ASSIGNMENT 1 – LINEAR ALGEBRA – MA1002

1. Given that

a) p is prime

b) ~~Field~~ $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ with arithmetic mod p.

(c) $\mathbb{Z}_p^n$ with coordinate addition and multiplication.

To check whether $(\mathbb{Z}_p^n, +, \cdot)$ is a field or not.

Consider $n = 1$,

$$\Rightarrow \mathbb{Z}_p^n = \mathbb{Z}_p$$

We already know that $\mathbb{Z}_p$ is a field when p is prime.

$\therefore \mathbb{Z}_p^n$ is a field for $n = 1$.

Now let us consider $n \geq 2$,

Take $u = (1, 0, 0, 0, \ldots, 0)$ and $v = (0, 1, 0, 0, \ldots, 0)$

We can observe that u and v are both elements of $\mathbb{Z}_p^n$ and are non-zero. Also,

$$u \cdot v = (1\cdot 0, 0 \cdot 1, 0, 0, \ldots, 0)$$

$$\Rightarrow u \cdot v = (0, 0, \ldots, 0)$$

We know that for a field if $a, b \in \mathbb{F}$, and $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

But, that is not the case here, both u and v are non-zero but their product is 0.

$\therefore \mathbb{Z}_p^n$ is not a field for $n \geq 2$.

So, $(\mathbb{Z}_p^n, +, \cdot)$ is a field if and only if $n = 1$.

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2 (a) The converse statement would be: Suppose two systems have the same solution set, then they are row-equivalent.

Consider System 1:

$$x+y=1$$
$$\cancel{0 \cdot x = 1}$$
$$x+y=0$$
$\Rightarrow$ no solution

Consider system 2:

$$2y+x=0,$$
$$2y+x=1$$
$\Rightarrow$ no solution.

The solution set for both systems is ϕ, but they are not row-equivalent as their RREF matrices are different.

$\therefore$ The converse is not true.

(b) Given:

→ $\mathbb{F}^{m \times n}$ is the set of all $m \times n$ matrices over a field $\mathbb{F}$

→ $A \sim B \Leftrightarrow A$ and B are row-equivalent.

To prove:

$\sim$ is an equivalence relation.

Proof:

An equivalence relation must satisfy reflexivity, symmetry and transitivity.

1. Reflexivity:

- Any matrix A is row-equivalent to itself (no row operations).
- Thus $A \sim A$.

2. Symmetry:

② • If $A \sim B$, then B is obtained from A by a sequence of operations.

- We know each row operation is reversible:

Swap can be swapped back, addition can be subtracted, multiplication by reciprocal.

- Hence we can reverse the sequence of operations to obtain A from B.

- Therefore, $A \sim B \Rightarrow B \sim A$.

3. Transitivity:

- If $A \sim B$ and $B \sim C$, by some sequence of operation we can obtain. B from A and another sequence to obtain C from B.

- Combining these sequences, we can obtain C from A.

- $(A \sim B) \wedge (B \sim C) \Rightarrow A \sim C$.

Hence, $\sim$ is equivalence relation.

3. Given the following system of linear equations:

$$x_1 - 2x_2 - x_5 = 3,$$
$$2x_2 - x_5 = -3,$$
$$2x_1 - 3x_2 + x_3 - 3x_4 + x_5 = -5.$$

(a) The augmented matrix is:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & -1 & 3 \\ 0 & 2 & 0 & 0 & -1 & -3 \\ 2 & -3 & 1 & -3 & 1 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & -1 & -3 \\ 0 & 1 & 1 & -3 & 3 & -11 \end{array}\right] \quad (R_1 \leftarrow R_1 + R_2,\ R_3 \leftarrow R_3 + (-2)R_1)$$

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$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & -3/2 \\ 0 & 1 & 1 & -3 & 3 & -11 \end{array}\right] \qquad (R_2 \leftarrow (\tfrac{1}{2})R_2)$$

$$\cancel{\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 7/2 & -3/2 \\ 0 & 0 & 1 & -3 & 9/2 & -29/2 \end{array}\right]} \qquad \cancel{R_3}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & -3/2 \\ 0 & 0 & 1 & -3 & 7/2 & -19/2 \end{array}\right]$$

This is the row-reduced augmented matrix.

(b) Pivot columns : 1, 2, 3 $\rightarrow$ variables x_1, x_2, x_3 are dependent.

Free variables : x_4, x_5

Let $x_4 = s$, $x_5 = t$ with $s, t \in \mathbb{R}$

(c) From row 1: $x_1 - 2x_5 = 0$

$\Rightarrow x_1 = 2t.$

From row 2:

$$x_2 - \frac{x_5}{2} = \frac{-3}{2}$$

$$\Rightarrow x_2 = -\frac{3}{2} + \frac{1}{2}t.$$

From row 3.

$$x_3 - 3x_4 + \frac{7}{2}x_5 = \frac{-19}{2}$$

$$\Rightarrow x_3 = \frac{-19}{2} + 3s - \frac{7t}{2}.$$

$\therefore x_1 = 2t \ , \ x_2 = \frac{-3}{2} + \frac{1}{2}t \ , \ x_3 = -\frac{19}{2} + 3s - \frac{7}{2}t \ , \ x_4 = s \ , \ x_5 = t.$

$\Rightarrow$ Solution set $S = \left\{ \left(2t, \frac{-3}{2} + \frac{1}{2}t, \frac{-19}{2} + 3s - \frac{7}{2}t, s, t\right) \mid s, t \in \mathbb{R} \right\}$

$$S = \left\{ \begin{array}{l} (0, -\frac{3}{2}, \frac{-19}{2}, 0, 0) \\ + (0, 0, 3, 1, 0)\, s \\ + (2, \frac{1}{2}, \frac{-7}{2}, 0, 1)\, t \end{array} \ \middle| \ s, t \in \mathbb{R} \right\}$$

4. The given system is

$$\begin{aligned} x + y + z + w &= 4 \\ x + 2y + 3z + 4w &= 3 \\ x + y + 2z &= 6 \\ y + 9z + aw &= b \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 1 & 1 & 2 & 0 & 6 \\ 0 & 1 & 9 & a & b \end{array}\right]$$

(a) $R_2 \leftarrow R_2 - R_1$
$R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 9 & a & b \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 7 & a-3 & b+1 \end{array}\right] \quad (R_4 \leftarrow R_4 - R_2)$$

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$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & a+4 & b-13 \end{array}\right] \quad (R_4 \leftarrow R_4 - 7R_3).$$

Now we perform backtracking:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{7a+3b+11}{a+4} \\ 0 & 1 & 0 & 0 & \frac{-5a-5b+45}{a+4} \\ 0 & 0 & 1 & 0 & \frac{2a+b-5}{a+4} \\ 0 & 0 & 0 & 1 & \frac{b-13}{a+4} \end{array}\right]$$

This is the RREF augmented matrix, where $a \neq -4$.

(b) <u>Case 1:</u>

$a \neq -4$

$\Rightarrow$ Pivot in every column $\rightarrow$ unique solution.

<u>Case 2:</u>

$a = -4$.

If $b \neq 13$, row 4 becomes $[0\,0\,0\,0 \mid \neq 0]$

Which is inconsistent $\Rightarrow$ No solution.

If $b = 13$, row 4 is all zeros $\Rightarrow$ rank $= 3$, unknown $= 4$.

$\Rightarrow$ Infinite solutions (1 free variable).

(i) No solution when, $a = -4$ and $b \neq 13$.

(ii) Unique solution when $a \neq -4$.

(iii) Infinite solution when $a = -4$ and $b = 13$.

(c) When $a = -4$, we have unique solution for the system and the solution is

$$x = \frac{7a + 3b - 11}{a+4}$$

$$y = \frac{-5a - 5b + 45}{a+4}$$

$$z = \frac{2a + b - 5}{a+4}$$

$$w = \frac{b-13}{a+4}$$

When $a = -4,\ b = 13$ (Infinite solutions).

The augmented matrix can be written as:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 + 3\frac{b-13}{a+4} \\ 0 & 1 & 0 & 0 & -5 - 5\frac{b-13}{a+4} \\ 0 & 0 & 1 & 0 & 2 + \frac{b-13}{a+4} \\ 0 & 0 & 0 & 1 & \frac{b-13}{a+4} \end{array}\right]$$

Let us say $w = t$, then the matrix becomes:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7+3t \\ 0 & 1 & 0 & 0 & -5-5t \\ 0 & 0 & 1 & 0 & 2+t \\ 0 & 0 & 0 & 1 & t \end{array}\right]$$

$\therefore$ The solution is $x = 7+3t,\ y = -5-5t,\ z = 2+t,\ w = t$.

Solution set, $S = \{(7+3t, -5-5t, 2+t, t) \mid t \in \mathbb{R}\}$.

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5. Given that

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

Let us consider the follow row operations:

$e_1 : R_3 \leftarrow R_3 + (-1) R_2$

$e_2 : R_2 \leftarrow R_2 + (-2) R_3$

$e_3 : R_1 \leftarrow R_1 + (-1) R_2$

$e_4 : R_1 \leftarrow R_1 + (-1) R_3$

$e_5 : R_1 \leftarrow (-1) R_1$

$e_6 : R_1 \leftrightarrow R_3$

$e_7 : R_2 \leftrightarrow R_3$

$$e_7(e_6(e_5(e_4(e_3(e_2(e_1(A))))))) = I_3 \quad \text{—— (1)}$$

We know that $e(I) A = e(A)$

So, eq. (1) becomes:

$$e_7(I)\, e_6(I)\, e_5(I)\, e_4(I)\, e_3(I)\, e_2(I)\, e_1(I)\, A = I_3$$

$$\Rightarrow E_1 = e_1(I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_2 = e_2(I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = e_3(I) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = e_4(I) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = e_5(I) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_6 = e_6(I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_7 = e_7(I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

6. Given that

$\mathbb{R}[x]$ be the set of all polynomials over the field $\mathbb{R}$.

Addition and multiplication are the usual polynomial operations:

$$\left(\sum_{k=0}^{n} a_k x^k\right) + \left(\sum_{k=0}^{m} b_k x^k\right) = \sum_{k=0}^{\max(m,n)} (a_k + b_k) x^k$$

$$\left(\sum_{k=0}^{n} a_k x^k\right) \cdot \left(\sum_{k=0}^{m} b_k x^k\right) = \sum_{j=0}^{n+m} \left(\sum_{k=0}^{j} a_k b_{j-k}\right) x^j .$$

- Polynomial addition and multiplication satisfies closure.
- Polynomial addition and multiplication satisfies Associativity and commutativity.

$$(f+g)+h = f+(g+h) \qquad f+g = g+f$$
$$(fg)h = f(gh) \qquad fg = gf$$

- Additive identity is $O(x) = 0$ and multiplicative identity is $P(x) = 1$.

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• Distributive property is also satisfied. $f(g+h) = fg + fh$.

Now let us see for existence of inverses:

Additive inverse is satisfied because for every polynomial f, $-f$ is also a polynomial, $(-f) \in \mathbb{R}[x]$ and $f + (-f) = 0$.

Take $f(x) = x \in \mathbb{R}[x]$

If $\mathbb{R}[x]$ were a field, there must exist some polynomial $g(x) \in \mathbb{R}[x]$ such that

$$f(x)\, g(x) = 1$$

$$\Rightarrow x\, g(x) = 1$$

But LHS has a degree ≥ 1 but RHS only has degree 0. Which is a contradiction.

$\therefore$ Non-constant polynomials have no multiplicative inverse in $\mathbb{R}[x]$

$\therefore$ $\mathbb{R}[x]$ is not a field.

7. Given the row reduced echelon form of matrix A

$$R = \begin{bmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We are also told that the 1st, 3rd and 6th columns of A are

$$A_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix} \quad \text{and} \quad A_6 = \begin{bmatrix} 3 \\ -9 \\ 2 \\ 5 \end{bmatrix}$$

From R, pivot columns are 1, 3 and 5.

non-pivot columns are 2, 4 and 6.

So, columns 2, 4 and 6 are linear combinations of pivot columns.

So, by reading the linear relations from R.

$$\text{Column 2 of } R = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -3R_1$$

$$\text{Column 4 of } R = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} = 4R_1 + 3R_2$$

$$\text{Column 6 of } R = \begin{bmatrix} 5 \\ 2 \\ -1 \\ 0 \end{bmatrix} = 5R_1 + 2R_2 - R_3$$

Where R_1, R_2, R_3 are the respective columns of R.

Thus in general:

$$A_2 = -3A_1, \quad A_4 = 4A_1 + 3A_3 \text{ and } A_6 = 5A_1 + 2A_3 - A_5$$

$$\Rightarrow A_2 = -3A_1 = -3\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 3 \\ -9 \end{bmatrix}$$

$$\Rightarrow A_4 = 4A_1 + 3A_3 = 4\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow A_5 = 5A_1 + 2A_3 - A_6 = 5\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 2 \end{bmatrix}$$

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Now we can construct full matrix A.

$$A = \begin{bmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{bmatrix}$$

8. (a) 1. Let $x \in U_1$. Since U_2 is a subspace of V, we have $0 \in W_2$. By definition the vector $x = x + 0$ is contained in $W_1 + W_2$. Similary let $y \in U_2$ and since U_1 is a subspace of V, $0 \in U_1$ and thus $y = y + 0$ (by definition) is contained in $U_1 + U_2$.

Since, $0 \in U_1$ and $0 \in U_2$, $0 + 0 = 0$ is contained in $U_1 + U_2$. Let $x_1 + y_1$ and $x_2 + y_2$ be elements of $U_1 + U_2$ with $x_1, x_2 \in U_1$ and $y_1, y_2 \in U_2$. Let $\lambda \in F$, Since U_1 and U_2 are subspaces we have:

$$\lambda x_1 \in W_1 \quad , \quad x_1 + x_2 \in U_1$$
$$\lambda y_1 \in U_2 \quad , \quad y_1 + y_2 \in U_2 .$$

$\Rightarrow \lambda(x_1 + y_1) = \lambda x_1 + \lambda y \in U_1 + U_2$ and $x_1 + y_1 + x_2 + y_2$ $= (x_1 + x_2) + (y_1 + y_2) \in U_1 + U_2$. This shows that $U_1 + U_2$ is a subspace of V, containing U_1 and U_2.

2. Let $W \subseteq V$ be a subspace such that $W_1 \subseteq W$ and $W_2 \subseteq W$. Take any $u + v \subseteq W_1 + W_2$ with $u \in W_1$ and $v \in W_2$. Since $u, v \in W$ and W is subspace, it is closed under addition.

$\Rightarrow u + v \in W$

So, $W_1 + W_2 \subseteq W$

Hence, proved.

(b) To prove:

$$\text{Span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$$

Take any $x \in \text{Span}(S_1 \cup S_2)$

Then x is linear combination of vectors from $S_1 \cup S_2$, ie

$$x = a_1 u_1 + \dots + a_m u_m + b_1 v_1 + \dots + b_n v_n,$$

where $u_i \in S_1$ and $v_i \in S_2$.

So, $x = \underbrace{(a_1 v_1 + \dots + a_m u_m)}_{\in \text{span}(S_1)} + \underbrace{(b_1 v_1 + \dots + b_n v_n)}_{\in \text{span}(S_2)}$

$\Rightarrow x \in \text{span}(S_1) + \text{span}(S_2)$

Hence, $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$ —— (1)

Let $x \in \text{span}(S_1) + \text{span}(S_2)$

Then $x = u + v$ where $u \in \text{span}(S_1)$, $v \in \text{Span}(S_2)$

but u is a linear combination of elements of S_1 and v is a linear combination of elements of S_2.

Hence, x is a linear combination of elements of $S_1 \cup S_2$.

So, $x \in \text{span}(S_1 \cup S_2)$

$\Rightarrow \text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$ —— (2)

From (1) and (2), $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$

Hence, proved.

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9 Since $W_i \subseteq M_{m \times n}(F)$, to show that it is a subspace we need to only prove that its closed under addition and scalar multiplication.

(a) If $A, B \in W_1$, then for $i > j$, $A_{ij} = B_{ij} = 0$,

So $(A+B)_{ij} = 0 \quad \forall\ i > j. \Rightarrow A+B \in W_1$

Hence its closed under addition.

For $\alpha \in F$, if $A_{ij} = 0$ for $i > j$, then $\alpha A_{ij} = 0$,

since $\alpha \cdot 0 = 0$

So, $\alpha A_{ij} = 0 \quad \forall\ i > j. \quad \Rightarrow \alpha A \in W_1$

Hence it is closed under scalar multiplication too.

And the 0 matrix is contained in W_1.

$\therefore W_1$ is a subspace. Hence, proved.

(b) Since $W_1 \subseteq M_{m \times n}(F)$, to show that it is a subspace, we need to only proove that its closed under addition and scalar multiplication.

If $A, B \in W_1$, then for $i \leq j$, $A_{ij} = B_{ij} = 0$,

So, $(A+B)_{ij} = 0$ for $i \leq j. \Rightarrow (A+B) \in W_2$

Hence, it is closed under addition.

For $\alpha \in F$, if $A_{ij} = 0 \ \forall\ i \leq j$. then $\alpha A_{ij} = 0$, since $\alpha . 0 = 0$

So, $\alpha A_{ij} = 0 \quad \forall\ i \leq j \quad \Rightarrow \alpha A \in W_2$

Hence, it is closed under scalar multiplication too.

0 matrix $\in W_2$.

$\therefore W_2$ is a subspace. Hence, prooved.

(c) $W_3 = \{A : A^T = A\}$

0 is symmetric matrix $\Rightarrow 0 \in W_3$.

If $A^T = A$ and $B^T = B$, and $A, B \in W_2$.

then $(A+B)^T = A^T + B^T = A + B$.

$\Rightarrow (A+B)^T = A+B$, So, $A+B \in W_2$.

Hence, closed under addition.

For $\alpha \in F$, $(\alpha A)^T = \alpha A^T = \alpha A$, where $A \in W_3$
$\Rightarrow A^T = A$.

So, $\alpha A \in W_3$

Hence, closed under scalar multiplication too.

$\therefore$ W_3 is a subspace. Hence, proved.

(d) $W_4 = \{A : A^T = -A\}$

0 matrix satisfies the condition $0^T = -0$, so, $0 \in W_4$.

If $A, B \in W_4 \Rightarrow A^T = -A$ and $B^T = -B$.

Then $(A+B)^T = A^T + B^T = -A - B = -(A+B)$

$\therefore$ $A+B \in W_4$

So, it is closed under addition.

For $\alpha \in F$, $(\alpha A)^T = \alpha A^T = -(\alpha A)$ where $A \in W_4$.

$\therefore$ $\alpha A \in W_4$

So, it is also closed under scalar multiplication.

$\therefore$ W_4 is a subspace. Hence, proved.

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(a) $M_{m\times n}(F) = W_1 \oplus W_2$

Every matrix $A \in M_{m\times n}(F)$ can be written as

$A = U + L$ (sum of ~~y~~ upper triangle and lower triangle)

where $U \in W_1$ (upper triangular part)

and $L \in W_2$ (strictly lower triangular part)

Specifically $U_{ij} = A_{ij}$ if $i \leq j$, else 0.

$L_{ij} = A_{ij}$ if $i > j$; else 0.

Then $A = U + L$.

Intersection : If $A \in W_1 \cap W_2$, then.

$A_{ij} = 0$ for $i > j$ (from W_1) and $A_{ij} = 0$ for $i < j$ (from W_2), thus ~~A~~ $A = 0$.

$\Rightarrow W_1 \cap W_2 = \{0\}$.

So, $M_{m\times n}(F) = W_1 \oplus W_2$.

(b) Any square matrix A can be decomposed as sum of symmetric and ~~sky~~ skew-symmetric matrix.

$$A = \underbrace{\tfrac{1}{2}(A + A^T)}_{\text{Symmetric}} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\text{Skew-symmetric}}$$

$\Rightarrow \frac{1}{2}(A + A^T) \in W_3$ and $\frac{1}{2}(A - A^T) \in W_4$

If $A \in W_3 \cap W_4$, then $A^T = A$ and $A^T = -A$

$\Rightarrow A = -A \Rightarrow A = 0$

$\Rightarrow W_3 \cap W_4 = \{0\}$ So, ~~...~~ $M_{m\times n}(F) = W_3 \oplus W_4$.

(c) For $A \in W_2$, $A_{ij} = 0 \ \forall \ i \leq j$ and $B \in W_3 \Rightarrow B^T \in W_3$

$$\Rightarrow B_{ij} = B_{ji}$$

$$(A+B)_{ij} = \begin{cases} B_{ij}, & i \leq j \\ A_{ij} + B_{ij}, & i > j \end{cases}$$

So, for the upper triangle we have freedom by choosing appropriate B_{ij} and for lower triangle we can choose appropriate A_{ij} that depend on the B_{ij} previously chosen. And thus any matrix can be formed.

And a matrix that is ~~both~~ both a lower triangle and symmetric is only 0 matrix.

$$\Rightarrow W_2 \cap W_3 = \{0\}$$

$$\therefore M_{m \times n}(F) = W_2 \oplus W_3.$$

10. (a) Suppose $v + W$ is a subspace of V, then it must contain the zero vector.

$$0 \in v + W$$

and $\exists \, w \in W$ such that

$$v + w = 0$$

$$\Rightarrow v = -w$$

But $-w \in W$ (subspace)

$$\Rightarrow v \in W$$

$\therefore$ If $v + W$ is a subspace of V, then $v \in W$.

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If $v \in W$, then

$$v + W = \{v + w : w \in W\} = W$$

(because W is closed under addition)

and since W is a subspace, $v + W$ is a subspace.

$\therefore$ If $v \in W$, $v + W$ is a subspace.

$\therefore$ $v + W$ is a subspace iff $v \in W$.

(b) Suppose $v_1 + W = v_2 + W$.

$$\Rightarrow v_1 \in v_2 + W$$

So, $\exists\, w \in W$ such that

$$v_1 = v_2 + w \quad \Rightarrow \quad w = v_1 - v_2 \in W$$

$\Rightarrow$ If $v_1 + W = v_2 + W$, then $v_1 - v_2 \in W$

Suppose $v_1 - v_2 \in W$. Let $v_1 - v_2 = w \in W$

For any $x \in v_1 + W$, we can write $x = v_1 + w_1$. Then

$$x = (v_2 + w) + w_1 = v_2 + (w + w_1)$$

Since W is closed under addition $\Rightarrow w + w_1 \in W$

$$\Rightarrow x \in v_2 + W$$

Since x is arbitrary and $x \in v_2 + W$ and $x \in v_1 + W$

$$\Rightarrow v_2 + W = v_1 + W$$

$\therefore$ If $v_1 - v_2 \in W$, then $v_2 + W = v_1 + W$

$\therefore$ $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$

(c) We define operations on $S = \{v + W : v \in V\}$

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W,$$

$$a(v + W) = (av) + W.$$

Closure is satisfied for addition and scalar multiplication.

$$((v_1 + W) + (v_2 + W)) + (v_3 + W) = (v_1 + v_2 + v_3) + W$$

$$= (v_1 + W) + ((v_2 + W) + (v_3 + W))$$

$\therefore$ Associativity for addition holds.

There is a zero element, ie, $0 + W$

$\Rightarrow$ For any $v + W$,

$$(v + W) + (0 + W) = (v + 0) + W = v + W$$

For every $(v + W)$ there is inverse $(-v + W)$ such that $(v + W) + (-v + W) = 0 + W$.

$\rightarrow$ Multiplicative axioms:

1. $1 \cdot (v + W) = v + W$

2. $(c_1 c_2)(v + W) = c_1 c_2 v + W$
$= c_1 (c_2 (v + W))$

3. $c[(v_1 + W) + (v_2 + W)] = c(v_1 + W) + c(v_2 + W)$

4. $(c_1 + c_2)(v + W) = (c_1 + c_2) v + W$
$= c_1 v + c_2 v + W$
$= c_1 v + W + c_2 v + W$
$= c_1 (v + W) + c_2 (v + W)$

Hence, all axioms are satisfied.

$\Rightarrow$ The collection $S = \{v + W : v \in V\}$ of all cosets of W forms a vector space.

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11. We are given $V = \{\langle x_n \rangle \mid x_n \in \mathbb{R}, \langle x_n \rangle \text{ converges in } \mathbb{R}\}$

Addition : $\langle x_n \rangle + \langle y_n \rangle = \langle x_n + y_n \rangle$

Scalar multiplication : $c \cdot \langle x_n \rangle = \langle c x_n \rangle$

- Zero element : The zero sequence $\langle 0, 0, \dots \rangle$ is in V.
- Closure under addition : If $x_n \to a$ and $y_n \to b$, then $x_n + y_n \to a+b$. So sum is convergent.
- Closure under scalar multiplication : If $x_n \to a$, then $cx_n \to ca$. So scalar multiples are convergent.
- Other axioms (associativity, distributivity, etc.) follow from real sequences.

So, $(V, \mathbb{R}, +, \cdot)$ is indeed a vector space.

For $\langle x_n \rangle \in V$ with limit $a = \lim_{n\to\infty} x_n$,

$$T(\langle x_n \rangle) = \langle y_n \rangle, \quad y_n = a - x_n .$$

(a) Take sequences $\langle x_n \rangle, \langle y_n \rangle \in V$. Let

$$\lim x_n = a, \quad \lim y_n = b$$

Then :

$$T(\alpha \langle x_n \rangle + \beta \langle y_n \rangle) = T(\langle \alpha x_n + \beta y_n \rangle)$$

$$\lim (\alpha x_n + \beta y_n) = \alpha a + \beta b$$

So, $T(\langle \alpha x_n + \beta y_n \rangle) = \langle (\alpha a + \beta b) - (\alpha x_n + \beta y_n) \rangle$

$= \langle \alpha (a - x_n) + \beta (b - y_n) \rangle$

$= \alpha \langle a - x_n \rangle + \beta \langle b - y_n \rangle$

$= \alpha T(\langle x_n \rangle) + \beta T(\langle y_n \rangle)$ Hence, proved.

(b) $S = \{\langle x_n \rangle : T(\langle x_n \rangle) = 0\}$

Let $\lim_{n \to \infty} x_n = a \in \mathbb{R}$

$\Rightarrow \quad T(\langle x_n \rangle) = \langle a - x_n \rangle = 0$

A sequence can be equated to 0 if each and every term is 0.

$\Rightarrow \langle a - x_n \rangle = 0$

$\Rightarrow \langle x_n \rangle = \langle a \rangle.$

So, $S = \{\langle a \rangle : a \in \mathbb{R}\}$

Let us take any $\langle x_n \rangle \in V$ with $\lim_{n \to \infty} x_n = a$. Then

$$T(\langle x_n \rangle) = \langle a - x_n \rangle$$

Since $x_n \to a$, we know $a - x_n \to 0$.

So, the image of T is the set of all convergent sequence with limit 0.

$$T(V) = \{\langle y_n \rangle \in V : \lim_{n \to \infty} y_n = 0\}.$$

12 Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, V_e be the subset of even functions, $f(-x) = f(x)$.

Let V_o be the subset of odd functions, $f(-x) = -f(x)$.

(a) Suppose f and g are even functions. Then for any scalar c,

$$(cf + g)(-x) = cf(-x) + g(-x)$$
$$= cf(x) + g(x) = (cf + g)(x)$$

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So $cf+g$ is also even and therefore V_e is a subspace of V.

Similarly, if f and g are both odd functions, then

$$\begin{aligned}(cf+g)(-x) &= cf(-x) + g(-x) \\ &= -cf(x) - g(x) \\ &= -(cf+g)(x)\end{aligned}$$

So, V_o is also a subspace.

(b) Let $f \in V$ be arbitrary. Let g be function in V defined by

$$g(x) = \frac{f(x)+f(-x)}{2}$$

and let h be the function given by

$$h(x) = \frac{f(x)-f(-x)}{2}$$

It is clear that $g(x) \in V_e$ and $h \in V_o$. Since $f(x) = g(x) + h(x)$ $\forall x$, we see that $V = V_e + V_o$.

(c) Suppose $f \in V_e \cap V_o$ and fix a particular $x \in \mathbb{R}$. Since f is even, $f(-x) = f(x)$. And since f is odd, $f(-x) = -f(x)$. Therefore, we have $f(x) = -f(x)$, which only possible if $f(x) = 0$. Since x was arbitrary, f must be the zero function. This shows that $V_e \cap V_o = \{0\}$.

13. Given A = a ~~matrix~~ $m \times n$ matrix and B an $n \times m$ matrix over a field $\mathbb{F}$

(a) Let $AB = C$ and C_{i*} = i^{th} row of C

From multiplication definition, $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

$$C_{i1} = A_{i1} B_{11} + A_{i2} B_{21} + \dots + A_{in} B_{n1}$$

$$C_{i2} = A_{i1} B_{12} + A_{i2} B_{22} + \dots + A_{in} B_{n2}$$

$$\vdots$$

$$C_{ip} = A_{i1} B_{1p} + A_{i2} B_{2p} + \dots + A_{in} B_{np}$$

Hence,

$$C_{i*} = A_{i1} B_{1*} + A_{i2} B_{2*} + \dots + A_{in} B_{n*}$$

~~Hence~~ So, ith row of AB is linear combination of the rows of B.

(b) Let $AB = C$ and C_{*i} = ith column of C

$$C_{*i} = \begin{bmatrix} C_{1i} \\ C_{2i} \\ C_{3i} \\ \vdots \\ C_{mi} \end{bmatrix}$$

From multiplication definition, $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

$$C_{1i} = A_{11} B_{1i} + A_{12} B_{2i} + \dots + A_{1n} B_{ni}$$

$$C_{2i} = A_{21} B_{1i} + A_{22} B_{2i} + \dots + A_{2n} B_{ni}$$

$$\vdots$$

$$C_{mi} = A_{m1} B_{1i} + A_{m2} B_{2i} + \dots + A_{mn} B_{ni}$$

Hence:

$$C_{*i} = B_{1i} A_{*1} + B_{2i} A_{*2} + \dots + B_{ni} A_{*n}$$

So, ith column of AB is linear combination of the ~~rows~~ columns of A.

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(c) A is $m \times n$, B is $n \times m$, So AB is $m \times m$

Let $m = 2$, $n = 3$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$ which is 2×2 singular matrix

(not invertible).

So, AB need not be invertible, if $n > m$.

14. Let A be an $n \times n$ upper-triangular matrix.

First, ~~s~~ suppose every entry on the main diagonal of A is non-zero and consider the homogeneous linear system $AX = 0$:

$$A_{11} x_1 + A_{12} x_2 + \cdots + A_{1n} x_n = 0$$
$$A_{22} x_2 + \cdots + A_{2n} x_n = 0$$
$$\vdots$$
$$A_{nn} x_n = 0$$

Since A_{nn} is nonzero, the last equation implies that $x_n = 0$, Then since $A_{n-1,n-1}$ is nonzero, the second to last equation implies that $x_{n-1} = 0$. Continuing in this way, we get $x_i = 0$ $\forall i \in \mathbb{N}, 1 \leq i \leq n$. Therefore $AX = 0$ has only the trivial solution, Hence A is invertible.

Conversely, suppose A is invertible. Then A cannot contain any zero rows nor can A be row equivalent to a matrix to a row of zeroes. This implies $A_{nn} \neq 0$. Consider ~~A~~ $A_{n-1,n-1} = 0$, Then by doing appropriate row operation on the last row, we can make row $n-1$ to be row equivalent to a row of zeros, which is a contradiction $\Rightarrow A_{n-1,n-1} \neq 0$. Similarly, we can show $A_{ii} \neq 0$ for each $i = 1, 2, \dots, n$.

Thus, all entries on the main diagonal of A are non-zero.

$\therefore$ An upper-triangular (square) matrix is invertible iff every entry on its main diagonal is different from zero.

Hence, proved.

Suppose A is upper-triangle and invertible.

Let $B = A^{-1}$. We want to prove that B is also upper-triangle

We know, $AB = I_n$

Let $B = [b_{ij}]$. Then for the (i,j)-entry of AB:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

But $a_{ik} = 0$ for $i > k$. So the sum effectively runs only over $k \geq$

Now, consider $i > j$:

$$(AB)_{ij} = \sum_{k=i}^{n} a_{ik} b_{kj}$$

But here $i > j$, $k \geq i \Rightarrow k > j$

Thus b_{kj} involves terms strictly below the main diagonal.

Working inductively from first column to the last, one finds that these entries must vanish in order for $AB = I$.

Hence $b_{ij} = 0$ whenever $i > j$.

So, B is upper-triangle.

Hence, proved.

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15. We are given

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ 1/2 & 1/3 & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{bmatrix}_{n \times n}.$$

This is called the Hilbert matrix $H_n = \left(\frac{1}{i+j-1}\right)$, $1 \le i, j \le n$

The determinant, $\det(H_n) = \frac{c_n^4}{(2n-1)!!^2}$, with $c_n = \prod_{k=1}^{n-1} k!$

So $\det(H_n) \ne 0$. Hence, A is invertible $\forall n$.

The explicit formula for A^{-1} is

$$(H_n^{-1})_{ij} = (-1)^{i+j} (i+j-1) \binom{n+i-j}{n-j} \binom{n+j-1}{n-i} \binom{i+j-2}{i-1}^2$$

Note that it is a product of binomial coefficients (which are integers), $(i+j-1)$ (which is another integer) and $(-1)^{i+j}$ (An integer for the sign).

$\therefore$ It is a product of integers, resulting in an integer.

$\therefore$ Each entry of A^{-1} is an integer.

Hence, proved.