Design and Analysis of Algorithms

Minimum Spanning Trees

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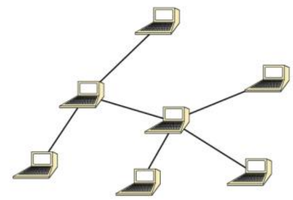
Minimum Spanning Trees

- Find a minimum-cost set of edges that connect all vertices of a graph
- Applications
  - Connect "nodes" with a minimum of "wire"
    - Networking
    - Circuit design



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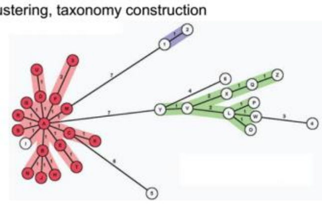
Minimum Spanning Trees

- Find a minimum-cost set of edges that connect all vertices of a graph
- Applications
  - Collect nearby nodes
    - Clustering, taxonomy construction



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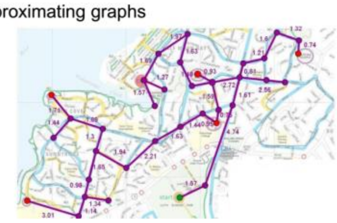
Minimum Spanning Trees

- Find a minimum-cost set of edges that connect all vertices of a graph
- Applications
  - Approximating graphs



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Minimum Spanning Trees

- A tree is an acyclic, undirected, connected graph
- A spanning tree of a graph is a tree containing all vertices from the graph
- A minimum spanning tree is a spanning tree, where the sum of the weights on the tree's edges are minimal

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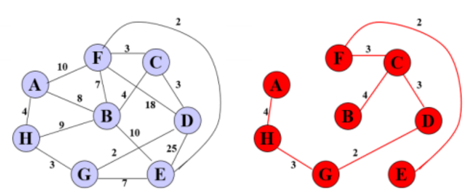
Minimum Spanning Trees



- A minimum spanning tree is a spanning tree, where the sum of the weights on the tree's edges are minimal

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Minimum Spanning Trees

- Problem formulation
  - Given an undirected, weighted graph $G = (V, E)$ with weights $w(u, v)$ for each edge $(u, v) \in E$
  - Find an acyclic subset $T \subseteq E$ that connects all of the vertices V and minimizes the total weight:

$$w(T) = \sum_{(u,v)\in T} w(u, v)$$

  - The minimum spanning tree is (V, T)
    - Minimum spanning tree may be not unique (can be more than one)

Minimum Spanning Trees

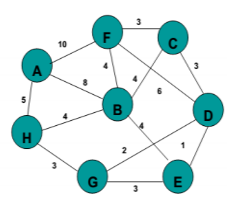
- **Kruskal's Algorithms** and
- **Prim's Algorithms**
- Both Kruskal's and Prim's Algorithms ***work with undirected graphs***
- Both are ***greedy algorithms*** that produce optimal solutions
  - The greedy strategy advocates making the choice that is the best at the moment.

Kruskal's algorithm

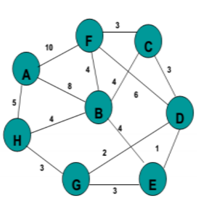
```
MST-KRUSKAL(G, w)
   A = ∅
   for each vertex v ∈ G.V
       MAKE-SET(v)
   sort the edges of G.E into nondecreasing order by weight w
   for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
       if FIND-SET(u) ≠ FIND-SET(v)
           A = A ∪ {(u, v)}
           UNION(u, v)
   return A
```


Kruskal's algorithm

- Two steps:
  - Sort edges by increasing edge weight
  - Select the first |V| - 1 edges that do not generate a cycle



Kruskal's algorithm: Example

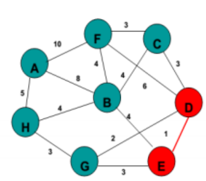


Sort the edges by increasing edge weight

edge	d_v	
(D,E)	1	
(D,G)	2	
(E,G)	3	
(C,D)	3	
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

Kruskal's algorithm: Example



Select first |V|–1 edges which do not generate a cycle

edge	d_v	
(D,E)	1	√
(D,G)	2	
(E,G)	3	
(C,D)	3	
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
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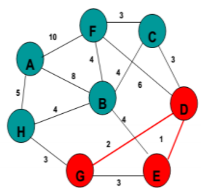
Kruskal’s algorithm: Example



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(B,F)	4	
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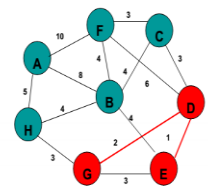
Kruskal’s algorithm: Example



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edge	d_v	
(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

Accepting edge (E,G) would create a cycle

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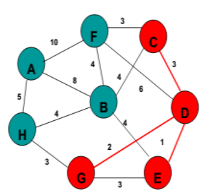
Kruskal’s algorithm: Example



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(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

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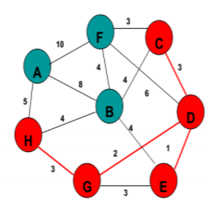
Kruskal’s algorithm: Example



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(G,H)	3	√
(C,F)	3	
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

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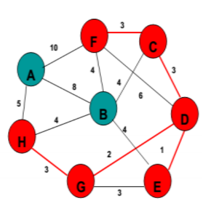
Kruskal’s algorithm: Example



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(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

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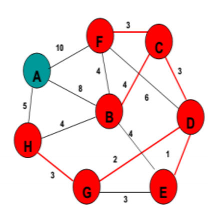
Kruskal’s algorithm: Example



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(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

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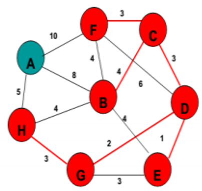
Kruskal's algorithm: Example



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(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	χ
(B,F)	4	
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

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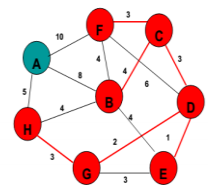
Kruskal's algorithm: Example



Select first |V|–1 edges which do not generate a cycle

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(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	χ
(B,F)	4	χ
(B,H)	4	
(A,H)	5	
(D,F)	6	
(A,B)	8	
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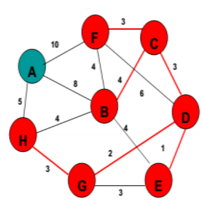
Kruskal's algorithm: Example



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(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	χ
(B,F)	4	χ
(B,H)	4	χ
(A,H)	5	
(D,F)	6	
(A,B)	8	
(A,F)	10	

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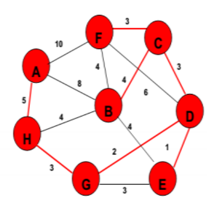
Kruskal's algorithm: Example



Select first |V|–1 edges which do not generate a cycle

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(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v	
(B,E)	4	χ
(B,F)	4	χ
(B,H)	4	χ
(A,H)	5	√
(D,F)	6	
(A,B)	8	
(A,F)	10	

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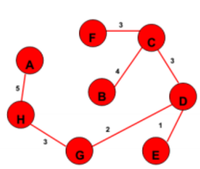
Kruskal's algorithm: Example



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(D,E)	1	√
(D,G)	2	√
(E,G)	3	χ
(C,D)	3	√
(G,H)	3	√
(C,F)	3	√
(B,C)	4	√

edge	d_v		
(B,E)	4	χ	
(B,F)	4	χ	
(B,H)	4	χ	
(A,H)	5	√	
(D,F)	6		not considered
(A,B)	8		
(A,F)	10		

Done

Total Cost = *21*

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Analysis of Kruskal's algorithm

- Depends on how we implement the disjoint-set data structure.

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Kruskal's algorithm

```
MST-KRUSKAL(G, w)
1  A = ∅
2  for each vertex v ∈ G.V
3      MAKE-SET(v)
4  sort the edges of G.E into nondecreasing order by weight w
5  for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
6      if FIND-SET(u) ≠ FIND-SET(v)
7          A = A ∪ {(u, v)}
8          UNION(u, v)
9  return A
```

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Analysis of Kruskal

- Lines 1-3 (initialization): O(V)
- Line 4 (sorting): O(E lg E)
- Lines 6-8 (set-operation): O(E log E)
- **Total:** O(E log E)

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Prim's algorithm

- Work with nodes (instead of edges)

Two steps

- Select node with minimum distance
- Update distances of adjacent, unselected nodes

•

the attribute $v.key$ is the minimum weight of any edge connecting v to a vertex in the tree; by convention, $v.key = \infty$ if there is no such edge.

attribute $v.\pi$ names the parent of v in the tree.

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Prim's algorithm

```
MST-PRIM(G, w, r)
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

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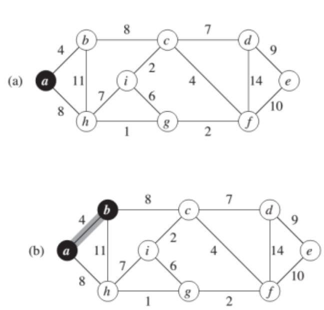
Prim's algorithm: Example



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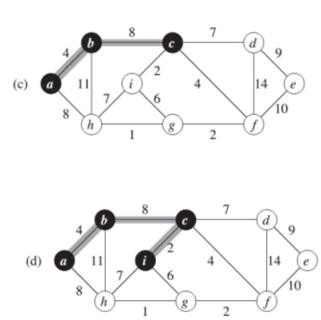
Prim's algorithm: Example



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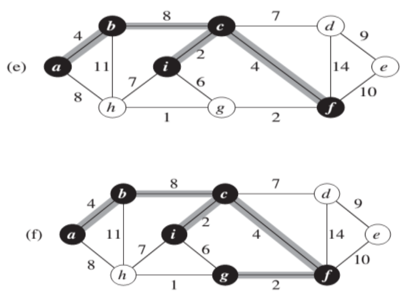
Prim's algorithm: Example



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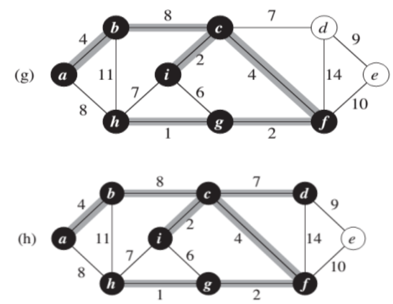
Prim's algorithm: Example



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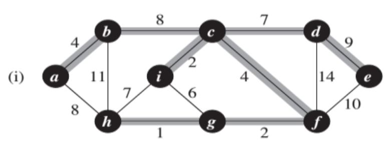
Prim's algorithm: Example



At the end, $\{(v, \pi[v])\}$ forms the MST.

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Analysis of Prim

- Extracting the vertex from the queue: $O(\lg n)$
- For each incident edge, decreasing the key of the neighboring vertex: $O(\lg n)$ where $n = |V|$
- The other steps are constant time.
- The overall running time is, where $e = |E|$

$$T(n) = \sum_{u \in V}(\lg n + \deg(u) \lg n) = \sum_{u \in V}(1 + \deg(u)) \lg n$$
$$= \lg n\,(n + 2e) = O((n + e) \lg n)$$

Essentially same as Kruskal's: $\boldsymbol{O((n+e) \lg n)}$ time

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