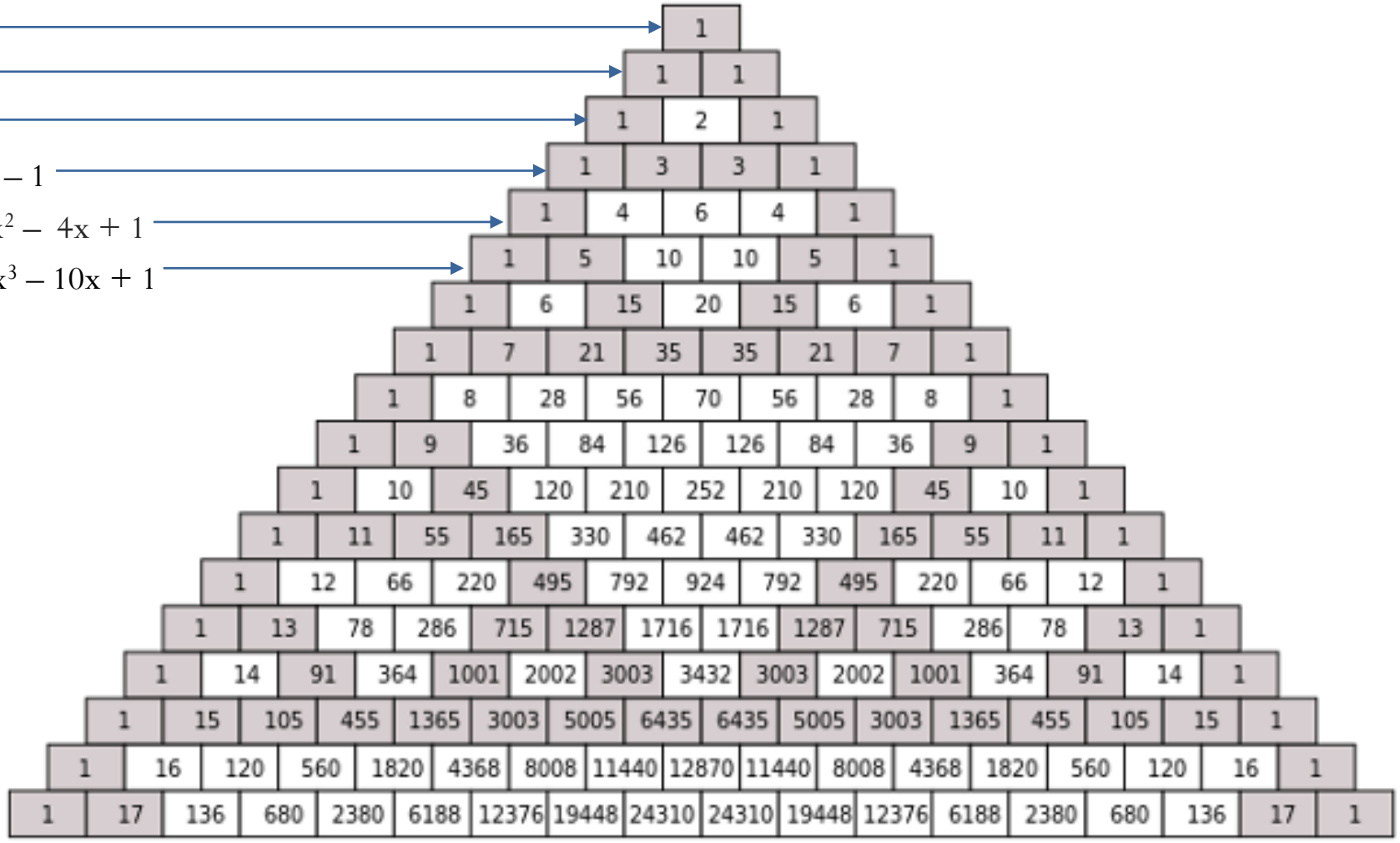
```
Input:  integer n > 1.
1.  If (n = a^b for a ∈ 𝒩 and b > 1), output COMPOSITE.
2.  Find the smallest r such that o_r(n) > log^2 n.
3.  If 1 < (a, n) < n for some a ≤ r, output COMPOSITE.
4.  If n ≤ r, output PRIME.[1]
5.  For a = 1 to ⌊√φ(r) log n⌋ do
        if ((X + a)^n ≠ X^n + a (mod X^r − 1, n)), output COMPOSITE;
6.  Output PRIME;
```

Agrawal, Manindra, Neeraj Kayal, and Nitin Saxena. "PRIMES is in P." *Annals of mathematics* (2004): 781-793.

https://en.wikipedia.org/wiki/AKS_primality_test

Pascal Triangle

- Coefficients of $(x - 1)^n$
  - $(x - 1)^0 = 1$
  - $(x - 1)^1 = x - 1$
  - $(x - 1)^2 = (x^2 - 2x + 1)$
  - $(x - 1)^3 = x^3 - 3x^2 + 3x - 1$
  - $(x - 1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$
  - $(x - 1)^5 = x^5 - 5x^4 + 10x^3 - 10x + 1$
  - and so on…



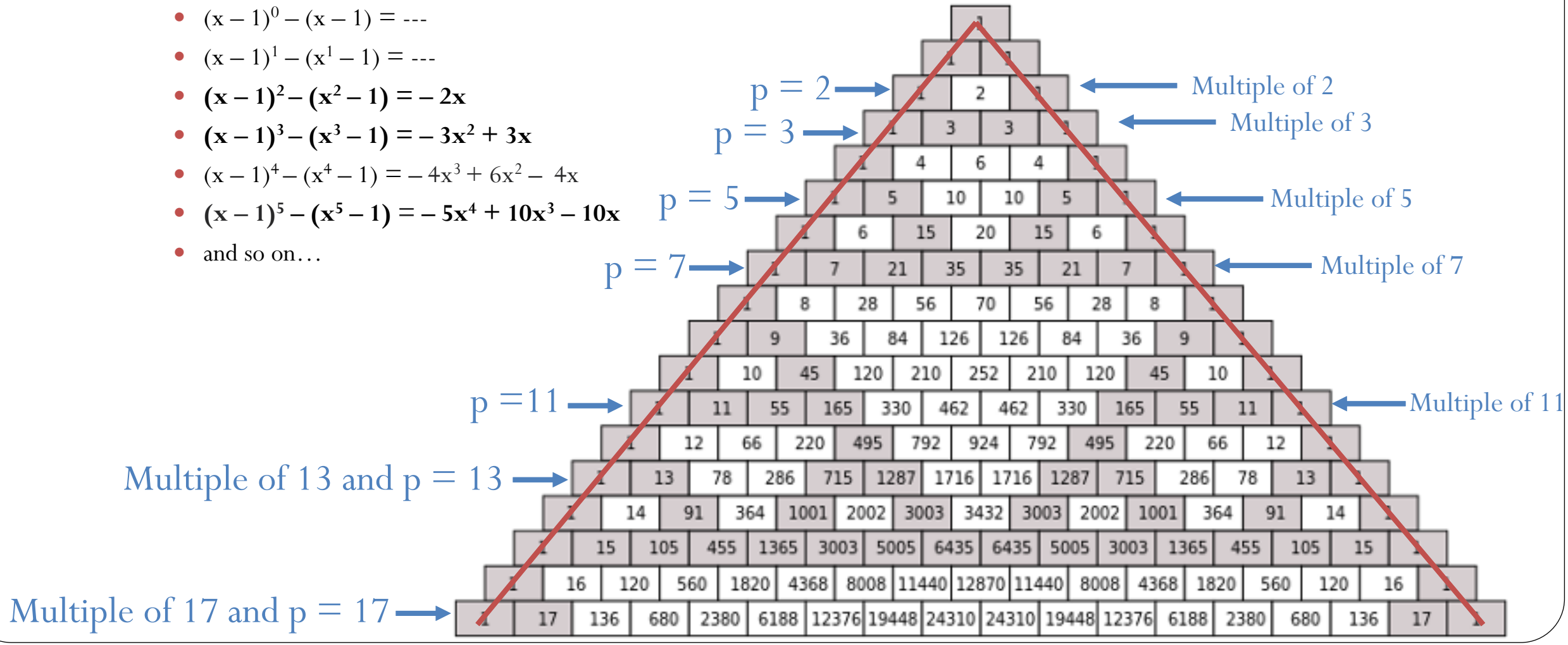
Pascal triangle trace back to Piṅgala

- The formal theory of Sanskrit meters formulated by Piṅgala in the 2nd century B.C.E. Halāyudha's construction of Pascal's triangle traces to Piṅgala.
- Piṅgala's calculation of the binomial coefficients, use of repeated partial sums of sequences and the formula for summing a geometric series became an integral part of Indian mathematics.
- Around 300 Sutras, proposed "**Meru Prastara**" now known as Pascal Triangle
- Meru means pyramid, Prastara means spreading "Pyramid Spreading"
- Pascal triangle construction can be traced to back to Piṅgala
- Pascal triangle is fixed 2D structure, **Meru Prastara** is generalized N-D structure

A HISTORY OF PIṄGALA'S COMBINATORICS, JAYANT SHAH Northeastern University, Boston, Mass,
https://web.northeastern.edu/shah/papers/Pingala.pdf
https://archive.org/details/ChhandaSutra-Pingala https://archive.org/details/halyudhaskavira00hellgoog

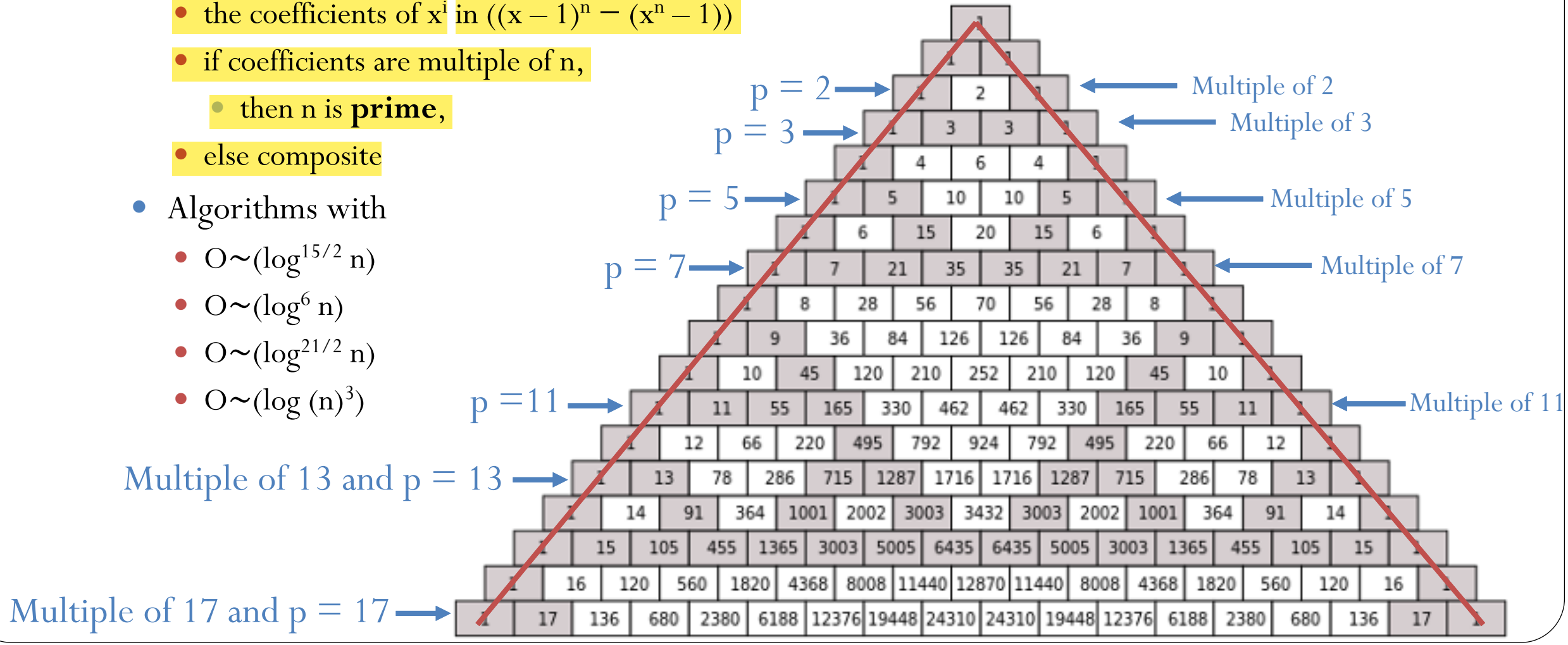
Prime and Pascals Triangle (Meru Prastara)

- Coefficients of $(x-1)^n - (x^n - 1)$
  - $(x-1)^0 - (x-1) =$ ---
  - $(x-1)^1 - (x^1 - 1) =$ ---
  - **$(x-1)^2 - (x^2 - 1) = -2x$**
  - **$(x-1)^3 - (x^3 - 1) = -3x^2 + 3x$**
  - $(x-1)^4 - (x^4 - 1) = -4x^3 + 6x^2 - 4x$
  - **$(x-1)^5 - (x^5 - 1) = -5x^4 + 10x^3 - 10x$**
  - and so on…



Prime and Pascals Triangle (Meru Prastara)

- It is possible to write a polynomial time algorithm to find
  - the coefficients of x^i in $((x-1)^n - (x^n - 1))$
  - if coefficients are multiple of n,
    - then n is **prime**,
  - else composite
- Algorithms with
  - $O\sim(\log^{15/2} n)$
  - $O\sim(\log^{6} n)$
  - $O\sim(\log^{21/2} n)$
  - $O\sim(\log (n)^3)$



More on Prime number

- Fast methods for primality test are available for Mersenne numbers $M_p = 2^p - 1$.
- As of December 2018, the largest known prime number is a Mersenne prime with 24,862,048 decimal digits.
- In 1975, Vaughan Pratt showed that there existed a certificate for primality that was checkable in polynomial time, and thus that PRIMES was in NP.
- It was long suspected but not proven that primality could be solved in polynomial time.
- AKS primality test finally settled this long-standing question and placed PRIMES in P

https://en.wikipedia.org/wiki/Prime_number
https://en.wikipedia.org/wiki/Prime_(disambiguation)
https://en.wikipedia.org/wiki/Primality_test

Millennium Problems

Millennium Problems

- The Millennium Prize Problems are seven problems in mathematics that were stated by the Clay Mathematics Institute on May 24, 2000.
- One of 7 Millennium Problems for which Clay Math Institute awards $1,000,000 i.e., US$1 million prize
- One-million dollar (*) question: $P = NP$?
- almost all researchers think $P \neq NP$

https://www.claymath.org/millennium-problems
https://www.claymath.org/millennium-problems/millennium-prize-problems
https://en.wikipedia.org/wiki/Millennium_Prize_Problems

Millennium Problems

- Yang–Mills and Mass Gap
- Riemann Hypothesis
- P vs NP Problem: If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.
- Navier–Stokes Equation
- Hodge Conjecture
- Poincaré Conjecture
- Birch and Swinnerton-Dyer Conjecture

https://www.claymath.org/millennium-problems

Millennium Problems

- To date, the only Millennium Prize problem to have been solved is the Poincaré conjecture,
- A century passed between its formulation in 1904 by Henri Poincaré and its solution by Grigoriy Perelman, announced in preprints posted on ArXiv.org in 2002 and 2003.
- Grigoriy Perelman is the Russian mathematician.
- He declined the prize money.
- Perelman was selected to receive the Fields Medal for his solution, but he declined the award.

https://www.claymath.org/millennium-problems/poincar%C3%A9-conjecture
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