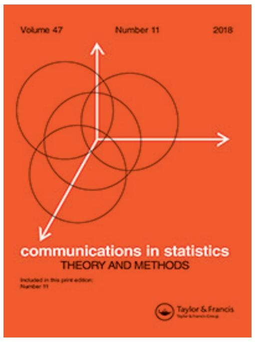

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Kolmogorov–Smirnov test for life test data with hybrid censoring

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ABSTRACT
This work considers goodness-of-fit for the life test data with hybrid censoring. An alternative representation of the Kolmogorov–Smirnov (KS) statistics is provided under Type-I censoring. The alternative representation leads us to approximate the limiting distributions of the KS statistic as a functional of the Brownian bridge for Type-II, Type-I hybrid, and Type-II hybrid censored data. The approximated distributions are used to obtain the critical values of the tests in this context. We found that the proposed KS test procedure for Type-II censoring has more power than the available one(s) in literature.

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Brownian bridge; censored data; goodness-of-fit; Kolmogorov–Smirnov test.

1. Introduction

The two most fundamental censoring schemes are Type-I and Type-II censoring schemes. The duration of a life-testing experiment in a Type-I censoring is predetermined, say X_0, and the number of failures in that time interval is a non negative integer-valued random quantity. On the contrary, in a Type-II censoring scheme, the experiment takes a random time to produce the required number of failures, say r, which is prespecified. So here the stopping time is a random variable, the rth order statistic, denoted by $X_{(r)}$. Hybrid censoring scheme is a mixture of Type-I and Type-II censoring schemes. The idea of hybrid censoring scheme was introduced by Epstein (1954) and has been extensively studied in subsequent years. In Type-I hybrid censoring scheme the experiment is terminated at X_0 or at the time of rth failure, whichever occurs first. On the other hand, in Type-II hybrid censoring scheme the experiment ends when X_0 and rth failure both take place.

When the data are complete and the distribution is prespecified, the Kolmogorov–Smirnov (KS) test statistic asymptotically follows the Kolmogorov distribution under the null hypothesis. The Kolmogorov distribution can be viewed as a supremum norm of the standard Brownian bridge on [0, 1]. Among many others, the most popular and worth performing goodness-of-fit tests with complete data are chi-square test, Cramér-von Mises test, and Anderson-Darling test. Csorgo and Faraway (1996) obtained the exact and asymptotic distributions of Cramér-von Mises statistics. Zhang (2002) proposed even more powerful

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test than the existing ones. For Type-I censored data, the KS test statistic has the asymptotic distribution similar to that of complete data on a suitable interval contained in [0,1] (see Koziol and Byar, 1975). For Type-II censored data, when the sample size (n) and the number of failures (r) be quite large such that r/n approaches to a constant, the distribution of KS test statistic has a similar behavior to that of the Type-I censored data (see Maag and Dicaire, 1971). The two-sided one-sample KS test is modified by Barr and Davidson (1973) to use it for the censored and truncated samples. According to their observation, the goodness-of-fit tests based on the modified test statistic are inappropriate when parameters of the hypothesized distribution are estimated from the data and used for the test. It reduces the power of the test. Some correction factors are also suggested by Dufour and Maag (1978) to the KS statistic obtained from Type-I and Type-II censoring schemes to make the statistic compatible with the tabulated critical values provided by Koziol and Byar (1975). A generalization of KS test was done by Fleming et al. (1980) for the one-sample and the two-sample problems of an arbitrarily right-censored data. Theoretical motivations are provided for asymptotic properties of the results. The size and the power of the generalized Smirnov two-sample procedure are evaluated for small and moderate sample sizes using Monte Carlo simulations. A new modified goodness-of-fit testing based on Type-II right censored data was proposed by Lin et al. (2008). The goodness-of-fit test for censored data from a location-scale distribution, especially for exponential distribution, has been discussed by Castro-Kuriss (2011).

In a review article on the hybrid censoring by Balakrishnan and Kundu (2013), it has been mentioned that no asymptotic result or methodology is available to perform a goodness-of-fit test for the hybrid censored data. It is noticed that none of the prevailing ideas can be readily extended to construct a modified KS test for the hybrid censored data. We study the asymptotic behavior of the KS test statistic when the data are coming from the above four censoring schemes. In this work, first we introduce an alternative representation to the KS test with Type-I censored data. It is easy to see from the construction that the new representation resembles the usual KS test when there is no time bound to complete the experiment with all of the observations. For Type-II censoring, this new representation is essential to obtain the supremum of the limiting process on an index set which does not depend on the data. We also show that the asymptotic power of the proposed test is more than that of Lin et al. (2008). We find that this construction can be generalized to approximate the limiting distribution of the KS test with hybrid censored data. We consider both Type-I and Type-II hybrid censoring schemes.

The rest of the article is organized as follows. The modified KS tests available in literature for Type-I and Type-II censoring schemes are discussed in Section 2. The cutoff points are also provided here under different scenarios. The alternative representation to the KS test statistic for the Type-I censoring scheme is introduced in Section 3. The alternative representation to the KS test statistic for Type-II censoring scheme is presented in Section 4. The approximated distributions of the KS test statistics for Type-I hybrid and Type-II hybrid data are provided in Sections 5 and 6, respectively. We discuss how to get the different percentage points through Monte Carlo simulation from the approximated distribution of the KS statistic for the above censoring schemes with any arbitrary proportion of censoring in Section 7. The power curves of modified KS statistics under different methods for Type-II censoring schemes are also compared in this section. A real-life data set on lifetime of aluminum coupon is analyzed for illustration in Section 8. We wind up with the concluding remarks in Section 9.

2. Modification to KS test

In one sample problem, if the hypothesized distribution is completely specified, KS statistic evaluates the supremum norm distance between the cumulative distribution function (CDF) and the empirical distribution function (EDF). Suppose, under the null hypothesis (H_0) the random samples $X_1, X_2, \ldots, X_n \sim F_0(\cdot)$, where the family and the parameter are already specified for the continuous distribution function $F_0(\cdot)$. So, it is immediate that the random variables $T_i = F_0(X_i)$ for $i = 1, \ldots, n$, independently and identically follow uniform distribution on $[0, 1]$. Hence, testing $H_0 : X \sim F_0(\cdot)$ against $H_1 : X$ does not follow $F_0(\cdot)$ is equivalent to testing $H_0 : T \sim U[0, 1]$ against $H_1 : T$ does not follow $U[0, 1]$, where X and T are the generic notations for the X_is and the T_is, respectively. Therefore, now onwards we will speak only about the random variable with $U[0, 1]$ distribution, unless mentioned separately. We define the difference function between the EDF and the CDF of uniform distribution for a particular $t \in [0, 1]$ as

$$D_n(t) = F_n(t) - t$$

and therefore the KS statistic is $\sup_{t\in[0,1]} |D_n(t)|$. It is well known that $\sqrt{n}D_n(t) \Rightarrow B_0(t)$ as $n \uparrow \infty$, where $B_0(t)$ is the standard Brownian bridge on $[0, 1]$ and $F_n(t)$ is the EDF defined as $F_n(t) = (\sum_{i=1}^{n} I_{\{T_i \le t\}})/n$. Here and in the rest of the text "$\Rightarrow$" stands for the weak convergence. So the asymptotic distribution of the KS statistic after $\sqrt{n}$ scaling , i.e., $KS_0 = \sup_{t\in[0,1]} \sqrt{n}|D_n(t)| = \sup_{t\in[0,1]} \sqrt{n}|F_n(t) - t|$, can be stated as

$$\sup_{t\in[0,1]} \sqrt{n}|D_n(t)| \Rightarrow \sup_{t\in[0,1]} |B_0(t)| \text{ as } n \uparrow \infty \tag{2.1}$$

If F_0 is continuous then under the null hypothesis $\sup_{t\in[0,1]} \sqrt{n}|D_n(t)|$ converges weakly to the Kolmogorov distribution, which does not depend on F_0. The goodness-of-fit test or the KS test is constructed by using the critical values from the Kolmogorov distribution. The null hypothesis is rejected at level α if $\sup_{t\in[0,1]} \sqrt{n}|D_n(t)| > K_\alpha$, where K_α is found from $P(K \le K_\alpha) = 1 - \alpha$, when $K = \sup_{t\in[0,1]} |B_0(t)|$ follows Kolmogorov distribution. In this article we discuss how the limiting distribution of $\sqrt{n}D_n(t)$ digresses from the standard Brownian bridge and so the distribution of KS statistic when the data are censored.

To perform the KS test, the sample size n is assumed to be sufficiently large so that the tabulated percentage values can be used as the critical value of the test. But in practice the sample size may be moderate or small. Stephens (1970) proposed some correction factors to the KS statistic for complete data to make the statistic compatible to the tabulated value for some commonly used percentage points provided in his paper. Percentage points are obtained by solving the distribution function of the modified statistic corresponding to the level of the test. Maag and Dicaire (1971) gave an approximate expression for the same by truncating the infinite series of the Kolmogorov distributions up to $O(n^{-1})$. Barr and Davidson (1973) suggested the two-sided one-sample KS "goodness-of-fit" test for censored and truncated sample. They also provided a table for the percentage points. Koziol and Byar (1975) calculated the percentage points for different truncation points of Type-I censoring and claimed that it will work equivalently well for Type-II censoring when $r/n \longrightarrow \lambda_0$, some fixed truncation point as used in Type-I censoring. Dufour and Maag (1978) suggest to use

$$\sqrt{n}D_{n:T_0} = \sqrt{n} \max_{i\le d} \left\{ \left| T_{(i)} - \frac{i}{n} \right|, \left| T_{(i)} - \frac{i-1}{n} \right|, \left| T_0 - \frac{d}{n} \right| \right\}$$

as the test statistic for Type-I censoring where $\{T_{(1)} < T_{(2)} < \cdots < T_{(d)} < T_{(d+1)}\}$ be the order statistics corresponding to T_is and $T_0 = F_0(X_0)$ satisfying $\{T_{(1)} < T_{(2)} < \cdots < T_{(d)} < T_0 < T_{(d+1)}\}$, where d is the number of failures. They also suggest

$$\sqrt{n}D_{n:r} = \sqrt{n}\max_{i\le r}\left\{\left|T_{(i)} - \frac{i}{n}\right|, \left|T_{(i)} - \frac{i-1}{n}\right|\right\}$$

for Type-II censoring as the test statistic. They argue to subtract $0.19/\sqrt{n}$ and $0.24/\sqrt{n}$ from the tabulated values in the paper by Koziol and Byar (1975) before comparing to $\sqrt{n}D_{n:T_0}$ and $\sqrt{n}D_{n:r}$, respectively. Though these approximations work very well, the tabulated values are available only for the truncation proportion $0.1, 0.2, \ldots, 0.9$ and it is not possible to extend this idea to the hybrid censoring. The method we propose in this work can provide the percentage points for any proportion of censoring. Moreover, the idea is extended to obtain the percentage point to perform the goodness-of-fit test for the hybrid censoring.

3. Type-I censoring

In Type-I censoring scheme, the experiment is terminated at time $T_0 = F_0(X_0) \in [0, 1]$. So it directly follows from the above construction [see Equation (2.1)] that the modified KS statistic, KS_I say, with $\sqrt{n}$ scaling, for Type-I censoring will be

$$KS_I = \sup_{t\in[0,T_0]} \sqrt{n}|D_n(t)| = \sup_{t\in[0,T_0]} \sqrt{n}|F_n(t) - t|$$

Here the distribution of the test statistic is immediate because the stopping time T_0 is independent of the process $B_0(t)$ (see Koziol and Byar, 1975). So the limiting distribution of KS_I can be stated as

$$KS_I = \sup_{t\in[0,T_0]} \sqrt{n}|D_n(t)| \Rightarrow \sup_{t\in[0,T_0]} |B_0(t)| \quad \text{when } n \uparrow \infty \tag{3.1}$$

3.1. An alternative representation

From Equation (3.1), it is immediate that KS_I converges in distribution to the supremum norm of $B_0(t)$ on the fixed and truncated interval $[0, T_0]$ instead of the entire path on $[0, 1]$. The points on the interval $[0, T_0]$ can be described as follows:

$$\{t|t \in [0, T_0]\} \equiv \{uT_0|u \in [0, 1] \text{ for given } T_0\} \tag{3.2}$$

Note that when $T_0 = 1$, it describes the entire $[0, 1]$ interval. We can give a physical interpretation to the above representation. Suppose a particle is moving from 0 to T_0 along real (or time) line with unit velocity. The left hand side of (3.2) indicates its absolute distance from 0 and right hand side of (3.2) describes the proportion of the target distance T_0 has traveled at the same time. Now we redefine KS_I depending on $u \in [0, 1]$, the proportion to the target distance traveled, instead of absolute time as

$$KS_I = \sup_{t\in[0,T_0]} \sqrt{n}|D_n(t)| \equiv \sup_{u\in[0,1]} \sqrt{n}|D_n^I(u)|$$

where $D_n^I(u) = D_n(uT_0)$. So the convergence result holds good as

$$\sup_{u\in[0,1]} \sqrt{n}|D_n^I(u)| \Rightarrow \sup_{u\in[0,1]} |B_0(uT_0)| \equiv KS_I^a \quad \text{when } n \uparrow \infty \tag{3.3}$$

When $T_0 = 1$, it is trivial to show that $\sqrt{n}D_n^I(u) = \sqrt{n}D_n(u) \Rightarrow B_0(u)$ as $n \uparrow \infty$. Describing the process with the index "u" will be more interesting and useful when the terminating time of the experiment is random. We will see it in the subsequent analyses for Type-II, Type-I hybrid, and Type-II hybrid censoring schemes.

4. Type-II censoring

In Type-II censoring scheme the experiment is stopped when the rth failure, i.e., $T_{(r)} = F_0(X_{(r)})$ takes place. We observe the realizations till the rth failure as $\{t_{(1)}, t_{(2)}, \ldots, t_{(r)}\}$. We also know that the stopping time $T_{(r)}$ follows the beta distribution $\mathbb{B}(r, n-r+1)$. Unlike the Type-I censoring scheme, here we have some difficulties to define the EDF $F_n(t) = (\sum_{i=1}^{n} I_{\{T_i \le t\}})/n \,\forall\, t \in [0,1]$. Note that $F_n(t)$ is well defined for $t \le t_{(r)}$, but not for $t > t_{(r)}$ because no data point will be observed in the interval $(t_{(r)}, t]$. Hence, it is not possible to define and study the function $F_n(t)$ appropriately for any arbitrarily chosen $t \in [0,1]$ because $T_{(r)}$ is random and contained in $[0,1]$ itself. But for KS test with Type-II censored data, it is possible to study the behavior of $D_n(t)$ for all $t \le t_{(r)}$. Here the alternative representation corresponding to Type-I censoring described in Section 3.1 will be advantageous to get rid of this problem. By this alternative representation, we always obtain the feature of $D_n(t)$ only on the random interval $[0, T_{(r)}]$ for different realizations of $T_{(r)}$. Consider $u \in [0,1)$ and define the distance function for Type-II censored data by

$$D_n^{II}(u) = D_n(uT_{(r)}) = F_n(uT_{(r)}) - uT_{(r)}$$

It can be easily shown that $F_n(uT_{(r)})$ and $uT_{(r)}$ are independent and $nF_n(uT_{(r)})$ follows the binomial distribution $Bin(r-1, u)$, which is invariant of $T_{(r)}$. Also, $(r-1)F_{r-1}(u)$ follows $Bin(r-1, u)$ when $F_{r-1}(u)$ is obtained from $r-1$ independent observations following $U[0,1]$. With the embedding of Skorohod topology and suitable use of Donsker's theorem, it is easy to note that (see Durrett, 2010, Chapter 8.9)

$$\sqrt{\frac{n^2}{r-1}}\left(F_n(uT_{(r)}) - \frac{r-1}{n}u\right) \stackrel{d}{=} \sqrt{r-1}\,(F_{r-1}(u) - u) \Rightarrow B_0(u) \quad \text{as } r \uparrow \infty \tag{4.1}$$

See **A1** in Appendix for details. Again from the definition of $D_n^{II}(u)$, using Glivenko–Cantelli theorem and the weak law of large numbers, we get

$$\sup_{u \in [0,1]} |D_n^{II}(u)| \longrightarrow 0 \quad \text{w.p. 1}, \quad \text{as } r \text{ and } n \uparrow \infty$$

See **A2** in Appendix for the proof. Now consider the distance function with $\sqrt{n}$ scaling, which gives

$$\begin{aligned}
\sqrt{n}D_n^{II}(u) &= \sqrt{n}(F_n(uT_{(r)}) - uT_{(r)}) \\
&\stackrel{d}{=} \sqrt{n}\left(\frac{r-1}{n}F_{r-1}(u) - uT_{(r)}\right) \\
&= \sqrt{n}\left(\frac{r-1}{n}(F_{r-1}(u) - u) + \frac{r-1}{n}u - uT_{(r)}\right) \\
&= \sqrt{\frac{r-1}{n}}[\sqrt{r-1}(F_{r-1}(u) - u)] - u\sqrt{n}\left(T_{(r)} - \frac{r-1}{n}\right)
\end{aligned} \tag{4.2}$$

We have already obtained the asymptotic behavior of the first component of (4.2) in (4.1). To examine the asymptotic behavior of the second component of (4.2), let us assume that both r

and n move to ∞ such that $r/n \to \lambda_0$. It is stated earlier that $T_{(r)} \sim \mathbb{B}(r, n-r+1)$. Let the sequence of random variables $W_1, W_2, \ldots$ be independently and identically distributed with $P(W_i > w) = e^{-w}$ and define the partial sum $Z_n = W_1 + W_2 + \cdots + W_n$. It is trivial to see that $T_{(r)} \stackrel{d}{=} Z_r/Z_{n+1}$. Again, with appropriate use of the Skorohod topology and the Donsker's theorem (see Durrett, 2010, Chapter 8.9), it is easy to get that

$$\sqrt{n}\left(T_{(r)} - \frac{r-1}{n}\right) \stackrel{d}{=} \sqrt{n}\left(\frac{Z_r}{Z_{n+a}} - \frac{r}{n}\right) + o_p(1) \Rightarrow B_0(\lambda_0) \quad \text{as } r \text{ and } \quad n \uparrow \infty$$

Now by Slutsky's theorem, it is immediate to see

$$\sqrt{n}D_n^{II}(u) \Rightarrow \sqrt{\lambda_0}B_0(u) - uB_0(\lambda_0)$$

Defining the KS statistic for Type-II censoring by

$$KS_{II} = \sup_{u\in[0,1]} \sqrt{n}|D_n^{II}(u)|$$

we conclude from the continuous mapping theorem that

$$KS_{II} \Rightarrow \sup_{u\in[0,1]} \left|\sqrt{\lambda_0}B_0(u) - uB_0(\lambda_0)\right| \quad \text{as } r \text{ and } n \uparrow \infty$$

Furthermore denote $D_{\lambda_0}(u) = \sqrt{\lambda_0}B_0(u) - uB_0(\lambda_0)$ and observe that $E(D_{\lambda_0}(u)) = 0$, $Var(D_{\lambda_0}(u)) = u\lambda_0(1 - u\lambda_0)$, and $Cov(D_{\lambda_0}(u), D_{\lambda_0}(v)) = u\lambda_0(1 - v\lambda_0)$ for $1 \leq u < v \leq 1$. Now it is interesting to observe that $\sup_{u\in[0,1]} D_{\lambda_0}(u) \stackrel{d}{=} \sup_{u\in[0,1]} |B_0(u\lambda_0)|$, which is nothing but the limiting distribution of $\sup_{u\in[0,1]} \sqrt{n}|D_n^I(u)|$ specifically when $T_0 = \lambda_0$. It is a mathematical justification of claim by Koziol and Byar (1975) that the limiting distribution of $\sqrt{n}D_n^{II}(u)$ coincides with that of $\sqrt{n}D_n^I(u)$ when r/n approaches to T_0 as $r, n \uparrow \infty$. Hence we can write

$$\begin{aligned} KS_{II} &\Rightarrow \sup_{u\in[0,1]} \left|\sqrt{\lambda_0}B_0(u) - uB_0(\lambda_0)\right| \\ &\stackrel{d}{=} \sup_{u\in[0,1]} |B_0(u\lambda_0)| \quad \text{as } r \text{ and } n \uparrow \infty \end{aligned}$$

Remark 1. When the sample size is moderately large, we can approximate the distribution as follows. Here we need to consider the case when $u = 1$ separately. Then the value of $F_n(T_{(r)}) = r/n$, which is a constant and $T_{(r)}$ itself has the beta distribution $\mathbb{B}(r, n-r+1)$. Note that

$$\lim_{u\to 1} E[F_n(t_{(r)}) - uF_n(t_{(r)})] = \frac{r}{n} - \frac{r-1}{n} = \frac{1}{n}$$

This is because $F_n(1.t_{(r)})$ is enumerated based on all of r observations when $u = 1$, but otherwise $F_n(ut_{(r)})$ is obtained based on first $(r-1)$ observations.

Therefore, for large values of n and r, we have

$$\sqrt{n}D_n^{II}(u) \stackrel{d}{\approx} \frac{r-1}{\sqrt{n}}u + \sqrt{\frac{r-1}{n}}B_0(u) - u\sqrt{n}\,\mathbb{B}(r, n-r+1) + \frac{1_{\{u=1\}}}{\sqrt{n}}$$

As defined above the KS statistic for Type-II censoring is given by

$$KS_{II} = \sup_{u\in[0,1]} \sqrt{n}|D_n^{II}(u)|$$

At the same time, we define

$$KS_{II}^{a} = \sup_{u\in[0,1]} \left| \frac{r-1}{\sqrt{n}}u + \sqrt{\frac{r-1}{n}}B_0(u) - u\sqrt{n}\,\mathbb{B}(r, n-r+1) + \frac{1_{\{u=1\}}}{\sqrt{n}} \right| \tag{4.3}$$

Both KS_{II} and KS_{II}^{a} have same asymptotic distributions. So we approximate the distribution of KS_{II} with that of the KS_{II}^{a} for moderate values of r and n.

5. Type-I hybrid censoring

According to the definition of the Type-I hybrid censoring, the experiment is terminated at T_0 or $T_{(r)}$ whichever comes up first. Here the stopping time is $\min\{T_0, T_{(r)}\} = T_{[m]}$, say. The probability that the experiment is stopped when $T_{(r)}$ is observed is same as the probability of occurring $\{T_{(r)} < T_0\}$ and therefore we have

$$P(T_{[m]} = T_{(r)}) = P(T_{(r)} \le T_0) = p_0, \quad \text{say}$$

On the other hand, the experiment is stopped at the fixed time T_0 with the probability $q_0 = 1 - p_0$. When $T_{(r)}$ occurs first, the situation is exactly same as the Type-II censoring. If $T_{(r)}$ comes after T_0, we observe the similar trajectory of $F_n(\cdot)$ for Type-II censoring but truncated at T_0. We define our process for $u \in [0, 1)$ as follows. Defining $D_n^{HI}(u) = D_n(uT_{[m]}) = F_n(uT_{[m]}) - uT_{[m]}$, it is sufficient to study the function $F_n(uT_{[m]})$ conditional on T_0 and $T_{(r)}$. It is easy to observe that

$$\begin{aligned} D_n^{HI}(u)|\{T_{[m]} = T_{(r)}\} &\sim D_n^{II}(u) \\ D_n^{HI}(u)|\{T_{[m]} = T_0\} &\sim D_n^{II}\left(\frac{uT_0}{T_{(r)}}\right) \end{aligned}$$

This is because, if $T_{[m]} = T_0$ then $D_n^{HI}(u) = F_n(uT_0) - uT_0 = F_n(\frac{uT_0}{T_{(r)}}T_{(r)}) - \frac{uT_0}{T_{(r)}}T_{(r)}$.

In other words $D_n^{HI}(\cdot)|\{T_{[m]} = T_0\}$ can be viewed as a process identical to $D_n^{II}(\cdot)$ but stopped in between at a random time $T_0/T_{(r)}$, i.e.,

$$D_n^{HI}(\cdot)|\{T_{[m]} = T_0\} \sim D_n^{II}\left(\min\left\{z, \frac{T_0}{T_{(r)}}\right\}\right)$$

where $z \in [0, 1]$ and $T_{(r)} > T_0$. Therefore, as a whole, for $u \in [0, 1]$ the distance function $D_n^{HI}(u)$ for the Type-I hybrid censoring can be approximated by a mixture of two Brownian bridges with a drift and scaling, which are conditionally independent, when r and n be large enough. The result can be stated as

$$\sqrt{n}D_n^{HI}(u) \overset{d}{\approx} D_*^{HI}(u) \qquad \text{for sufficiently large } r \text{ and } n$$

where

$$\begin{aligned} D_*^{HI}(u) = &\left[\frac{r-1}{\sqrt{n}}u + \sqrt{\frac{r-1}{n}}B_0(u) - u\sqrt{n}\,T_{(r)} + \frac{1_{\{u=1\}}}{\sqrt{n}}\right]\mathbf{1}_{\{T_{(r)}\le T_0\}} \\ &+ \left[\frac{r-1}{\sqrt{n}}\left(\frac{uT_0}{T_{(r)}}\right) + \sqrt{\frac{r-1}{n}}B_0\left(\frac{uT_0}{T_{(r)}}\right) - u\sqrt{n}T_0\right]\mathbf{1}_{\{T_{(r)}>T_0\}} \end{aligned} \tag{5.1}$$

and $T_{(r)} \sim \mathbb{B}(r, n-r+1)$.

Now, we define the KS statistic for the Type-I hybrid censoring by

$$KS_{HI} = \sup_{u \in [0,1]} \sqrt{n} \left| D_n^{HI}(u) \right|$$

So the asymptotic distribution of KS_{HI} can be well approximated by the distribution of

$$\sup_{u \in [0,1]} \sqrt{n} \left| D_*^{HI}(u) \right| = KS_{HI}^a, \quad \text{say} \tag{5.2}$$

6. Type-II hybrid censoring

According to the definition of the Type-II hybrid censoring, the experiment is terminated at T_0 or $T_{(r)}$ whichever comes later. Stopping time in a Type-II hybrid censoring is $\max\{T_0, T_{(r)}\} = T_{[M]}$, say, which is a random quantity. The probability that the experiment is stopped when $T_{(r)}$ is observed after T_0 is the same as the probability of $T_{(r)} > T_0$ and therefore we have

$$P(T_{[M]} = T_{(r)}) = P(T_{(r)} > T_0) = q_0$$

On the other hand, the experiment is stopped at the fixed time T_0 with the probability p_0. When $T_{(r)}$ occurs after T_0, the situation is exactly same as the Type-II censoring. But if $T_{(r)}$ occurs before T_0, we are supposed to observe the similar trajectory of $F_n(\cdot)$ for Type-I censoring which is terminated at T_0. We define our process for $u \in [0, 1)$ as follows. Defining $D_n^{HII}(u) = D_n(uT_{[M]}) = F_n(uT_{[M]}) - uT_{[M]}$, it is sufficient to study the function $F_n(uT_{[M]})$ conditional on T_0 and $T_{(r)}$. It is easy to observe that

$$D_n^{HII}(u) | \{T_{[M]} = T_{(r)}\} \sim D_n^{II}(u)$$

$$D_n^{HII}(u) | \{T_{[M]} = T_0\} \sim \sum_{k=r}^{n} D_n\,(uT_0|\,k) \mathbf{1}_{\{K=k\}}$$

where $K \sim bin(n, T_0 | K \geq r)$ and hence, $P(K = k) = \binom{n}{k} T_0^k (1 - T_0)^k / p_0$. This is because, when $T_{[M]} = T_0$, there are at least r observations with lifetimes less than T_0, which can be represented by the random variable K. We can use the approximation as

$$\sqrt{n} D(uT_0|K) \overset{d}{\approx} \frac{uK}{\sqrt{n}} + B_0(u)\sqrt{\frac{K}{n}} - \sqrt{n}\,uT_0$$

where $K \sim bin(n, T_0 | K \geq r)$.

Hence, for $u \in [0, 1]$, the distance function $D_n^{HII}(u)$ can be approximated by a mixture of two Brownian bridges, which are conditionally independent, up to a proper adjustment of location and a scale when r and n are sufficiently large. The result can be stated as

$$\sqrt{n} D_n^{HII}(u) \overset{d}{\approx} D_*^{HII}(u) \text{ for sufficiently large } r \text{ and } n$$

where

$$\begin{aligned} D_*^{HII}(u) = {} & \left[\frac{r-1}{\sqrt{n}} u + \sqrt{\frac{r-1}{n}} B_0(u) - u\sqrt{n}\,T_{(r)} + \frac{1_{\{u=1\}}}{\sqrt{n}} \right] \mathbf{1}_{\{T_{(r)} > T_0\}} \\ & + \left[\sum_{k=r}^{n} \left(\frac{uk}{\sqrt{n}} + B_0(u)\sqrt{\frac{k}{n}} - \sqrt{n}\,uT_0 \right) \mathbf{1}_{\{K=k\}} \right] \mathbf{1}_{\{T_{(r)} \leq T_0\}} \end{aligned}$$

and $T_{(r)} \sim \mathbb{B}(r, n - r + 1)$.

Now we define the KS statistic for the Type-II hybrid censoring by

$$KS_{HII} = \sup_{u\in[0,1]} \sqrt{n}\left|D_n^{HII}(u)\right|$$

So the asymptotic distribution of KS_{HII} can be well approximated by the distribution of

$$\sup_{u\in[0,1]} \sqrt{n}|D_*^{HII}(u)| \equiv KS_{HII}^a, \quad \text{say} \tag{6.1}$$

Remark 2. It is evident from Sections 5 and 6 that hybrid censoring schemes depend on the values of $T_{(r)}$ and T_0. But $T_{(r)}$ approaches to λ_0 with probability 1 as r and n both increase unbounded above such that $r/n \to \lambda_0$. So the events $\{T_{(r)} > T_0\}$ or $\{T_{(r)} \leq T_0\}$ become almost deterministic. Then the hybrid censoring takes place as a mixture of Type-I censoring and Type-II censoring when $\sqrt{n}|\lambda_0 - T_0|$ is bounded.

7. Simulation findings

An extensive simulation study is conducted to obtain the approximated distribution of KS test statistic under different censoring schemes to be used for testing the goodness-of-fit (see Table 1). For the illustrative purpose, we consider $n = 100$ and get the 95th percentile points. For Type-I censoring, we take $T_0 = 0.40, 0.45, 0.50, \ldots, 0.85, 0.90,$ and 0.95, and for Type-II censoring, $r = 40, 45, 50, \ldots, 85, 90,$ and 95. To obtain the exactly simulated distributions of the KS test statistic, we use the expressions of $\sqrt{n}D_{n:T_0}$ and $\sqrt{n}D_{n:r}$, for Type-I and Type-II censoring schemes, respectively, discussed in Section 2. The same are approximated by the distributions of KS_I^a and KS_{II}^a as proposed in Equations (3.3) and (4.3), respectively. To obtain each of the distribution and get its 95th percentile point, we generate $n = 100$ i.i.d. samples from $U[0, 1]$ and repeat the same 50,000 times. The entire process is repeated for 1000 times and we take the median of those estimated 95th percentile points as a final approximation. For KS_I^a and KS_{II}^a, the supremum (maximum) of the Brownian bridge is obtained through a grid search over $n \times 10$ equidistant points on $[0, 1]$. The 95th percentage points are compared between the exact simulation and the method proposed in the article. One more comparison is done with the tabulated values for each of Type-I and Type-II censoring schemes available in the paper by Koziol and Byar (1975) after the adjustments suggested

Table 1. 95th percentage points of the distributions of KS statistic for Type-I and Type-II censoring schemes by the proposed methodology and their comparison with exact simulation and tabulated values by Koziol and Byar (1975) (*Values are taken from Koziol and Byar, 1975).

$n = 100$	Type-I (T_0)			$n = 100$	Type-II (r)		
T_0	$\sqrt{n}D_{n:T_0}$	KS_I^a	Tabulated*	r	$\sqrt{n}D_{n:r}$	KS_{II}^a	Tabulated*
0.40	1.1809	1.1854	1.1785	40	1.1776	1.1836	1.1735
0.45	1.2196	1.2184	—	45	1.2173	1.2204	—
0.50	1.2548	1.2538	1.2541	50	1.2503	1.2525	1.2491
0.55	1.2830	1.2813	—	55	1.2767	1.2787	—
0.60	1.3019	1.3019	1.3021	60	1.2970	1.2972	1.2971
0.65	1.3175	1.3175	—	65	1.3123	1.3134	—
0.70	1.3283	1.3274	1.3281	70	1.3234	1.3240	1.3231
0.75	1.3348	1.3348	—	75	1.3310	1.3304	—
0.80	1.3383	1.3378	1.3378	80	1.3356	1.3359	1.3328
0.85	1.3394	1.3395	—	85	1.3381	1.3391	—
0.90	1.3401	1.3396	1.3391	90	1.3398	1.3397	1.3341
0.95	1.3399	1.3386	—	95	1.3399	1.3395	—

Table 2. 95th percentage points of the distributions of KS statistic by the proposed methodology for Type-I Hybrid and Type-II Hybrid censoring schemes.

$n = 100$		Type-I Hybrid (T_0, r)		Type-II Hybrid (T_0, r)	
r	T_0	KS^E_{HI}	KS^a_{HI}	KS^E_{HII}	KS^a_{HII}
40	0.40	1.1222	1.1326	1.2278	1.2261
45	0.45	1.1736	1.1775	1.2608	1.2651
50	0.50	1.2124	1.2176	1.2880	1.2879
55	0.55	1.2482	1.2514	1.3081	1.3100
60	0.60	1.2765	1.2783	1.3219	1.3227
65	0.65	1.2987	1.2997	1.3315	1.3330
70	0.70	1.3141	1.3159	1.3368	1.3377
75	0.75	1.3262	1.3267	1.3392	1.3398
80	0.80	1.3338	1.3340	1.3392	1.3402
85	0.85	1.3377	1.3388	1.3399	1.3407
90	0.90	1.3397	1.3390	1.3402	1.3395
95	0.95	1.3399	1.3395	1.3399	1.3394

by Dufour and Maag (1978) (see "Tabulated" columns in Table 1). The tabulated percentage points are available only when T_0 (or r/n) $= 0.1, 0.2, 0.3, \ldots, 0.7, 0.8, 0.9,$ and 1.0. The methods developed in this article are advantageous over the existing ones in the sense that we can obtain the 95th percentage points for any possible values of T_0 and r/n. It is observed that our approximations are very close to the exact simulations and the tabulated values as suggested by Dufour and Maag (1978) (see Section 2).

For the hybrid censoring schemes, we consider the pairs $(T_0, r) = (0.40, 40)$, $(0.45, 45), \ldots, (0.90, 90)$, and $(0.95, 95)$. Note that Type-I hybrid and Type-II hybrid censoring schemes take place according to the occurrence of the event $\{T_{(r)} > T_0\}$ or its complement. Therefore, a mixture of the distributions of $\sqrt{n}D_{n:T_0}$ and $\sqrt{n}D_{n:r}$ gives the exactly simulated distribution of the KS statistic for Type-I hybrid and Type-II hybrid censoring schemes accordingly. Let us denote the exactly simulated distribution for the KS statistic for Type-I hybrid censoring by KS^E_{HI} and that for Type-II hybrid censoring by KS^E_{HII}. We compare the 95th percentile of the distributions of KS^E_{HI} and KS^E_{HII} with that of KS^a_{HI} and KS^a_{HII}, obtained from the methods developed in this article (see Equations (5.2) and (6.1), respectively). Our methodology approximates very closely to the 95th percentage points of the exactly simulated distributions of KS test for Type-I hybrid and Type-II hybrid censored data (see Table 2). Next, using the cutoff values reported in Tables 1 and 2, the level of the test is cross verified for different sample sizes. The results are reported in Table 3. From Table 3, it is clear that given level of the test is achieved for large sample size.

Lin et al. (2008) proposed a modified KS test for the Type-II censoring scheme. According to the initials of the surname of the authors, we name it the method by LHB. We compare

Table 3. Level of the proposed test under the null hypothesis for the different censoring schemes.

$n = 100$		Using T_0	Using r	Using (T_0, r)	
r	T_0	Type-I (%)	Type-II (%)	Type-I Hybrid (%)	Type-II Hybrid (%)
40	0.4	4.89	4.85	4.72	5.05
50	0.5	5.03	4.95	4.86	5.00
60	0.6	5.01	4.99	4.95	4.98
70	0.7	5.03	4.99	4.95	4.98
80	0.8	5.02	5.00	5.00	5.00
90	0.9	5.02	5.01	5.03	5.03

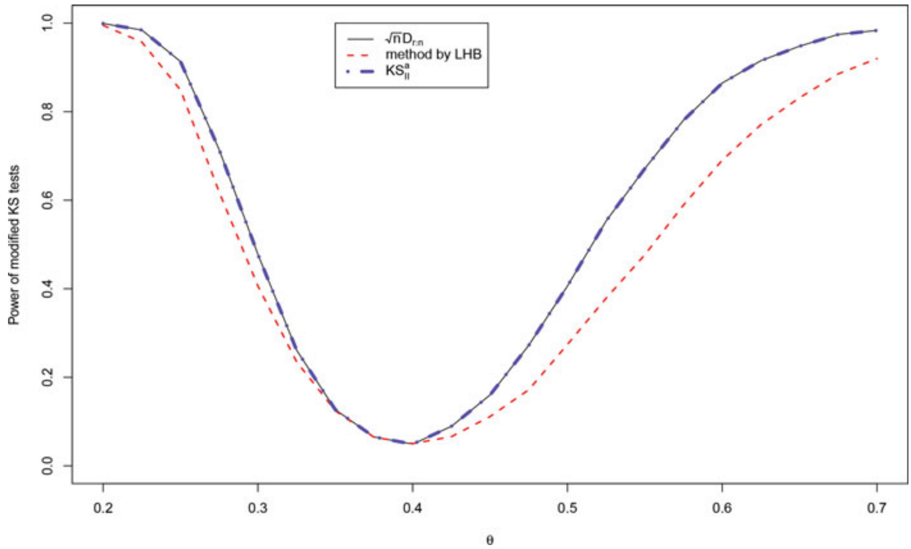

Figure 1. Power curves for modified KS tests under Type-II censoring for $n = 100$, $r = 60$ with exponential distribution with scale parameter $\theta = 0.4$ under the null hypothesis.

the power curves for the goodness-of-fit tests for Type-II censored data using the method by LHB, the new method proposed in this article, and the method by exact simulation. We assume that the data are coming from exponential distribution with scale parameter $\theta = 0.4$ (i.e., mean $1/\theta$) under the null hypothesis. We take $(n, r) = (100, 60)$ and $(100, 80)$. When $r = 60$, to test for the both sided alternative at 5% level of significance, the cutoff value for the exact simulation is 1.2970 and that for the proposed test is 1.2972. Cutoff value for the same according to the methodology by LHB is 1.3581 (see Stephens, 1970). It is observed that the power curve obtained from our methodology almost overlaps with that of the exact simulation results. Moreover, the power curve from our method dominates the power curve obtained from the method by LHB (see Figure 1). When $r = 80$, the cutoff values with 5% level become 1.3356 and 1.3359 for exact simulation method and proposed method, respectively. But it remains unchanged as 1.3581 (see Stephens, 1970) for the method by LHB. It is because the test statistic proposed by Lin et al. (2008) adjusts the value of r in such a way that the cutoff value does not depend on r. We observe a repetition of the same features among the power curves as in the previous case (see Figure 2).

We also conduct similar simulation studies for Type-II censored data with $(n, r) = (120, 200)$ for Weibull distribution with scale parameter $\lambda = 1$ and shape parameter $\theta = 0.5$ under the null hypothesis and compare the power curves, in Figure 3, for the tests with exact simulation, proposed method, and the method by Lin et al. (2008). Another simulation is conducted Type-II censored data with $(n, r) = (160, 200)$ for Weibull distribution with scale parameter $\lambda = 1$ and shape parameter $\theta = 0.5$ under the null hypothesis. A comparison with the methodology by Lin et al. (2008) is done in terms of the power curves which are provided in Figure 4.

8. Data analysis

Birnbaum and Saunders (1958) reported an application of the gamma distribution to the lifetime of aluminum coupon. In their study, 17 sets of six strips were placed in a specially

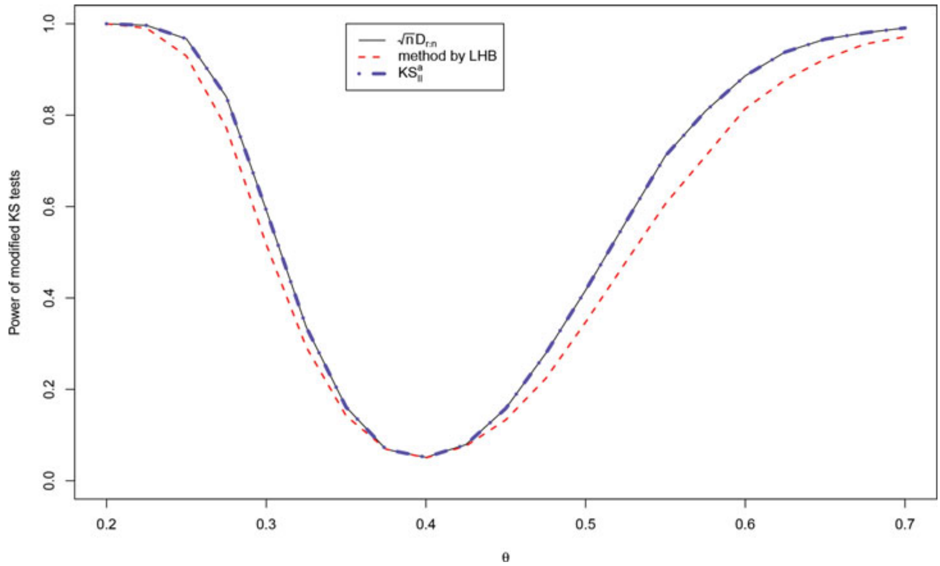

Figure 2. Power curves for modified KS tests under Type-II censoring for $n = 100$, $r = 80$ with exponential distribution with scale parameter $\theta = 0.4$ under the null hypothesis.

designed machine. The 102 strips were run until all of them failed. One of the 102 strips tested had to be discarded for an extraneous reason, yielding 101 observations. The lifetime data of 101 aluminum coupons are reported in (Lee and Wang, 2003, Ch. 6, p. 153). We consider the gamma distribution with density function $f(t) = \Gamma(\alpha)^{-1}\lambda(\lambda t)^{\alpha-1}e^{-\lambda t}$, $t > 0, \alpha > 0, \lambda > 0$. The maximum likelihood estimates of α and λ based on complete data are $\hat{\alpha} = 11.8$ and $\hat{\lambda}= 1/(118.76)$ (Lee and Wang, 2003, Ch. 6, p 153), respectively. We considered the data in a scale of 10^3. The parametric and non parametric estimates of survival functions based

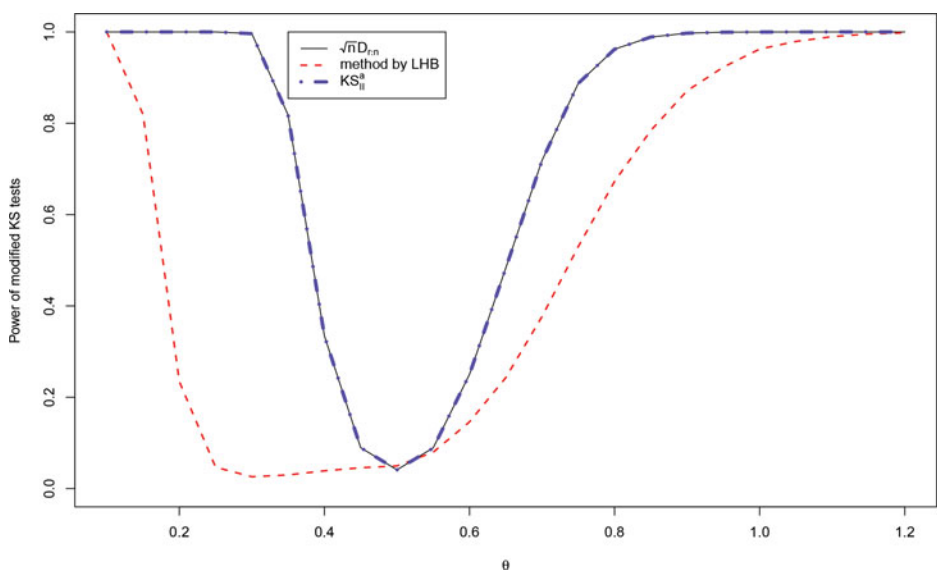

Figure 3. Power curves for modified KS tests under Type-II censoring for $n = 200$, $r = 120$ with Weibull distribution with scale parameter $\lambda = 1$ and shape parameter $\theta = 0.5$ under the null hypothesis.

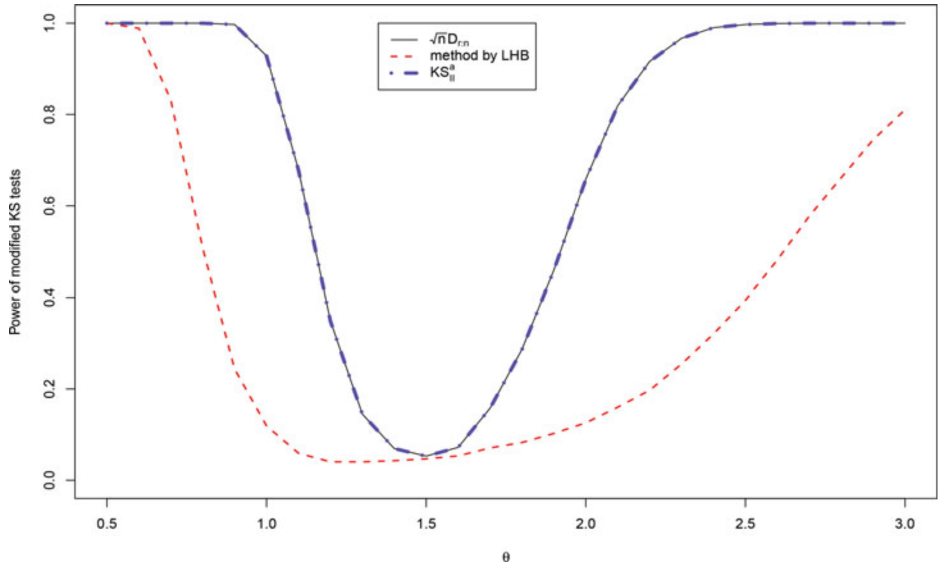

Figure 4. Power curves for modified KS tests under Type-II censoring for $n = 200$, $r = 160$ with Weibull distribution with scale parameter $\lambda = 1$ and shape parameter $\theta = 1.5$ under the null hypothesis.

on complete data are provided in Figure 5 with vertical lines indicating the stopping times under the Type-I and Type-II censoring schemes. Next, we generate Type-I censored data from the complete data by taking $X_0 = 1413$, that is, $T_0 = 0.55$, and Type-II censored data by taking $r = 55$, i.e., $X_{(r)} = 1450$. The Type-I hybrid and Type-II hybrid data are generated from the complete data by taking same X_0 and r. The cutoff values, observed test statistic, and the corresponding p-values are reported in Table 4. In each case, p-value is too high to reject the test.

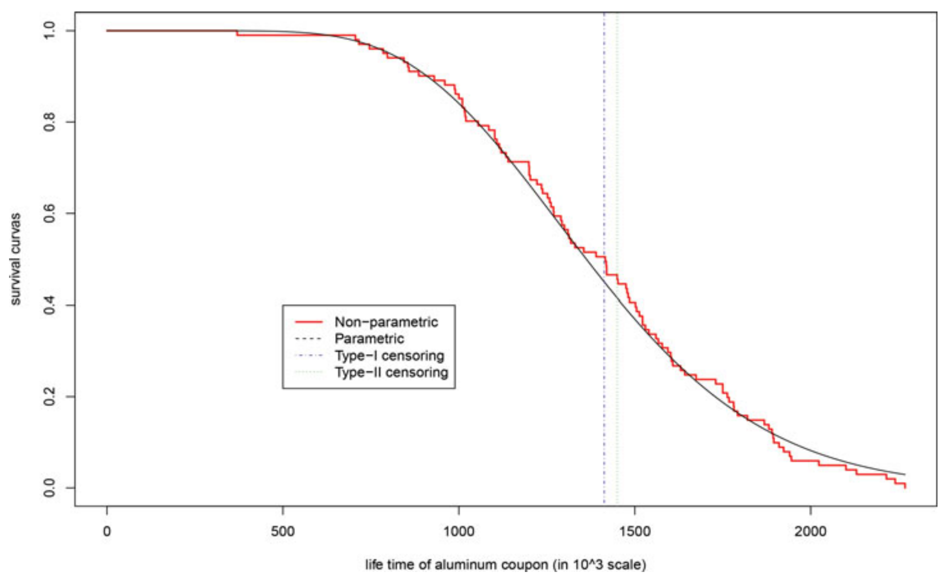

Figure 5. Survival functions of aluminum coupons lifetime data.

Table 4. Goodness-of-fit test of aluminum coupons lifetime data to Gamma (11.8,1/118.76) distribution.

$n = 101$		Using T_0	Using r	Using (T_0, r)	
$r = 55$	$T_0 = 0.55$	Type-I	Type-II	Type-I Hybrid	Type-II Hybrid
95% cut-off values		1.2808	1.2760	1.2490	1.2917
Observed test statistic		0.5522	0.5809	0.5522	0.5809
p-Values		0.7849	0.7308	0.7653	0.7522

9. Conclusion

In this article we study the extension of the KS test for the censored data. We give an alternative representation to the KS test for Type-I censored data and obtain the limiting distribution of the KS test statistic. The limiting distribution of the KS test statistic for Type-II censoring scheme is derived using the proposed alternative representation corresponding to Type-I censoring scheme. Subsequently, we get the approximated distributions of the KS statistic for Type-I hybrid and Type-II hybrid data. It is easy to obtain the cutoff value of the goodness-of-fit test from the approximated distributions through Monte Carlo simulation for any presumed values of T_0 and r. Through Monte Carlo simulation, it is observed that the proposed methodology for testing the goodness-of-fit outperforms the method provided by Lin et al. (2008) for moderate and large sample sizes under Type-II censoring scheme.

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Appendix

A1

Consider for $0 < u < v < 1$ and we get the following immediately: $E(F_n(uT_{(r)})) = \frac{r-1}{n}u$, $E(F_n(uT_{(r)})) = \frac{r-1}{n}u$, and $Cov(F_n(uT_{(r)}), F_n(vT_{(r)})) = \frac{r-1}{n^2}u(1-v)$. Hence

$$\begin{aligned}
&\sqrt{\frac{n^2}{r-1}}\left(F_n(uT_{(r)}) - \frac{r-1}{n}u\right) \\
&\stackrel{d}{=} \sqrt{\frac{n^2}{r-1}}\left(\frac{r-1}{n}F_{r-1}(u) - \frac{r-1}{n}u\right) \\
&= \sqrt{r-1}\,(F_{r-1}(u) - u) \\
&\Rightarrow B_0(u) \quad \text{as } r \uparrow \infty
\end{aligned}$$

A2

To show $\sup_{u\in[0,1]} |D_n^{II}(u)| \longrightarrow 0$ w.p. 1, as r and $n \uparrow \infty$, let us consider $u \in [0, 1)$ and get

$$\begin{aligned}
\sup_{u\in[0,1)} |D_n^{II}(u)| &= \sup_{u\in[0,1)} |F_n(uT_{(r)}) - uT_{(r)}| \\
&\stackrel{d}{=} \sup_{u\in[0,1]} \left|\frac{r-1}{n}F_{r-1}(u) - uT_{(r)}\right| \\
&= \sup_{u\in[0,1]} \left|\frac{r-1}{n}(F_{r-1}(u) - u) + \frac{r-1}{n}u - uT_{(r)}\right| \\
&\le \left|\frac{r-1}{n}\right| \sup_{u\in[0,1]} |(F_{r-1}(u) - u)| + \sup_{u\in[0,1]} |u| \left|\frac{r-1}{n} - T_{(r)}\right|
\end{aligned}$$

The first component of the right hand side goes to zero almost surely for sufficiently large r by Glivenko–Cantelli theorem and the second component converges to 0 with probability 1 by the use of Chebyshev's inequality. But when $u = 1$, the value of $F_n(T_{(r)}) = r/n$ and so $|\frac{r-1}{n} - T_{(r)}| \longrightarrow 0$ with probability 1 for large values of n and r. Hence the result follows.