Unit-3

Chapter-5 Stochastic integrals & Differential Eq

Stochastic calculus is the area of mathematics that deals with the processes containing a stochastic component (random) and thus allows modelling of random system. Many stochastic processes are based on fn which are continuous but nowhere differentiable. A theory of integration is reqd. where integrals eqn's do not need the direct definition of derivative terms. In quantitative finance the theory is known as Itô.

The main use of stochastic calculus in finance is through modelling the random motion of an asset price in the Black Scholes model. The physical process of brownian motion is used as a model of the asset price via Wiener Process. The process is represented by stochastic differential eqn.

The fundamental diff. b/w stochastic calculus & ordinary calculus is that the stochastic calculus allows the derivative to have a random component determine by brownian motion.

The derivative of a random variable has both deterministic & a random component which is normally distributed.

Let us consider an example of stock market

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

above eqn describes the price of the stock where S_t is the price of the stock

dS_t — change in the stock price.
dt — change in time.
μ — Average expected return of the stock over time.
σ — the volatility of the stock.
dW_t — change in the brownian motion.

The equation indicates that the stock prices changes in relation to two terms.
$\mu S_t\, dt$, it denotes the long term expected return of the stock times the current price of the stock times length of time.

$\mu = 3\%$, $S_t = 100$, $dt = 1$

The second term, $\sigma S_t\, dW_t$ models random up & down of the stocks. The ups and downs are represented by brownian motion. dW_t represents the change in the brownian motion over the period of time.
In order to calculate the actual price of the stock we solve the above eqn using stochastic calculus.
On solving the eqn on using stochastic calculus we would not get a definite price of the stock 1 year from now But we will obtain a random variable with its mean & variance.
Brownian motion is the random motion of the particles suspended in a fluid resulting from the collision of the particles.

wiener
Weiner process, It is the ~~constant~~ continuous time stochastic process named in honour of Norbert Weiner. It is often called standard brownian motion or brownian motion due to its historical pro

connection with physical process known as brownian of motion

$$X_t^{\Delta} = \sum_{k=1}^{n} Z_k \Delta x \qquad X_0 = 0$$

E

$$= X_n \cdot \Delta x.$$

$$E(X_t^{\Delta}) = E(X_n) \cdot \Delta x.$$
$$= 0$$

$$\Delta t \to 0, \quad \Delta x \to c\sqrt{\Delta t}$$

$$V(X_t^{\Delta}) = V(X_n \Delta x)$$
$$= (\Delta x)^2 V(X_n)$$
$$= (\Delta x)^2 n \qquad = t \cdot \frac{(\Delta x)^2}{\Delta t}$$
$$= (c\sqrt{\Delta t})^2 n$$
$$= c^2 \Delta t\, n$$
$$= c^2 t$$

X_t^{Δ} limiting $\Delta t \to 0$ X_t

$$L(X_t^{\Delta}) \to N(0, c^2 t)$$

→ Show that $X_t - X_s$ follows a $N(0, c^2(t-s))$, $s < t$

$$E(X_t - X_s) = E(X_t) - E(X_s)$$
$$= 0$$

$$V(X_t - X_s) = V(X_t) + V(X_s) - 2\,cov(X_t, X_s)$$
$$= c^2 t + c^2 s - 2\,cov(X_t - X_s + X_s, X_s)$$
$$= c^2(t+s) - 2\,cov(X_t - X_s, X_s) - 2V(X_s)$$
$$= c^2(t+s) - 0 - 2c^2 s$$
$$= c^2(t-s)$$

Distribution of W_t given $W_s = x$ → standard wiener process

$$W_t - W_s \sim N(0, t-s)$$
$$X_t \sim N(0, c^2 t), \quad W_t \sim N(0, t)$$

$$E(W_t \mid W_s = x) = E[W_t - W_s + W_s \mid W_s = x]$$
$$= E[W_t - W_s \mid W_s = x] + E[W_s \mid W_s = x]$$
$$= 0 + x$$
$$= x.$$

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$$\Rightarrow V(W_t \mid W_s = x) = V(W_t - W_s \mid W_s = x) + V(W_s \mid W_s = x)$$

$$= V(W_t - W_s \mid W_s = x)$$

$$= V(W_t - W_s)$$

$$= t - s$$

When $E(X_n) = n(2p-1)$

$$E(X_t^{\Delta}) = n(2p-1)\,\Delta x$$

$$= \frac{t}{\Delta t}\,\Delta x\,(2p-1)$$

$$= t(2p-1)\,\frac{\Delta x}{\Delta t}$$

$$V(X_t^{\Delta}) = (\Delta x)^2\, V(X_n)$$

→ Let $\{W_t, t \geq 0\}$ be a standard wiener process and $\mu(\ ,\ )$, $\sigma(\ ,\ )$ are specific ~~function~~ function

$$X_{t+\Delta t} - X_t = \mu(t, X_t)\,\Delta t + \sigma(t, X_t)\,\Delta W_t \quad \text{— (1)}$$

$$X_0 = 0$$

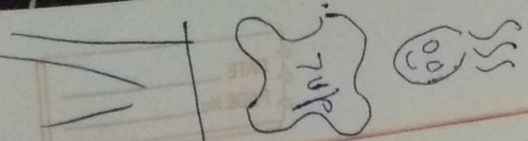
divide eqn (1) by Δt and let $\Delta t \to 0$

$$\dot{x}(t) = \mu(t, x_t) + \sigma(t, X_t)\,\frac{\Delta W_t}{\Delta t}$$

$$\dot{X}(t) = \mu(t, X_t) + \sigma(t, X_t)\,V(t)$$

where $V(t) = \frac{\Delta W_t}{\Delta t}$

It can be shown the process $V(t)$ is not well defined as the wiener process is nowhere differentiable although it is continuous



Under the mean square manner ~~drift~~ derivative of the wiener process W_t is not the derivative process $V(t) = \frac{\Delta W_t}{\Delta t}$

→ Another proof to show that wiener process is not differentiable

$$X_{t+\Delta t} - X_t = \mu(t, X_t)\Delta t + \sigma(t, X_t)\Delta W_t \quad (2)$$

without dividing eqn (2) by Δt and $\Delta t \to 0$

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \quad (3)$$

and $X_0 = x$

On defining $\lim_{\Delta t \to 0} X_{t+\Delta t} - X_t = dX_t$

$$\lim_{\Delta t \to 0} \mu(t, X_t)\Delta t = \mu(t, X_t)\,dt$$

$$\lim_{\Delta t \to 0} \sigma(t, X_t)\Delta W_t = \sigma(t, X_t)\,dW_t$$

The eqn (3) is an eqn in differential form.

i.e. $X_t = x_0 + \int \mu(t, X_t)\,dt + \int \sigma(t, X_t)\,dW_t$

$$= x + \int_0^t \mu(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s$$

↓ Riemann integral

↓ Riemann Stieltjes integral (divergent integral unbounded)

Riemann Stieltje's integral of a real value fn $f()$ of real variable w.r.t a real fn $g()$ is $\int_a^b f(x)\,dg(x)$

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$\S$:. we have to integrate w.r.t. wiener process, we observe the following points

(i) As a wiener process has an unbounded variation it will eventually hit every real value no matter how large or ... once a wiener process hits the value it immediately hits it again infinitely and then again and again time to time in future

Even if the scale of wiener process change i.e. either increase or decrease it will look just the same. This property is known as self similarity and in mathematics it is known as fractals

→ to obtain the sol^n a stochastic differential eq^n we shall make use of ITO's integral i.e. a stochastic integral w.r.t. a wiener process.

In the ITO integral instead of integrating w.r.t. a deterministic function we shall integrate w.r.t. a random function viz. the path of the wiener process since the integrant can be a we have to analyse the mutual dependences and the integrant of

Let $\{Y_t, t \geq 0\}$ be the process to be integrated and $\{W_t, t \geq 0\}$ be a standard wiener process According to the definition of stochastic integral we assume that $\{Y_t, t > 0\}$ is ~~not~~ non-anticipating (does not anticipate any future information) Thus it means the process upto the time s doesnot contain any information about the future increments $W_t - W_s$, $t > s$ of wiener process

General form of stochastic integral
we define a stochastic integral keeping in mind the property of the wiener process.

$$I(t, \omega) = \int_0^t g(s, \omega)\, dW_s$$

where $g(t, \omega)$ is some suitably chosen smooth function of the following scheme.

① Partition the time interval 0 to t in 'n' subintervals of equal length $n\,[t_k, t_{k+1}]$

$K = 1, 2, \ldots n-1$

$0 = t_0 < t_1 < t_2 \ldots < t_n = t$

② Define for each trajectory of ω and appropriate integral.

$$I_n(t, \omega) = \sum_{k=0}^{n-1} g(\tau_k ; \omega)\left[W(t_{k+1}, \omega) - W(t_k, \omega)\right]$$

where τ_k is some arbitrarily chosen time in the interval $(t_k$ and $, t_{k+1})$

③ Let $\lim_{n \to \infty}$ then if the above sum converge to some limit say I then I will be the sol^n to the differential eq^n.

Let us now consider the Itô integral with integrand $\{Y_t, t \geq 0\}$ and $\{W_t, t \geq 0\}$ be the standard wiener process

The Itô's integral w.r.t. wiener process is defined as the limit of the sum of the randomly weighted random f^n $\{Y_t, t \geq 0$

$t(n,k)$

$I_n = \sum_{k=1}^{n} Y_{(k-1)\Delta t}(W_{k\Delta t} - W_{(k-1)\Delta t})$ where $t = n\Delta t$

$\lim_{n\to\infty} I_n = \int_0^t Y_s \, dW_s$

where the lt is in terms of the mean square error i.e., $E\left[\left(\int_0^t Y_s \, dW_s - I_n\right)^2\right]$. we shall now show that $E(\ \ u\ ) \to 0$ as $n \to \infty$

The integral depends criticelly on the point of the integral $((k-1)\Delta t, k\Delta t)$ at which the random variable Y_s evaluted.

Consider a situation where $Y_t = W_t$

Consider the value of Y_t i.e. W_t at any arbitrary point in the interval $(k-1)\Delta t, k\Delta t$ so that the integral $\int_0^t Y_s \, dW_s = \int_0^t W_s \, dW_s$

$= \lim_{n\to\infty} \sum_{k=1}^{n} W_{t(n,k)} (W_{k\Delta t} - W_{(k-1)\Delta t})$

consider the expected value of summation

$$E\left[\sum_{k=1}^{n} W_t (W_{k\Delta t} - W_{(k-1)\Delta t})\right] =$$

$$= \sum_{k=1}^{n} E[(W_t (W_{k\Delta t} - W_{(k-1)\Delta t})]$$

$$= \sum_{k=1}^{n} \text{cov}(W_t, W_{k\Delta t} - W_{(k-1)\Delta t})$$

$$= \sum_{k=1}^{n} \{t(n,k) - (k-1)\Delta t\} \to E\{\int_0^t W_s \, dW_s\}$$

case I when $t(n,k) = (k-1)\Delta t$

$$E\left[\int_0^t W_s \, dW_s\right] = 0$$

(Itô integral)

When $t(n,k) = k\Delta t$

$$E\left[\int_0^t W_s\, dW_s\right] = \sum_{k=1}^{n} \Delta t$$

$$= n\Delta t = t$$

$$\therefore E\left[\int_0^t W_s\, dW_s\right] = \sum\left(k\Delta t - (k-1)\Delta t\right)$$

$$= \sum k\Delta t - \cancel{k\Delta t} + \Delta t$$

$$= \sum \Delta t$$

$$= t$$

For suitably chosen sequences $t(n,k)$ we can obtain, for the expectation of the stochastic integral any value between 0 and t {In order to assign a unique value to $\int_0^t W_s\, dW_s$, we have to agree on a certain sequence $t(n,k)$}

$$E\left[\sum_{k=1}^{n} \underline{W_{t(n,k)}}\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right)\right] \to E\left[\int_0^t W_s\, dW_s\right]$$

To obtain the value of $\int_0^t W_s\, dW_s$

Consider $\frac{1}{2}\left[\sum\left(W_{k\Delta t}^2 - W_{(k-1)\Delta t}^2\right)\right]$

$$= \frac{1}{2}\left[\cancel{W_{\Delta t}^2} + W_0^2 + W_{2\Delta t}^2 - \cancel{W_{\Delta t}^2} + \dots + W_{n\Delta t}^2 - \cancel{W_{(n-1)\Delta t}^2}\right]$$

$$= \frac{1}{2}\left[W_{n\Delta t}^2 - W_0^2\right]$$

$\therefore n\Delta t = t$

$$= \frac{1}{2}\left[W_t^2 - W_0^2\right]$$

$$\frac{1}{2}\left[W_t^2 - W_0^2\right] = \frac{1}{2}\left\{\Sigma\left(W_{k\Delta t} + W_{(k-1)\Delta t}\right)\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right)\right\}$$

$$= \frac{1}{2}\left[\Sigma\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right)^2\right] + \boxed{\Sigma\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right) W_{(k-1)\Delta t}}$$

(reqd. expression)

On taking expectation, we get

$$\frac{1}{2}\left(W_t^2 - W_0^2\right) = \frac{1}{2} E\left(\sum_{k=1}^{n}\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right)^2\right) + E\left(\sum_{k=1}^{n}\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right) W_{(k-1)\Delta t}\right)$$

where

$E\left[\sum_{k=1}^{n}\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right) W_{(k-1)\Delta t}\right]$ converges to $\int_0^t W_s \, dW_s$ as $n \to \infty$

and $\frac{1}{2} E\left(\sum_{k=1}^{n}\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right)^2\right)$ is sum of n iid r.v.'s and hence for $n \to \infty$

$$\frac{1}{2} E\left(\Sigma\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right)^2\right) = \frac{1}{2} n\left[k\Delta t - (k-1)\Delta t\right]$$

$$= \frac{1}{2} n \Delta t = \frac{t}{2}$$

$$\therefore \int_0^t W_s \, dW_s = \frac{1}{2}\left(W_t^2 - W_0^2\right) - \frac{1}{2} t$$

→ **Stochastic differential Equation:**

Let us assume the existence of prob. space (Ω, F, P) where F is σ-algebra on the sample space Ω of possible outcomes and (Ω, F) is a measurable space and $P: F \to [0,1]$ is prob. measure.

Let the drift be 'μ' and the diffusion be 'σ' be Borel measurable functions and assume that $X_t : \Omega \to \mathbb{R}$ is a solution to time homogeneous Itô stochastic differential

$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)\, dw_t$

$X(0) = x_0$ (initially)

where $\{w_t, t \geq 0\}$ is a standard weiner process defined on a prob space (Ω, F, P)

Ques: Consider the weiner process $dX_t = \sigma dw_t$ where σ is s.d of process & x_0 is the deterministic initial condition. Show that the weiner process is a divergent process.

Sol: $dX_t = \sigma dw_t$ is a short form for

$X(t) = x_0 + \int_0^t \sigma dw_s$ (Random walk) → increments in the form of integral

$\cancel{X(t) = x_0 + \sigma(w_t - w_0)}$ [P1]

$E(X(t)) = x_0$

$V(X(t)) = V\left(\int_0^t \sigma dw_s\right)$

$= \sigma^2 E\left[\left(\int_0^t dw_s\right)^2\right]$ — [P2]

$V(X(t)) = \sigma^2 t \to \infty$ as $t \to \infty$

$X(t)$ is approx $N(x_0, \sigma^2 t)$

Thus weiner is a divergent process.

→ weiner process with non-zero drift $dX_t = \mu dt + \sigma dw_t$

This is short form for $X_t = x_0 + \int_0^t \mu ds + \int_0^t \sigma dw_s$

$\mu, \sigma \to$ constants

$E(X(t)) = x_0 + \mu t$

$V(X(t)) = V\left(\int_0^t \mu ds\right) + V\left(\int_0^t \sigma dw_s\right)$

$= \mu^2 V\left(\int_0^t ds\right) + \sigma^2 t$

$$= \mu^2\left[E\left(\int_0^t ds\right)^2 - \left(E\left(\int_0^t ds\right)\right)^2\right] + \sigma^2 t$$

$$= \mu^2(t^2 - t^2) + \sigma^2 t = \sigma^2 t$$

Thus mean of X_t has a linear trend or drift

$$dX_t = \mu X_t dt + \sigma dW_t$$

This stochastic differential eq^n is used to describe unlimited growth in biological systems or a stochastic money market account

This stochastic d.E. is short form for

$$X_t = x_0 + \int_0^t \mu X_s ds + \int_0^t \sigma dW_s$$

$$E(X_t) = x_0 + \mu\int_0^t E(X_s)\, ds + 0 \quad \text{--- (1)}$$

using Fubini's theorem, $E\left[\int_0^t X_s ds\right] = \int_0^t E(X_s)\, ds$

let $m_t = E(X_t)$

$$\frac{dm_t}{dt} = m'(t)$$

Also $\frac{dm_t}{dt} = \mu m(t)$ (from (1))

$$\therefore \frac{d}{dt}\left[x_0 + \mu\int_0^t E(X_s)\, ds\right] = \mu E(X_t) = \cancel{\quad}\ \mu m_t$$

$$\frac{1}{m_t}\frac{d m_t}{dt} = \mu$$

$$m_t = ce^{\mu t}$$

$m_0 = c$ [where c will be determined using initial condition]

$$m_t = m_0 e^{\mu t}$$

$$m_t = E(X_t) = E(X_0)e^{\mu t} \to \infty \text{ as } t \to \infty$$

$$\Rightarrow m_t \to \infty \text{ as } t \to \infty$$

The stock price as a Stochastic Process

$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t$

$\frac{dS_t}{S_t} = \frac{\mu(S_t, t)dt}{S_t} + \frac{\sigma(S_t, t)}{S_t}dW_t$

$E\left(\frac{dS_t}{S_t}\right) = \frac{1}{S_t}\mu(S_t, t)dt + \frac{\sigma(S_t, t)}{S_t}E(dW_t)$

$\mu dt = \frac{\mu(S_t, t)}{S_t}dt$

$\mu S_t = \mu(S_t, t)$

$E\left(\frac{dS_t}{S_t}\right) = \frac{\mu(S_t, t)dt}{S_t}$

$E(dS_t) = \mu(S_t, t)dt$

5.5 Itô's Lemma

Let $\{X_t, t \geq 0\}$ be an Itô's process given by

$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$ — (1)

Let $Y_t = g(X_t)$ be a fn of X_t
$\{Y_t, t \geq 0\}$ is again a stochastic process and the stochastic differential eqn for Y_t is given by

$dY_t = d(g(X_t)) = \left[\frac{d}{dx}g(X_t)\mu(X_t, t) + \frac{1}{2}\frac{d^2 g(X_t)}{dx^2}\sigma^2(X_t, t)\right]dt + \frac{d}{dx}g(X_t)\,\sigma(X_t, t)dW_t$

where g is assumed to differentiable as many times as reqd.

Squaring eqn (1) we get

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$$(dX_t)^2 = \mu^2(X_t,t)(dt)^2 + \sigma^2(X_t,t)(dW_t)^2 + 2\mu(X_t,t)\,\sigma(X_t,t)\,dt\,dW_t$$

as $dt \to 0$, the first & the 3rd term are of size $(dt)^2$, $dt\,dW_t = dt\sqrt{dt}$ resp. which becomes insign

$\because$ $dW_t = W_{t+\Delta t} - W_t$ which is of the size of standard deviation $\sqrt{dt}$

$E[(dW_t)^2] = dt$

$E(W_{t+dt} - W_t)^2 = dt$

$\therefore$ $V(W_{t+\Delta t} - W_t) = dt$

$$(dX_t)^2 = \sigma^2(X_t,t)(dW_t)^2 = \sigma^2(X_t,t)\,dt$$

consider the Taylor's expansion to $\{Y_t, t \geq 0\}$

$$Y_{t+dt} - Y_t = g(X_t + dX_t) - g(X_t)$$

$f(X_t + dt)$

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \dots \qquad \boxed{a = X_t}$$

$x - a = X_t + dX_t - X_t$

$$= \left[\cancel{g(X_t)} + \frac{dg(X_t)}{dx}dX_t + \frac{1}{2!}\frac{d^2g(X_t)}{dX^2}(dX_t)^2 + \dots\right] - \cancel{g(X_t)}$$

$$= \frac{dg(X_t)}{dX}dX_t + \frac{1}{2!}\frac{d^2(g(X_t))}{dX^2}(dX_t)^2$$

$$= \frac{dg(X_t)}{dX}\left(\mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t\right) + \frac{1}{2!}\frac{d^2(g(X_t))}{dX^2}\left(\mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t\right)^2$$

$$= \left[\frac{dg(X_t)}{dX}\mu(X_t,t) + \frac{1}{2!}\frac{d^2(g(X_t))}{dX^2}\right]dt + \frac{d}{dX}g(X_t)\,\sigma(X_t,t)\,dW_t$$

$dS_t = \mu S_t dt + S_t \sigma dW_t$

$g(x) = \log S_t \qquad \frac{1}{S_t} dS_t + \frac{1}{2} \times \frac{-1}{2S_t^2} \times (dS_t)^2$

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Ex. 5.3

Show that $\{Y_t, t \geq 0\}$ follows generalised weiner process

Sol. Consider $g(x) = \log x$

$$\frac{dg}{dx} = \frac{1}{x} \quad ; \quad \frac{d^2 g}{dx^2} = -\frac{1}{x^2}$$

$$\mu(x, t) = \mu x$$
$$\sigma(x, t) = \sigma x$$

$$dY_t = \left(\frac{1}{\cancel{S_t}} \mu \cancel{S_t} + \frac{1}{2} \left(\frac{-1}{\cancel{S_t^2}} \right) \sigma^2 \cancel{S_t^2} \right) dt + \frac{1}{\cancel{S_t}} (\sigma \cancel{S_t}) dW_t$$

$$= \underbrace{\left(\mu - \frac{1}{2}\sigma^2 \right)}_{\mu^*} dt + \sigma dW_t$$

$Y_t = \log S_t \qquad Y_t \sim N(\cdot, \cdot)$

$S_t = e^{Y_t} \qquad S_t \sim$ lognormal

Ques. 4 / Calculate $E\left(\int_0^{2\pi} W_s \, dW_s \right)$

$$\int_0^t W_s dW_s = \frac{1}{2}\left[W_t^2 - t \right]$$

$$\int_0^{2\pi} W_s \, dW_s = \frac{1}{2}\left[W_{2\pi}^2 - 2\pi \right]$$

$$E\left[\quad \right] = \frac{1}{2}\left[E(W_{2\pi}^2) - 2\pi \right]$$

$$= \frac{1}{2}(2\pi - 2\pi) = 0$$

~~Ex~~ example 5.4 imp.

Q. 5.1

(a) $X_t = c^{-1/2} W_{ct}$ for $c > 0$

$\begin{cases} W_0 = 0 \\ E(W_t) = 0 \\ W_t - W_s \text{ indep.} \\ \text{Normally distributed} \end{cases}$

$V(W_t - W_s) = t - s$

(unbounded)

$X_0 = 0$

$E(X_t) = 0$

$V(X_t) = c^{-1} V(W_{ct})$

$= \frac{1}{c}(ct) = t$

W_{ct} has ind. increments $W_{ct} \sim N(0, ct)$

$X_t - X_s \sim N(0, V(X_t - X_s))$

$t > s$

$$V(X_t - X_s) = V(X_t) + V(X_s) - cov(X_t, X_s)$$
$$= V(c^{-1/2} W_{ct} - c^{-1/2} W_{cs})$$
$$= c^{-1} V(W_{ct} - W_{cs})$$
$$= c^{-1}(ct - cs)$$
$$= (t - s)$$

(b) $Y_t = W_{T+t} - W_T$

$Y_0 = W_T - W_T = 0$

$E(Y_t) = E(W_{T+t} - W_T) = 0$

$V(Y_t) = V(W_{T+t} - W_T)$

$= T + t - T = t$

$V(Y_t) = t$

$Y_t \sim N(0, t)$

$$V(Y_t - Y_s) = V(W_{T+t} - \cancel{W_T} - W_{T+s} + \cancel{W_T})$$
$$= V(W_{T+t} - W_{T+s})$$
$$= T + t - (T + s) = t - s$$