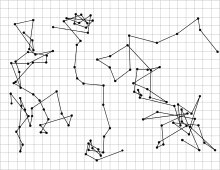
WIKIPEDIA:

This motion is named after the [botanist](https://en.wikipedia.org/wiki/Botany) [Robert Brown](https://en.wikipedia.org/wiki/Robert_Brown_(botanist,_born_1773)), who first described the phenomenon in [1827](https://en.wikipedia.org/wiki/1827), while looking through a [microscope](https://en.wikipedia.org/wiki/Microscope) at [pollen](https://en.wikipedia.org/wiki/Pollen) of the [plant](https://en.wikipedia.org/wiki/Plant) [*Clarkia pulchella*](https://en.wikipedia.org/wiki/Clarkia_pulchella) immersed in water. In 1905, almost eighty years later, [theoretical physicist](https://en.wikipedia.org/wiki/List_of_theoretical_physicists) [Albert Einstein](https://en.wikipedia.org/wiki/Albert_Einstein) published [a paper](https://en.wikipedia.org/wiki/%C3%9Cber_die_von_der_molekularkinetischen_Theorie_der_W%C3%A4rme_geforderte_Bewegung_von_in_ruhenden_Fl%C3%BCssigkeiten_suspendierten_Teilchen) where he modeled the motion of the pollen as being moved by individual water molecules, making one of his first big contributions to science.[[](https://en.wikipedia.org/wiki/Brownian_motion#cite_note-3)

**History**

The Roman philosopher [Lucretius](https://en.wikipedia.org/wiki/Lucretius)' scientific poem "[On the Nature of Things](https://en.wikipedia.org/wiki/On_the_Nature_of_Things)" (c. 60 BC) has a remarkable description of Brownian motion of [dust](https://en.wikipedia.org/wiki/Dust) particles in verses 113–140 from Book II. He uses this as a proof of the existence of atoms:

[](https://en.wikipedia.org/wiki/File:PerrinPlot2.svg)

Reproduced from the book of [Jean Baptiste Perrin](https://en.wikipedia.org/wiki/Jean_Baptiste_Perrin), *Les Atomes*, three tracings of the motion of colloidal particles of radius 0.53 µm, as seen under the microscope, are displayed. Successive positions every 30 seconds are joined by straight line segments (the mesh size is 3.2 µm).[[5]](https://en.wikipedia.org/wiki/Brownian_motion#cite_note-5)

"Observe what happens when sunbeams are admitted into a building and shed light on its shadowy places. You will see a multitude of tiny particles mingling in a multitude of ways... their dancing is an actual indication of underlying movements of matter that are hidden from our sight... It originates with the atoms which move of themselves [i.e., spontaneously]. Then those small compound bodies that are least removed from the impetus of the atoms are set in motion by the impact of their invisible blows and in turn cannon against slightly larger bodies. So the movement mounts up from the atoms and gradually emerges to the level of our senses so that those bodies are in motion that we see in sunbeams, moved by blows that remain invisible."

Although the mingling motion of dust particles is caused largely by air currents, the glittering, tumbling motion of small dust particles is, indeed, caused chiefly by true Brownian dynamics.

While [Jan Ingenhousz](https://en.wikipedia.org/wiki/Jan_Ingenhousz) described the irregular motion of [coal](https://en.wikipedia.org/wiki/Coal) [dust](https://en.wikipedia.org/wiki/Dust) particles on the surface of [alcohol](https://en.wikipedia.org/wiki/Ethanol) in 1785, the discovery of this phenomenon is often credited to the botanist [Robert Brown](https://en.wikipedia.org/wiki/Robert_Brown_(botanist,_born_1773)) in 1827. Brown was studying [pollen](https://en.wikipedia.org/wiki/Pollen) grains of the plant [*Clarkia pulchella*](https://en.wikipedia.org/wiki/Clarkia_pulchella) suspended in water under a microscope when he observed minute particles, ejected by the pollen grains, executing a jittery motion. By repeating the experiment with particles of inorganic matter he was able to rule out that the motion was life-related, although its origin was yet to be explained.

The first person to describe the mathematics behind Brownian motion was [Thorvald N. Thiele](https://en.wikipedia.org/wiki/Thorvald_N._Thiele) in a paper on the method of [least squares](https://en.wikipedia.org/wiki/Least_squares) published in 1880. This was followed independently by [Louis Bachelier](https://en.wikipedia.org/wiki/Louis_Bachelier) in 1900 in his PhD thesis "The theory of speculation", in which he presented a stochastic analysis of the stock and option markets. The Brownian motion model of the [stock market](https://en.wikipedia.org/wiki/Stock_market) is often cited, but [Benoit Mandelbrot](https://en.wikipedia.org/wiki/Benoit_Mandelbrot) rejected its applicability to stock price movements in part because these are discontinuous.[[6]](https://en.wikipedia.org/wiki/Brownian_motion#cite_note-6)

[Albert Einstein](https://en.wikipedia.org/wiki/Albert_Einstein) (in one of his [1905 papers](https://en.wikipedia.org/wiki/%C3%9Cber_die_von_der_molekularkinetischen_Theorie_der_W%C3%A4rme_geforderte_Bewegung_von_in_ruhenden_Fl%C3%BCssigkeiten_suspendierten_Teilchen)) and [Marian Smoluchowski](https://en.wikipedia.org/wiki/Marian_Smoluchowski) (1906) brought the solution of the problem to the attention of physicists, and presented it as a way to indirectly confirm the existence of atoms and molecules. Their equations describing Brownian motion were subsequently verified by the experimental work of [Jean Baptiste Perrin](https://en.wikipedia.org/wiki/Jean_Baptiste_Perrin) in 1908.

MARTINGALE

[*martingale*](https://en.wikipedia.org/wiki/Martingale_(betting_system)) referred to a class of [betting strategies](https://en.wikipedia.org/wiki/Betting_strategy) that was popular in 18th-century [France](https://en.wikipedia.org/wiki/France).[[1]](https://en.wikipedia.org/wiki/Martingale_(probability_theory)#cite_note-1)[[2]](https://en.wikipedia.org/wiki/Martingale_(probability_theory)#cite_note-2) The simplest of these strategies was designed for a game in which the [gambler](https://en.wikipedia.org/wiki/Gambler) wins their stake if a coin comes up heads and loses it if the coin comes up tails. The strategy had the gambler double their bet after every loss so that the first win would recover all previous losses plus win a profit equal to the original stake. As the gambler's wealth and available time jointly approach infinity, their probability of eventually flipping heads approaches 1, which makes the martingale betting strategy seem like a [sure thing](https://en.wikipedia.org/wiki/Almost_surely). However, the [exponential growth](https://en.wikipedia.org/wiki/Exponential_growth) of the bets eventually bankrupts its users due to finite bankrolls. [Stopped Brownian motion](https://en.wikipedia.org/wiki/Stopped_process#Brownian_motion), which is a martingale process, can be used to model the trajectory of such games.

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Translation invariance. {Bt − B0, t \_ 0} is independent of B0 and has the same

distribution as a Brownian motion with B0 = 0.

The Brownian scaling relation. If B0 = 0 then for any t > 0,

{Bst, s \_ 0} d=

{t1/2Bs, s \_ 0}

A second equivalent definition of Brownian motion starting from B0 = 0, that we

will occasionally find useful is that Bt, t \_ 0, is a real-valued process satisfying

(a0) B(t) is a Gaussian process (i.e., all its finite dimensional distributions are

multivariate normal).

(b0) EBs = 0 and EBsBt = s ^ t.

(c0) With probability one, t ! Bt is continuous.

L2 weak law. Let X1,X2, . . . be uncorrelated random variables

with EXi = μ and var (Xi) \_ C < 1. If Sn = X1 + . . . + Xn then as n ! 1,

Sn/n ! μ in L2 and in probability.

If An is a sequence of subsets of , we let

lim supAn = lim

m!1

[1n=mAn = {! that are in infinitely many An}

(the limit exists since the sequence is decreasing in m) and let

lim inf An = lim

m!1

\1n=mAn = {! that are in all but finitely many An}

Motivation

Much of probability theory is devoted to describing the *macroscopic picture* emerging in random

systems de¯ned by a host of *microscopic random e®ects*. Brownian motion is the macroscopic

picture emerging from a particle moving randomly in *d*-dimensional space. On the microscopic

level, at any time step, the particle receives a random displacement, caused for example by

other particles hitting it or by an external force, so that, if its position at time zero is *S*0, its

position at time *n* is given as *Sn* = *S*0 +

P*n*

*i*=1 *Xi;* where the displacements *X*1*;X*2*;X*3*; : : :* are

assumed to be independent, identically distributed random variables with values in R*d*. The

process *fSn* : *n ¸* 0*g* is a random walk, the displacements represent the microscopic inputs.

When we think about the macroscopic picture, what we mean is questions such as:

*²* Does *Sn* drift to in¯nity?

*²* Does *Sn* return to the neighbourhood of the origin in¯nitely often?

*²* What is the speed of growth of max*fjS*1*j; : : : ; jSnjg* as *n ! 1*?

*²* What is the asymptotic number of windings of *fSn* : *n ¸* 0*g* around the origin?

It turns out that not all the features of the microscopic inputs contribute to the macroscopic

picture. Indeed, if they exist, only the *mean* and *covariance* of the displacements are shaping

the picture. In other words, all random walks whose displacements have the same mean and

covariance matrix give rise to same macroscopic process, and even the assumption that the

displacements have to be independent and identically distributed can be substantially relaxed.

This e®ect is called *universality*, and the macroscopic process is often called a *universal object*.

It is a common approach in probability to study various phenomena through the associated

universal objects.

Any continuous time stochastic process *fB*(*t*) : *t ¸* 0*g* describing the macroscopic features of a

random walk should have the following properties:

(1) for all times 0 *· t*1 *· t*2 *· : : : · tn* the random variables

*B*(*tn*) *¡ B*(*tn¡*1)*; B*(*tn¡*1) *¡ B*(*tn¡*2)*; : : : ; B*(*t*2) *¡ B*(*t*1)

are independent; we say that the process has *independent increments*,

(2) the distribution of the increment *B*(*t* + *h*) *¡ B*(*t*) does not depend on *t*; we say that

the process has *stationary increments*,

(3) the process *fB*(*t*) : *t ¸* 0*g* has almost surely continuous paths.

It follows (with some work) from the central limit theorem that these features imply that there

exists a vector *¹ 2* R*d* and a matrix § *2* R*d£d* such that

(4) for every *t ¸* 0 and *h ¸* 0 the increment *B*(*t* + *h*) *¡ B*(*t*) is multivariate normally

distributed with mean *h¹* and covariance matrix *h*§§T.

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Hence any process with the features (1)-(3) above is characterised by just three parameters,

*²* the *initial distribution*, i.e. the law of *B*(0),

*²* the *drift vector ¹*,

*²* the *di®usion matrix* §.

We call the process *fB*(*t*) : *t ¸* 0*g* a *Brownian motion* if the drift vector is zero, and the

di®usion matrix is the identity. If *B*(0) = 0, i.e. the motion is started at the origin, we use the

term *standard Brownian motion*.

Suppose we have a standard Brownian motion *fB*(*t*) : *t ¸* 0*g*. If *X* is a random variable with

values in R*d*, *¹* a vector in R*d* and § a *d£d* matrix, then it is easy to check that *f*~*B*

(*t*) : *t ¸* 0*g*

given by

~*B*

(*t*) = ~*B*

(0) + *¹t* + §*B*(*t*)*;* for *t ¸* 0*;*

is a process with the properties (1)-(4) with initial distribution *X*, drift vector *¹* and di®usion

matrix §. Hence the macroscopic picture emerging from a random walk can be fully described

by a standard Brownian motion.

0 50 100 150 200

−140

−120

−100

−80

−60

−40

−20

0

Figure 1. The range *fB*(*t*) : 0 *· t ·* 1*g* of a planar Brownian motion

In *Chapter 1* we start exploring Brownian motion by looking at dimension *d* = 1. Here Brownian

motion is a random continuous function and we ask about its *regularity*, for example:

*²* For which parameters *®* is the random function *B*: [0*;* 1] *!* R *®*-HÄolder continuous?

*²* Is the random function *B*: [0*;* 1] *!* R di®erentiable?

The surprising answer to the second question was given by Paley, Wiener and Zygmund in 1933:

Almost surely, the random function *B*: [0*;* 1] *!* R is *nowhere* di®erentiable! This is particularly

interesting, as it is not easy to construct a continuous, nowhere di®erentiable function without

the help of randomness. We will give a modern proof of the Paley, Wiener and Zygmund

theorem, see Theorem 1.30.

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In *Chapter 2* we move to general dimension *d*. We shall explore the strong Markov property,

which roughly says that at suitable random times Brownian motion starts afresh. Among the

facts we derive are: Almost surely,

*²* the set of all points visited by Brownian motion in *d* = 2 has area zero,

*²* the set of times when Brownian motion in *d* = 1 revisits the origin is uncountable.

Besides these sample path properties, the strong Markov property is also the key to some

fascinating distributional identities. It enables us to understand, for example,

*²* the process *fM*(*t*) : *t ¸* 0*g* of the running maxima *M*(*t*) = max0*·s·t B*(*s*) of a one-

dimensional Brownian motion,

*²* the process *fTa* : *a ¸* 0*g* of the ¯rst hitting times *Ta* = inf*ft ¸* 0: *B*(*t*) = *ag* of level *a*

of a one-dimensional Brownian motion,

*²* the process of the vertical ¯rst hitting positions by a two-dimensional Brownian motion

of the lines *f*(*x; y*) *2* R2 : *x* = *ag*, as a function of *a*.

Properties::

cov( Ws , Wt )=min(s,t)

corr(Ws,Wt)= cov(Ws,Wt)/ sd(s).sd(t) = min(s,t)/root(st) =root( min(s,t)/max(s,t) )

<https://tex.stackexchange.com/questions/129088/create-a-frame-for-a-title-page>

Brownian motion is the random motion of the particles suspended in the fluid resulting from the collision of the particles.

**gauss markov process**

Every Gauss–Markov process *X*(*t*) possesses the three following properties:

1. If *h*(*t*) is a non-zero scalar function of *t*, then *Z*(*t*) = *h*(*t*)*X*(*t*) is also a Gauss–Markov process
2. If *f*(*t*) is a non-decreasing scalar function of *t*, then *Z*(*t*) = *X*(*f*(*t*)) is also a Gauss–Markov process
3. There exists a non-zero scalar function *h*(*t*) and a non-decreasing scalar function *f*(*t*) such that *X*(*t*) = *h*(*t*)*W*(*f*(*t*)), where *W*(*t*) is the standard [Wiener process](https://en.wikipedia.org/wiki/Wiener_process)

a **Gaussian process** is a [stochastic process](https://en.wikipedia.org/wiki/Stochastic_process) (a collection of random variables indexed by time or space), such that every finite collection of those random variables has a [multivariate normal distribution](https://en.wikipedia.org/wiki/Multivariate_normal_distribution), i.e. every finite [linear combination](https://en.wikipedia.org/wiki/Linear_combination) of them is normally distributed.

**Brownian motion**

A one-dimensional Brownian motion is a real-valued process Bt, t \_ 0 that has the

following properties:

(a) If t0 < t1 < . . . < tn then B(t0),B(t1) − B(t0), . . . ,B(tn) − B(tn−1) are independent.

(b) If s, t \_ 0 then P(B(s + t) − B(s) 2 A) =

Z (2pi t)−1/2 exp(−x2/2t) dx

(c) With probability one, t -> Bt is continuous.

Thinking of Brown’s pollen grain (c) is certainly reasonable. (a) and (b) can be

justified by noting that the movement of the pollen grain is due to the net effect of

the bombardment of millions of water molecules, so by the central limit theorem, the

displacement in any one interval should have a normal distribution, and the displacements

in two disjoint intervals should be independent.

The Brownian scaling relation. If B0 = 0 then for any t > 0,

{Bst, s \_ 0} d= {t1/2Bs, s \_ 0}

**Donsker’s Theorem**

the random walk can be weakly approximated

by Brownian motion.

**Brownian Bridge**

http://www.randomservices.org/random/brown/Bridge.html