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Journal of Statistical Planning and Inference 70 (1998) 35–49

journal of statistical planning and inference

Some properties of progressive censored order statistics from arbitrary and uniform distributions with applications to inference and simulation

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Received 14 April 1995; received in revised form 29 September 1997

Abstract

In this paper, we first establish three properties of progressive Type-II censored order statistics from arbitrary continuous distributions. These properties are then used to develop an algorithm to simulate general progressive Type-II censored order statistics from any continuous distribution, by generalizing the algorithm given recently by Balakrishnan and Sandhu (Sankhya Series B58 (1995), 1–9). We then establish an independence result for general progressive Type-II censored samples from the uniform (0,1) population, which generalizes a result given by Balakrishnan and Sandhu (1995) for progressive Type-II right censored samples. This result is used in order to obtain moments for general progressive Type-II censored order statistics from the uniform (0,1) distribution. This independence result also gives rise to a second algorithm for the generation of general progressive Type-II censored order statistics from any continuous distribution. Finally, best linear unbiased estimators (BLUEs) for the parameters of one- and two-parameter uniform distributions are derived, and the problem of maximum-likelihood estimation is discussed.

Keywords: Progressive Type-II censored samples; Computer simulation; Markov chain; Truncated distribution; Uniform distribution; Best linear unbiased estimators; Maximum-likelihood estimators

1. Introduction

Progressive Type-II censored sampling is an important method of obtaining data in lifetime studies. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter, and a compromise can be achieved between time consumption and the observation of some extreme values. Little work has been done on the properties of progressive censored order statistics (see Balakrishnan and Sandhu, 1995), although numerous authors have discussed inference problems for a wide range of distributions under this sampling scheme. See, for example, Cohen (1963), (1966), (1975),

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PII S0378-3758(97)00173-0

(1976), Cohen and Helm (1973), Mann (1971), Thomas and Wilson (1972), Nelson (1982), Lemon (1975), Cohen and Norgaard (1977), Cohen and Whitten (1988), Cohen (1991), and Viveros and Balakrishnan (1994).

Let us consider the following general Type-II censoring scheme: Suppose N randomly selected units were placed on a life test; the failure times of the first r units to fail were not observed; at the time of the $(r+1)$th failure, R_{r+1} number of surviving units are withdrawn from the test randomly, and so on; at the time of the $(r+i)$th observed failure, R_{r+i} number of surviving units are randomly withdrawn from the test; finally, suppose $X_{r+1,N} \leqslant X_{r+2,N} \leqslant \cdots \leqslant X_{m,N}$ are the lifetimes of the completely observed units to fail, and $R_{r+1}, R_{r+2}, \ldots, R_m$ are the number of units withdrawn from the test at these failure times, respectively. It is clear that $N = m + R_{r+1} + R_{r+2} + \cdots + R_m$. If the failure times are from a continuous population with cumulative distribution function $F(x)$ and probability density function $f(x)$, the joint probability density function for $X_{r+1,N}, \ldots, X_{m,N}$, the general progressive Type-II censored order statistics, is given by

$$f_{X_{r+1,N},\ldots,X_{m,N}}(x_{r+1},\ldots,x_m) = c[F(x_{r+1})]^r \prod_{i=r+1}^{m} f(x_i)[1-F(x_i)]^{R_i},$$

where

$$\begin{aligned} c &= \binom{N}{r}(N-r)(N-r-R_{r+1}-1)(N-r-R_{r+1}-R_{r+2}-2)\cdots \\ &\quad \times (N-r-R_{r+1}-R_{r+2}-\cdots-R_{m-1}-(m-r)+1) \\ &= \binom{N}{r}(N-r)\prod_{j=r+2}^{m}\left[N-\sum_{i=r+1}^{j-1} R_i - j + 1\right]. \end{aligned} \tag{1.1}$$

In this paper, we establish three useful results for these general progressive Type-II censored order statistics $X_{r+1,N}, X_{r+2,N}, \ldots, X_{m,N}$. Further, by assuming that such a general progressive Type-II censored sample is available from uniform distributions, we derive exact expressions for the single moments, variances and covariances of general progressive Type-II censored order statistics. These moments are then used to derive BLUEs for the parameters in an explicit form and also their variances. We also consider the problem of maximum-likelihood estimation in one- and two-parameter cases. Two algorithms are given to generate general progressive Type-II censored samples from arbitrary continuous distributions.

2. Properties of general progressive censored order statistics

Theorem 2.1.(i) *The distribution of $X_{r+i,N}$ independent of $R_{r+i}, R_{r+i+1}, \ldots, R_m$.*

(ii) *$X_{r+1,N}, X_{r+2,N}, \ldots, X_{r+i,N}$ form a general progressive Type-II censored sample of size i from N items put on test with censoring scheme r, $R_{r+1}, \ldots, R_{r+i-1}, N - R_{r+1} - \cdots - R_{r+i-1} - r - i$.*

Proof. An algebraic proof is unnecessary, if we just consider the definition of general progressive Type-II censored order statistics given in the Introduction. Whether we withdraw all or no surviving items after observing $X_{r+i,N}$ will certainly not affect its distribution. Thus, we can regard $X_{r+1,N}, X_{r+2,N}, \ldots, X_{r+i,N}$ as a general progressive Type-II censored sample of size i with the censoring scheme given in the statement of the theorem, that is,

$$\begin{aligned}
&f_{X_{r+1,N},\ldots,X_{r+i,N}}(x_{r+1},\ldots,x_{r+i})\\
&\quad=\binom{N}{r}(N-r)(N-r-R_{r+1}-1)(N-r-R_{r+1}-R_{r+2}-2)\cdots\\
&\qquad\times(N-r-R_{r+1}-\cdots-R_{r+i-1}-i+1)\prod_{j=r+1}^{r+i-1} f(x_j)[1-F(x_j)]^{R_j}\\
&\qquad\times[F(x_{r+1})]^r f(x_{r+i})[1-F(x_{r+i})]^{N-R_{r+1}-R_{r+2}-\cdots-R_{r+i-1}-r-i}. \qquad \square
\end{aligned} \tag{2.1}$$

Remark 1. Theorem 2.1 will be useful in establishing some recurrence relations for the moments of progressive Type-II censored order statistics. These results, which will generalize the corresponding results on usual order statistics are being developed, and we hope to present them in a future paper.

Theorem 2.2. *The general progressive Type-II censored order statistics from an arbitrary continuous distribution form a Markov chain; that is, given* $X_{r+i,N}, X_{r+j,N}$ $(j > i)$ *is independent of* $X_{r+1,N}, X_{r+2,N}, \ldots, X_{r+i-1,N}$.

Proof. We would like to show that

$$f_{X_{r+j,N}|X_{r+i,N},X_{r+i-1,N},\ldots,X_{r+1,N}}(x_{r+j}\,|\,x_{r+i},x_{r+i-1},\ldots,x_{r+1}) = f_{X_{r+j,N}\,|\,X_{r+i,N}}(x_{r+j}|x_{r+i}).$$

It is sufficient to show that

$$\begin{aligned}
&f_{X_{r+i+1,N}|X_{r+i,N},X_{r+i-1},\ldots,X_{r+1,N}}(x_{r+i+1}|x_{r+i},x_{r+i-1},\ldots,x_{r+1})\\
&\quad= f_{X_{r+i+1,N}\,|\,X_{r+i,N}}(x_{r+i+1}\,|\,x_{r+i}).
\end{aligned} \tag{2.2}$$

Using Eq. (2.1), the left-hand side of Eq. (2.2) becomes

$$\begin{aligned}
&(N-r-R_{r+1}-\cdots-R_{r+i}-i)\left[\frac{1-F(x_{r+i+1})}{1-F(x_{r+i})}\right]^{N-r-R_{r+1}-\cdots-R_{r+i}-i-1}\frac{f(x_{r+i+1})}{[1-F(x_{r+i})]}\\
&\quad= f_{X_{r+i+1,N}|X_{r+i,N}}(x_{r+i+1}|x_{r+i}).
\end{aligned}$$

Hence, the general progressive Type-II censored order statistics $X_{r+1,N},\ldots,X_{m,N}$ form a Markov chain. $\square$

Remark 2. It should be mentioned here that Theorem 2.2 is a generalization of a well-known result for usual order statistics given, for example, in David (1981) and Arnold et al. (1991, p. 24).

Theorem 2.3. *Given $X_{r+i,N} = x_{r+i}, X_{r+i+1,N}, \ldots, X_{m,N}$ $(i \geqslant 1)$ are jointly distributed as a progressive Type-II right censored sample of size $m-r-i$ from $N - R_{r+1} - R_{r+2} - \cdots - R_{r+i} - r - i$ identically distributed random variables from the density $f(\bullet)$ left-truncated at x_{r+i}, that is, with density $f(x)/[1-F(x_{r+i})]$, and with progressive censoring scheme $R_{r+i+1}, \ldots, R_m$.*

Proof. Given $X_{r+i,N} = x_{r+i}$, the conditional joint distribution of $X_{r+i+1,N}, \ldots, X_{m,N}$ is given by, upon invoking the Markov property established in Theorem 2.2,

$$
\begin{aligned}
& f_{X_{r+i+1,N},\ldots,X_{m,N} \mid X_{r+i,N}}(x_{r+i+1},\ldots,x_m|x_{r+i}) \\
&= f_{X_{r+i+1,N},\ldots,X_{m,N} \mid X_{r+i,N},X_{r+i-1,N},\ldots,X_{r+1,N}}(x_{r+i+1},\ldots,x_m \mid x_{r+i},x_{r+i-1},\ldots,x_{r+1}) \\
&= \frac{f_{X_{r+1,N},\ldots,X_{m,N}}(x_{r+1},\ldots,x_m)}{f_{X_{r+1,N},\ldots,X_{r+i,N}}(x_{r+1},\ldots,x_{r+i})}.
\end{aligned}
\tag{2.3}
$$

The joint density in the numerator of Eq. (2.3) is given in Eq. (1.1), and the joint density in the denominator is given in Eq. (2.1). Thus, upon simplification, we obtain the conditional distribution to be

$$
\begin{aligned}
& f_{X_{r+i+1,N},\ldots,X_{m,N}|X_{r+i,N}}(x_{r+i+1},\ldots,x_m|x_{r+i}) \\
&= \left[\prod_{j=r+i+1}^{m}\left(N - \sum_{i=r+1}^{j-1} R_i - j + 1\right)\right]\left[\prod_{k=r+i+1}^{m} \frac{f(x_k)}{1-F(x_{r+i})}\left(\frac{1-F(x_k)}{1-F(x_{r+i})}\right)^{R_k}\right].
\end{aligned}
$$

This is seen simply to be the joint density of $m-r-i$ progressive Type-II right censored order statistics from $N - R_{r+1} - R_{r+2} - \cdots - R_{r+i} - r - i$ identically distributed random variables with density $f(x)/[1-F(x_{r+i})]$, and with progressive censoring scheme $R_{r+i+1}, \ldots, R_m$. □

Using this result, one may, for example, generalize the result given by Balakrishnan and Sandhu (1995) for the generation of progressive Type-II right censored samples by considering first the uniform (0,1) distribution, where the $(r+1)$th-order statistic from the full sample of size N is known to have a Beta $(r+1, N-r)$ distribution, and then using the order preserving probability integral transformation to generate a general progressive Type-II censored sample from an arbitrary continuous distribution. Two algorithms for simulating general progressive Type-II censored samples are presented in Section 7.

Remark 3. It should be mentioned here that Theorem 2.3 is a generalization of a result presented originally by Scheffé and Tukey (1945); also see David (1981) and Balakrishnan and Cohen (1991).

3. Moments of general progressive Type-II censored order statistics from the uniform (0,1) distribution

Let us denote the general progressive Type-II censored order statistics for the case of the uniform (0,1) distribution by $U_{r+1,N}, U_{r+2,N}, \ldots, U_{m,N}$. Then, from Eq. (1.1), the joint probability density function of $U_{r+1,N}, U_{r+2,N}, \ldots, U_{m,N}$ is given by

$$f_{U_{r+1,N},U_{r+2,N},\ldots,U_{m,N}}(u_{r+1}, u_{r+2}, \ldots, u_m) = c(u_{r+1})^r \prod_{i=r+1}^{m} (1-u_i)^{R_i},$$

$$0 \leqslant u_{r+1} \leqslant u_{r+2} \leqslant \cdots \leqslant u_m \leqslant 1, \tag{3.1}$$

where the constant c is as given after Eq. (1.1).

Now, consider the following one-to-one transformation, which is a generalization of the transformation considered by Balakrishnan and Sandhu (1995):

$$V_{r+i} = \frac{1 - U_{m-i+1,N}}{1 - U_{m-i,N}}, \quad i = 1, \ldots, m-r-1, \quad V_m = 1 - U_{r+1,N}.$$

Then

$$U_{r+i,N} = 1 - \prod_{j=m-i+1}^{m} V_j, \quad i = 1, 2, \ldots, m-r \tag{3.2}$$

and the Jacobian of this transformation is

$$\prod_{i=2}^{m-r} V_{r+i}^{i-1}.$$

Thus, the joint distribution of V_{r+i}, $i = 1, 2, \ldots, m-r$, is obtained from Eq. (3.1) as

$$f_{V_{r+1},V_{r+2},\ldots,V_m}(v_{r+1}, v_{r+2}, \ldots, v_m) = c(1-v_m)^r \prod_{i=1}^{m-r} v_{r+i}^{i-1+R_{m-i+1}+R_{m-i+2}+\cdots+R_m},$$

$$0 < v_{r+1}, v_{r+2}, \ldots, v_m < 1 \tag{3.3}$$

from which we readily find that $V_{r+1}, V_{r+2}, \ldots, V_m$ are independent beta random variables, distributed as follows:

$$V_{r+i} \sim \text{Beta}\left(i + \sum_{j=m-i+1}^{m} R_j, 1\right), \quad i = 1, \ldots, m-r-1,$$

$$V_m \sim \text{Beta}\left(m - r + \sum_{j=r+1}^{m} R_j, r+1\right). \tag{3.4}$$

Thus, using Eqs. (3.2) and (3.4), we can obtain the moments of $U_{r+1,N},\ldots,U_{m,N}$. In doing so, we adopt the following notation:

(1) $\quad a_{i+r} = i + \sum_{j=m-i+1}^{m} R_j, \quad i = 1,\ldots,m-r,$

(2) $\quad \alpha_j = \dfrac{a_j}{1+a_j}, \quad j = r+1,\ldots,m-1, \qquad \alpha_m = \dfrac{a_m}{1+r+a_m},$

(3) $\quad \beta_j = \dfrac{1}{(a_j+2)(a_j+1)}, \quad j = r+1,\ldots,m-1,$

$\quad \beta_m = \dfrac{r+1}{(a_m+r+2)(a_m+r+1)},$

(4) $\quad \gamma_j = \alpha_j + \beta_j, \quad j = r+1,\ldots,m. \qquad (3.5)$

After some algebra, the following expressions are obtained:

$$E(U_{r+i,N}) = 1 - \prod_{j=m-i+1}^{m} \alpha_j, \quad i = 1,\ldots,m-r,$$

$$\mathrm{Var}(U_{r+i,N}) = \left[\prod_{j=m-i+1}^{m} \alpha_j\right]\left[\prod_{j=m-i+1}^{m} \gamma_j - \prod_{j=m-i+1}^{m} \alpha_j\right], \quad i = 1,\ldots,m-r,$$

$$\mathrm{Cov}(U_{r+i,N}, U_{r+k,N}) = \left[\prod_{j=m-i+1}^{m} \alpha_j\right]\left[\prod_{j=m-k+1}^{m} \gamma_j - \prod_{j=m-k+1}^{m} \alpha_j\right],$$
$$k < i, \quad i = 2,\ldots,m-r. \qquad (3.6)$$

Upon making use of the above expressions, we see that the $(m-r)\times(m-r)$ variance–covariance matrix is of the special form $\sigma_{ij} = s_i t_j$ which can be inverted explicitly as given in Graybill (1983, pp. 198). The elements of the symmetric tri-diagonal inverted matrix (c_{ij}) are as follows:

$$c_{ii} = \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j}, \quad i = 1,2,\ldots,m-r-1,$$

$$c_{m-r,m-r} = \frac{1}{\beta_{r+1}\prod_{j=r+1}^{m}\alpha_j\prod_{j=r+2}^{m}\gamma_j},$$

$$c_{i,i+1} = \frac{-1}{\beta_{m-i}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j}, \quad i = 1,2,\ldots,m-r-1. \qquad (3.7)$$

Remark 4. In the special case when there is no censoring (that is, $m = N, r = R_1 = \cdots = R_N = 0$), the following expressions are obtained:

$$\alpha_i = \frac{i}{i+1}, \qquad \gamma_i = \frac{i+1}{i+2}, \qquad \beta_i = \frac{1}{(i+2)(i+1)}, \quad i = 1,2,\ldots,N, \tag{3.8}$$

so that

$$c_{ii} = 2(N+1)(N+2), \quad i = 1,\ldots,N,$$

$$c_{i,i+1} = -(N+1)(N+2), \quad i = 1,\ldots,N-1,$$

a well-known result; see Arnold et al. (1992).

4. Best linear unbiased estimation: One-parameter case

Let us denote the general progressive Type-II censored order statistics from the uniform $(0,\theta)$ distribution by $U_{r+1,N}, U_{r+2,N}, \ldots, U_{m,N}$. Using the results of Balakrishnan and Cohen (1991) and David (1981), and the above expression of the inverted variance–covariance matrix for general progressive Type-II censored samples from the uniform (0,1) distribution, we can obtain the exact best linear unbiased estimate (BLUE) of θ. Explicit expressions of the BLUE and its variance are as follows:

$$\theta^* = \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} U_{r+i,N} \right.$$

$$\left. + \frac{(1-\alpha_{r+1})}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} U_{m,N} \right\}$$

$$\times \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} \right.$$

$$\left. \times \left(1 - \prod_{j=m-i+1}^{m} \alpha_j\right) + \frac{(1-\alpha_{r+1})\left(1-\prod_{j=r+1}^{m}\alpha_j\right)}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\}^{-1} \tag{4.1}$$

and

$$\mathrm{Var}(\theta^*) = \theta^2 \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} \times \left(1 - \prod_{j=m-i+1}^{m}\alpha_j\right) + \frac{(1-\alpha_{r+1})\left(1-\prod_{j=r+1}^{m}\alpha_j\right)}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\}^{-1}. \tag{4.2}$$

Remark 5. It is of interest to note that the precision of θ^* in Eq. (4.2) does depend on the censoring scheme used, unlike in the one-parameter exponential case; see Balakrishnan and Sandhu (1996).

Remark 6. In the case of no censoring, upon using Eq. (3.8), the above expressions reduce to the well-known results

$$\theta^* = \frac{N+1}{N} U_{N,N} \quad \text{and} \quad \mathrm{Var}(\theta^*) = \frac{\theta^2}{N(N+2)}.$$

5. Best linear unbiased estimation: Two-parameter case

Exact expressions for the BLUEs of μ and σ in the two-parameter uniform $(\mu, \mu+\sigma)$ distribution can also be obtained, as well as their variances and covariance. Let us denote the general progressive Type-II censored order statistics from the uniform $(\mu, \mu+\sigma)$ distribution by $U_{r+1,N}, U_{r+2,N}, \ldots, U_{m,N}$. Letting

$$\begin{aligned}\Delta = {} & \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} \times \left(1 - \prod_{j=m-i+1}^{m}\alpha_j\right) + \frac{(1-\alpha_{r+1})(1-\prod_{j=r+1}^{m}\alpha_j)}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\} \\ & \times \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - 2\beta_{m-i+1}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} + \frac{1}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\} \\ & - \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} + \frac{1-\alpha_{r+1}}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\}^2,\end{aligned} \tag{5.1}$$

we obtain

$$\sigma^* \Delta = \sum_{i=2}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\beta_m \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} U_{r+i,N}$$

$$+ \frac{1-\alpha_{r+1}}{\beta_{r+1}\beta_m\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} U_{m,N}$$

$$- \left\{ \sum_{i=2}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\beta_m \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} \right.$$

$$\left. + \frac{1-\alpha_{r+1}}{\beta_{r+1}\beta_m\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\} U_{r+1,N} \tag{5.2}$$

and

$$\mathrm{Var}(\sigma^*) = \frac{\sigma^2}{\Delta} \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - 2\beta_{m-i+1}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} \right.$$

$$\left. + \frac{1}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\}, \tag{5.3}$$

$$\mu^* \Delta = \left\{ \sum_{i=2}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\beta_m \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} \right.$$

$$\left. \times \left(1 - \prod_{j=m-i+1}^{m}\alpha_j\right) + \frac{(1-\alpha_{r+1})\left(1 - \prod_{j=r+1}^{m}\alpha_j\right)}{\beta_{r+1}\beta_m\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\} U_{r+1,N}$$

$$- \left\{ \sum_{i=2}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\beta_m \prod_{j=m-i+1}^{m} \alpha_j\gamma_j} U_{r+i,N} \right.$$

$$\left. + \frac{1-\alpha_{r+1}}{\beta_{r+1}\beta_m\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} U_{m,N} \right\} (1-\alpha_m) \tag{5.4}$$

and

$$\mathrm{Var}(\mu^*) = \frac{\sigma^2}{\Delta} \left\{ \sum_{i=1}^{m-r-1} \frac{\gamma_{m-i}\gamma_{m-i+1} - \alpha_{m-i}\alpha_{m-i+1} - \beta_{m-i+1} - \alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j} \right.$$

$$\left. \times \left(1 - \prod_{j=m-i+1}^{m}\alpha_j\right) + \frac{(1-\alpha_{r+1})\left(1 - \prod_{j=r+1}^{m}\alpha_j\right)}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j} \right\}. \tag{5.5}$$

Finally,

$$\mathrm{Cov}(\mu^*,\sigma^*) =$$

$$-\frac{\sigma^2}{\Delta}\left\{\sum_{i=1}^{m-r-1}\frac{\gamma_{m-i}\gamma_{m-i+1}-\alpha_{m-i}\alpha_{m-i+1}-\beta_{m-i+1}-\alpha_{m-i+1}\gamma_{m-i+1}\beta_{m-i}}{\beta_{m-i}\beta_{m-i+1}\prod_{j=m-i+1}^{m}\alpha_j\gamma_j}\right.$$

$$\left.+\frac{1-\alpha_{r+1}}{\beta_{r+1}\alpha_{r+1}\prod_{j=r+2}^{m}\alpha_j\gamma_j}\right\} \tag{5.6}$$

Remark 7. For the case of no censoring, upon using Eq. (3.8), the expressions given above simplify to the well-known results

$$\sigma^* = \frac{N+1}{N-1}(U_{N,N}-U_{1,N}), \qquad \mu^* = \frac{1}{N-1}(NU_{1,N}-U_{N,N}),$$

$$\mathrm{Var}(\sigma^*) = \frac{2\sigma^2}{(N+2)(N-1)}, \qquad \mathrm{Var}(\mu^*) = \frac{N\sigma^2}{(N+1)(N+2)(N-1)},$$

$$\mathrm{Cov}(\mu^*,\sigma^*) = \frac{-\sigma^2}{(N+2)(N-1)}.$$

5.1. Numerical example

Using the expressions for BLUEs of the location and scale parameters and their variances and covariance, we can obtain the coefficients of observed general progressive Type-II censored order statistics corresponding to BLUEs for any sample size and any censoring scheme, as well as the variances and covariance of the BLUEs. As a numerical illustration, censoring schemes as given in Table 1 were considered. In each scheme, exactly 5 full-failure times are observed, and $N = 20$. Schemes are given as $(r, R_{r+1}, R_{r+2}, R_{r+3}, R_{r+4}, R_{r+5})$. Coefficients are given in order of observed failures. All calculations were done using Fortran with double-precision accuracy.

Remark 8. Notice that censoring scheme [1] is the conventional Type-II right censoring scheme, censoring scheme [4] is the conventional Type-II left censoring scheme, and censoring scheme [6] is a conventional Type-II double censoring scheme.

Remark 9. Suppose a practitioner is interested in employing a general progressive Type-II censoring scheme for an experiment in which failure times are assumed to be uniformly distributed. For fixed values of N and $m - r$ (number of observed failures), the practitioner may decide upon a scheme by choosing from all possible general progressive Type-II censoring schemes with these values of N and $m - r$ the one which has the minimum trace or minimum determinant of the variance–covariance matrix of the BLUEs. If $N = 20$ and $m - r = 5$, there are 15 504 possible censoring schemes. Of these, scheme [5] gives both the minimum trace and the minimum determinant of

Table 1

Scheme No.	Coefficients (μ^*)	Coefficients (σ^*)	$\mathrm{Var}(\mu^*)/\sigma^2$	$\mathrm{Var}(\sigma^*)/\sigma^2$	$\mathrm{Cov}(\mu^*,\sigma^*)/\sigma^2$
[1] (0, 0, 0, 0, 0, 15)	1.250, 0.000, 0.000, 0.000, −0.250	−5.250, 0.000, 0.000, 0.000, 5.250	0.0027	0.1932	−0.0114
[2] (5, 2, 2, 2, 2, 2)	2.131, −0.122, −0.149, −0.198, −0.662	−3.958, 0.426, 0.521, 0.695, 2.316	0.0290	0.1511	−0.0561
[3] (10, 0, 0, 5, 0, 0)	2.908, 0.000, −0.561, 0.000, −1.347	−3.643, 0.000, 1.071, 0.000, 2.571	0.0692	0.1201	−0.0867
[4] (15, 0, 0, 0, 0, 0)	5.000, 0.000, 0.000, 0.000, −4.000	−5.250, 0.000, 0.000, 0.000, 5.250	0.1732	0.1932	−0.1818
[5] (0, 15, 0, 0, 0, 0)	1.063, 0.000, 0.000, 0.000, −0.063	−1.312, 0.000, 0.000, 0.000, 1.313	0.0024	0.0440	−0.0043
[6] (7, 0, 0, 0, 0, 8)	3.000, 0.000, 0.000, 0.000, −2.000	−5.250, 0.000, 0.000, 0.000, 5.250	0.0519	0.1932	−0.0909

the variance–covariance matrix of the BLUEs. The efficiency gained over conventional Type-II right censoring (scheme [1]) as measured by the trace is 422%, and as measured by the determinant, the gain in efficiency is 457%. The censoring scheme which performs most poorly with respect to trace is scheme [4] (53% efficiency with respect to scheme [1]) and with respect to determinant scheme [6] performs most poorly (22% efficiency with respect to scheme [1]).

6. Maximum likelihood estimation

In the general progressive Type-II censored sample considered, the MLE of θ in the one-parameter uniform $(0,\theta)$ distribution does not exist in an explicit form and has to be determined from the likelihood equation by a numerical method. The equation to be solved numerically for θ is given by

$$\sum_{i=r+1}^{m} \frac{R_i U_{i,N}}{\theta - U_{i,N}} = m, \quad \theta \geqslant U_{m,N}, \tag{6.1}$$

where $U_{r+1,N}, U_{r+2,N}, \ldots, U_{m,N}$ denote the general progressive Type-II censored order statistics from the uniform $(0,\theta)$ distribution. Notice that at $\theta = \max(U_{i,N}$ such that $R_i > 0)$, the left-hand side of Eq. (6.1) is infinite, while as θ goes to infinity, the left-hand side goes to 0. Furthermore, the left-hand side of Eq. (6.1) is monotonically decreasing in θ between $\max(U_{i,N}$ such that $R_i > 0)$ and ∞. Thus, there must be exactly one solution to this equation in (max($U_{i,N}$ such that $R_i > 0$), ∞). If this solution is in the interval $[U_{m,N}, \infty)$, then it is the maximum likelihood estimate of θ. If, however, this solution is in the interval (max($U_{i,N}$ such that $R_i > 0$), $U_{m,N}$), then one must consider the nature of the likelihood (or log-likelihoood) function.

The derivative of the log-likelihood function is given by

$$\frac{1}{\theta}\sum_{i=r+1}^{m}\frac{R_i U_{i,N}}{\theta - U_{i,N}} - \frac{m}{\theta}. \tag{6.2}$$

Now, since the left-hand side of Eq. (6.1) is monotonically decreasing in θ over the interval (max($U_{i,N}$ such that $R_i > 0$), ∞), it follows that Eq. (6.2) is negative, and therefore, the log-likelihood function is decreasing, for values of θ larger than the solution of Eq. (6.1). If this solution is in (max($U_{i,N}$ such that $R_i > 0$), $U_{m,N}$), then the maximum-likelihood estimate of θ must be $U_{m,N}$.

In the two-parameter uniform $(\mu, \mu+\sigma)$ distribution, the two equations to be solved simultaneously are

$$\sum_{i=r+1}^{m}\frac{R_i}{\sigma+\mu-U_{i,N}} = \frac{r}{U_{r+1,N}-\mu}, \quad \mu \leqslant U_{r+1,N},$$

$$\sum_{i=r+1}^{m}\frac{R_i U_{i,N}}{\sigma+\mu-U_{i,N}} = \frac{\mu r}{U_{r+1,N}-\mu} + m, \quad \mu+\sigma \geqslant U_{m,N},$$

where $U_{r+1,N}, U_{r+2,N}, \ldots, U_{m,N}$ denote the general progressive Type-II censored order statistics from the uniform $(\mu, \mu+\sigma)$ distribution. In the special case when $r = 0$, however, the MLE of μ is given by $\hat{\mu} = U_{1,N}$ and the following equation must then be solved σ:

$$\sum_{i=2}^{m}\frac{R_i(U_{i,N}-U_{1,N})}{\sigma-(U_{i,N}-U_{1,N})} = m, \quad \sigma \geqslant U_{m,N}-U_{1,N},$$

which is very similar to Eq. (6.1). Using similar arguments to those above, we conclude that there must be exactly one solution to this equation in ((max($U_{i,N}$ such that $R_i > 0$) $- U_{1,N}$), ∞). If this solution is in $[(U_{m,N}-U_{1,N}), \infty)$ then it is the maximum likelihood estimate of σ. Otherwise, the maximum-likelihood estimate of σ is $U_{m,N}-U_{1,N}$.

7. Computer generation

One obvious way of generating a general progressive Type-II censored sample is to generate the entire sample of size N, and then perform the left censoring and progressive censoring scheme after sorting the sample. However, even with quick sort algorithms, this can be very time consuming and inefficient for large values of N. As such, it would be advantageous to develop an algorithm which generates the required progressive censored sample directly and without sorting. Many authors, including Schucany (1972), Lurie and Hartley (1972), Lurie and Mason (1973), Ramberg and Tadikamalla (1978) and Horn and Schlipf (1986) have discussed various efficient algorithms for generating either a complete or a partial set of order statistics without requiring routines. Balakrishnan and Sandhu (1995) recently presented a direct method to simulate a progressive Type-II right-censored sample (with no left censoring) which also requires no sorting.

By considering a general progressive Type-II censored sample from the uniform $(0,1)$ distribution and using the probability integral transformation, one can generate a general progressive Type-II censored sample from any continuous distribution with cumulative distribution function $F(x)$. This can be done efficiently and without sorting by generalizing the result of Balakrishnan and Sandhu (1995). We present two ways in which general progressive Type-II censored samples may be generated as follows.

7.1. *Conditioning method*

Let us denote the $m-r$ observations from a general progressive Type-II censored sample from the uniform $(0,1)$ distribution by $U_{r+1,N},\dots,U_{m,N}$. Then, using Theorem 2.3 with $i=1$, we know that given $U_{r+1,N}=u_{r+1}$, $U_{r+2,N},\dots,U_{m,N}$ are jointly distributed as a progressive Type-II right-censored sample of size $m-r-1$ from $N-R_1-r-1$ uniform $(u_{r+1},1)$ variates with progressive censoring scheme $R_{r+2},\dots,R_m$.

We can therefore generate a general progessive Type-II censored sample from an arbitrary continuous distribution with cumulative distribution function $F(x)$ using the following algorithm:

(1) Generate $U_{r+1,N}$ as a Beta $(r+1, N-r)$ random variate. This may be done by using any efficient Beta-generating algorithm; see, for example, Johnson et al. (1995).

(2) Generate a progressive Type-II right-censored sample from the uniform $(0,1)$ distribution of size $m-r-1$, $u_{(r+2)},\dots,u_{(m)}$, with removal pattern $R_{r+2},\dots,R_m$ using the algorithm given by Balakrishnan and Sandhu (1995).

(3) Set $U_{i,N}=u_{(i)}(1-U_{r+1,N})+U_{r+1,N}$, $i=r+2,\dots,m$.

(4) Set $X_{i,N}=F^{-1}(U_{i,N}), i=r+1,\dots,m$. This is the desired general progressive Type-II censored sample.

7.2. *Uniform method*

Let us denote the $m-r$ observations from a general progressive Type-II censored sample for the uniform $(0,1)$ distribution by $U_{r+1,N},\dots,U_{m,N}$. Using the results of Section 3, that is,

$$V_{r+i}\sim\text{Beta}\left(i+\sum_{j=m-i+1}^{m}R_j,1\right),\quad i=1,\dots,m-r-1,$$

$$V_m\sim\text{Beta}\left(i+\sum_{j=r+1}^{m}R_j,r+1\right),$$

we see that

$$Z_{r+i} = V_{r+i}^{a_{r+i}}, \quad i = 1, 2, \ldots, m - r - 1$$

are i.i.d. uniform $(0, 1)$ random variables, also independent of V_m.

We can therefore generate a general progressive Type-II censored sample from an arbitrary continuous distribution with cumulative distribution function $F(x)$ using the following algorithm:

(1) Generate a Beta $(N - r, r + 1)$ random variable V_m.

(2) Independently generate $m-r-1$ independent uniform $(0, 1)$ observations $Z_{r+1}, \ldots, Z_{m-1}$.

(3) Set $V_{r+i} = Z_{r+i}^{1/a_{r+i}}, \quad i = 1, \ldots, m - r - 1$.

(4) Set $U_{r+i,N} = 1 - V_{m-i+1}\, V_{m-i+2} \cdots V_m, \ i = 1, 2, \ldots, m - r$.

(5) Set $X_{i,N} = F^{-1}(U_{i,N}), \ i = r + 1, \ldots, m$. This is the desired general progressive Type-II censored sample.

Acknowledgements

The authors thank the Natural Sciences and Engineering Research Council of Canada for funding this research.

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