DS Assignment

1.) Mathematical Induction

Q 1 Conjecture : $\sum_{k=1}^{n} (2k-1) = n^2 \quad n \geq 1$

Basis of Induction : for $n = 1$

$$2(1) - 1 = (1)^2$$

Induction step : Let us assume it true for $n = k$

$1 + 3 + 5 + 7 + \ldots (2k-1) = (k)^2$

adding $(2k+1)$ on both sides,

$2k + 1 = 2(k+1) - 1$

$1 + 3 + 5 + 7 + \ldots (2k-1) + 2(k+1) - 1 = k^2 + 2(k+1) - 1$

$1 + 3 + 5 + 7 + \ldots 2(k+1) - 1 = (k+1)^2$

$\sum_{k=1}^{k=n} (2k-1) = (k+1)^2$

As it is true for $n = 1$ as for every $n = k$ it is true for $n = k+1$, $\therefore$ the eqn is true for $n \geq 1$

Q 2 $1 + 2 + 2^2 + \ldots 2^n = 2^{n+1} - 1 \quad n \geq 0$

$$\sum_{k=0}^{k=n} 2^k = 2^{n+1} - 1$$

Basis of Induction : $n = 0$

$2^0 = 2^{0+1} - 1$

$1 = 1$

Induction step : Let us assume true for $n = k$

$1 + 2 + 2^2 + \ldots 2^k = 2^{k+1} - 1$

add 2^{k+1} on both sides

$1 + 2 + 2^2 + \ldots 2^k + 2^{k+1} = 2^{k+1} - 1$

$$\sum_{k=0}^{k+1} (2^k) = 2(2^{k+1}) - 1 = 2^{(k+1)+1} - 1$$ (2)

As eq^n is true for $n = 0$ & for $n = k$ it is also true for $n = k + 1$, hence it is true for all $n \geq 0$

$$1^2 + 3^2 + 5^2 + \cdots (2n+1)^2 = [(n+1)(2n+1)(2n+3)]/3 \qquad n \geq 0$$

$$\sum_{k=0}^{k=n} (2k+1)^2 = ((n+1)(2n+1)(2n+3))/3$$

Basis of Induction : $n = 0$

$$(2(0)+1)^2 = [(0+1)(0+1)(0+3)]/3$$

$$1 = 1$$

Induction step : Let us assume true for $n = k$

$$1^2 + 3^2 + 5^2 + \cdots (2k+1)^2 = ((k+1)(2k+1)(2k+3))/3$$

adding $(2(k+1)+1)^2$;

$$1 + 3^2 + 5^2 + \cdots (2k+1)^2 + (2(k+1)+1)^2 = (k+1)(2k+1)(2k+3)/3 + (2(k+1)+1)^2$$

$$1^3 + 3^2 + 5^2 + \cdots (2(k+1)+1)^2 = \frac{(2k+3)}{3}\left[\frac{(k+1)(2k+1)(2k+3) +}{3(2k+3)}\right]$$

$$'' = \frac{(2k+3)}{3}[2k^2 + 9k + 10]$$

$$'' = \frac{2k+3}{3}(2k^2 + 4k + 5k + 10)$$

$$= (2k+3)(k+2)(2k+5)/3$$

$$'' = \frac{1(2(k+1)+1)((k+1)+1)(2(k+1)+3)}{3}$$

$\therefore$ As this is true for $n = 0$ & $n = k$, it is true for $n = k+1$

Hence true for all $n \geq 0$.

$(n^3 + 2n)$ divisible by 3 for all $n \geq 1$

$(n^3 + 2n) = 3a \qquad n \geq 1$

Basis of Induction : for $n = 1$

$(1 + 2) = 3 \qquad 3 = 3a$

$a = 1$

Hence for $n = 1$ it is divisible

Induction step : Let it be true for $n = k$

$k^3 + 2k = k(k^2 + 2) = 3a$

For $n = k + 1$

$LHS = (k+1)^3 + 2(k+1)$

$= k^3 + 3k(k+1) + 1 + 2(k+1)$

$= (k^3 + 2k) + 3k(k+1) + 3$

$= 3a + 3k(k+1) + 3$

$= 3(a + k(k+1) + 1)$

$LHS = 3A$ ∴ Hence LHS is also divisible by 3

∵ As it is true for $n = 1$, $n = k$, it is true for $n = k+1$

∴ it is true for all $n \geq 1$.

~~$(2^n * 2^{n+1})$~~

5 $(2^n * 2^n - 1)$ is divisible by 3 c $n \geq 1$

$(2^n * 2^n - 1) = 3a$ $\quad n \geq 1$

Basis of Induction for $n = 1$

$(2 \times 2 - 1) = 3$ Hence for $n = 1$ is divisible by 3

Induction step : Let it be true for $n = k$

$(2^k * 2^k - 1) = 3a$

For $n = k+1$

$LHS = (2^{k+1} * 2^{k+1} - 1)$

$= (2 * 2^k) * (2 * 2^k) - 1$

$= 4 * 2^k * 2^k - 4 + 3$

$= 4 * (2^k * 2^k - 1) + 3$

$= 4 * 3a + 3$

$LHS = 3(4a+1) = 3A$ $\therefore$ LHS is divisible by 3.

$\because$ It is true for $n = 1$ & $n = k$, it is true for $n = k+1$

$\therefore$ it is true for all $n \geq 1$

6 $1.1! + 2.2! + 3.3! + \ldots n.n! = (n+1)! - 1$

Basis of Induction : $n = 1$

$1.1! = (1+1)! - 1$

$1 = 2 - 1$ Hence true for $n = 1$.

Induction step : Let it be true for $n = k$

$1.1! + 2.2! + \ldots k.k! = (k+1)! - 1$

on adding $(k+1)$ & $(k+1)!$

$1.1! + 2.2! + \ldots k.k! + (k+1)(k+1)! = (k+1)! - 1 + (k+1) \times (k+1)!$

RHS ~~KHKS~~ $= (k+1)!\left[1+1+k\right]-1$

RHS $= (k+1)!\left[k+2\right]-1$

RHS ~~LH~~ $= (k+2)!-1$

RHS $= ((k+1)+1)!-1$

$\therefore$ as it is true for $n=1$, $n=k$ & even $n=k+1$,

$\therefore$ it is true for all $n \geq 1$

~~Q7~~ ~~$1+2+2^2+\dots 2^n$~~

Q 7 same as Q 2

Q 8 $n^5 - n$ is divisible by 5 for all $n \geq 0$

$n^5 - n = 5a$

Basis of Induction: $n = 0$

$0 = 0$ $\therefore$ true for $n=0$

Induction step: Let it be true for $n = k$

$k^5 - k = 5a$

For $n = k+1$

LHS $= (k+1)^5 - (k+1)$

$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$

$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$

$= 5a + 5(k^4 + 2k^3 + 2k^2 + k)$

LHS $= 5[A]$

Hence LHS also divisible by 5

$\therefore$ as it is true for $n=0$, & $n=k$, it is true for $n = k+1$

$\therefore$ true for all $n \geq 0$.

$n^2 - n$ is divisible by 2 for $n \geq 0$

Basis of Induction : $n = 0$

$0 - 0 = 0$ $\therefore$ divisible by 2

Induction Step : Let it be true for $n = k$

$k^2 - k = 2a$

For $n = k+1$

$LHS = (k+1)^2 - (k+1)$

$= k^2 + 2k + 1 - k - 1$

$= (k^2 - k) + 2k$

$= 2(a + k) = 2A$ Hence divisible by 2

$\therefore$ As it is true for $n = 0$, $n = k$, it is true for $n = k+1$.

$\therefore$ It is true for all $n \geq 0$.

$1.2.3 + 2.3.4 + \cdots n(n+1)(n+2) = [n(n+1)(n+2)(n+3)]/4$

Basis of Induction : $n = 1$

$1.2.3 = 1.2.3.4/4$ $\therefore$ n is true for $n = 1$

Induction steps $n = k$

$1.2.3 + 2.3.4 + \cdots k(k+1)(k+2) = [k(k+1)(k+2)(k+3)]/4$

adding $(k+1)(k+2)(k+3)$

$1.2.3 + \cdots k(k+1)(k+2) + (k+1)(k+2)(k+3) = [k(k+1)(k+2)(k+3)]/4 + (k+1)(k+2)(k+3)$

$RHS = \dfrac{(k+1)(k+2)(k+3)}{4}\left[k + 4\right]$

$RHS = (k+1)((k+1)+1)((k+1)+2)((k+1)+3)/4$

$\therefore$ as it is true for $n = 1$ & for $n = k$, it is true for $n = k+1$ $\therefore$ it is true for all $n \geq 1$.

Set Theory

Q1 a.) $\phi \in \{\phi\}$ → True

b.) $\phi \in \{\phi, \{\phi\}\}$ → True

(c) $\{\phi\} \in \{\phi\}$ → False

(d) $\{\phi\} \in \{\{\phi\}\}$ → True

(e) $\{\phi\} \subset \{\phi, \{\phi\}\}$ → True

(f) $\{\{\phi\}\} \subset \{\phi, \{\phi\}\}$ → ~~False~~ True

(g) $\{\{\phi\}\} \subset \{\{\phi\}, \{\phi\}\}$ → False

a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$
→ $\{1, -1\}$

(b) $\{x \mid x \text{ is a positive integer less than } 12\}$
→ $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

(c) $\{x \mid x \text{ is square of integer and } x < 100\}$
→ $\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$

(d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$
→ ~~[illegible]~~, ~~[illegible]~~ $\{\phi\}$ (as $\pm\sqrt{2}$ is not an integer)

(a) Show $\phi \times A = A \times \phi = \phi$

we know $|A \times B| = |A| \times |B|$

$|\phi \times A| = |A \times \phi| = |A| \cdot |\phi| = |A| \cdot 0 = 0$

As the cardinality of the resulting set is zero,

$\therefore \quad \phi \times A = A \times \phi = \phi$

(b) $A \times B \times C$ & $(A \times B) \times C$ are not same

$A \times B = \{ (x, y) \mid x \in A \wedge y \in B \}$

$A \times B \times C = \{ (x, y, z) \mid x \in A \wedge y \in B \wedge z \in C \}$.

The cartesian product $A \times B \times C$ contains triplets.

$(A \times B) \times C = \{ (n, z) \mid n \in A \times B \wedge z \in C \}$

$= \{ ((x, y), z) \mid (x \in A \wedge y \in B) \wedge z \in C \}$

The cartesian product $(A \times B) \times C$ contain doublets whose first value is itself a doublet.

$\therefore$ $A \times B \times C$ & $(A \times B) \times C$ are not same.

Q4 $A = \{ a, b, c \}$ $\quad B = \{ x, y \}$ $\quad C = \{ 0, 1 \}$

(A) $A \times B \times C = \{ (a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1), (c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1) \}$

(b) $C \times B \times A = \{ (0, x, a), (0, x, b), (0, x, c), (0, y, a), (0, y, b), (0, y, c), (1, x, a), (1, x, b), (1, x, c), (1, y, a), (1, y, b), (1, y, c) \}$

(c) $C \times A \times B = \{ (0, a, x), (0, a, y), (0, b, x), (0, b, y), (0, c, x), (0, c, y), (1, a, x), (1, a, y), (1, b, x), (1, b, y), (1, c, x), (1, c, y) \}$

(d) $B \times B \times B = \{ (x, x, x), (x, x, y), (x, y, x), (x, y, y)$

$= , (y, x, x) , (y, x, y) , (y, y, x) , (y, y, y) \}$

5 (a) Find Power set.

(a) $\{a\} \Rightarrow P = \{ \phi, \{a\} \}$

(b) $\{a, b\} \Rightarrow P = \{ \{a, b\}, \{a\}, \{b\}, \cancel{\{\phi\}} \}$

(c) $\{\phi, \{\phi\}\} \Rightarrow P = \{ \phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\} \}$

(b) Find cardinality

(a) ϕ $\quad |S| = 0$

(b) $\{\phi\}$ $\quad |S| = 1$

(c) $\{\phi, \{\phi\}\}$ $\quad |S| = 2$

(d) $\{\phi, \{\phi, \{\phi\}\}, \{\phi\}\}$ $\quad |S| = 3$

A: set of sophomores at school

B: set of students in discrete maths

(a) $A \cap B$

(b) $A \cap \overline{B}$

(c) $A \cup B$

(d) $\overline{A} \cup \overline{B}$

7 (a) $A = \{1, 2, 3, 4, 5\}$ $\quad B = \{0, 3, 6\}$

a) $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$

b) $A \cap B = \{3\}$

c) $A - B = \{1, 2, 4, 5\}$

d) $B - A = \{0, 6\}$

(b) $A = \{a, b, c, d, e\}$ $\quad B = \{a, b, c, d, e, f, g, h\}$

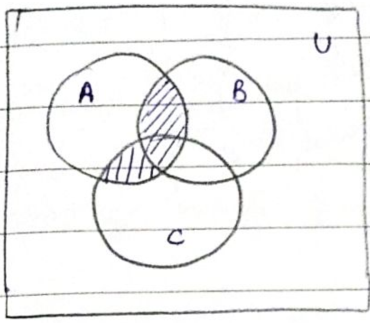
a) $A \cup B = \{a, b, c, d, e, f, g, h\}$

b) $A \cap B = \{a, b, c, d, e\}$

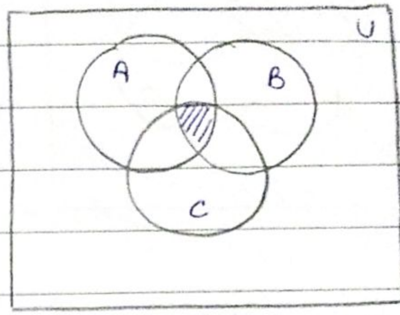
c) $A - B = \phi$

(d) $B - A = \{f, g, h\}$

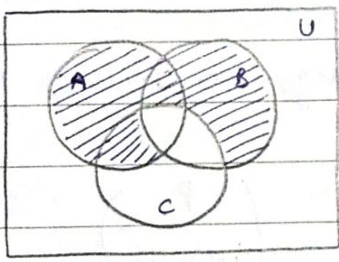
(a) $A \cap (B \cup C)$



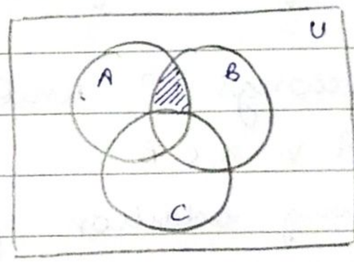
(b) $A \cap B \cap C$



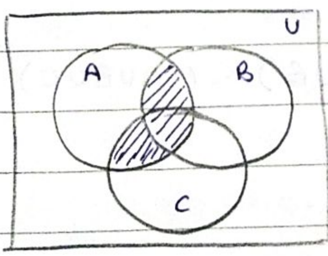
c) $(A - B) \cup (A - C) \cup (B - C)$



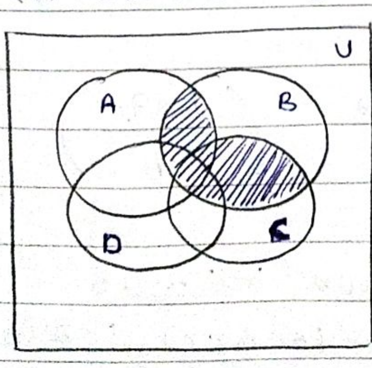
(d) $A \cap (B - C)$



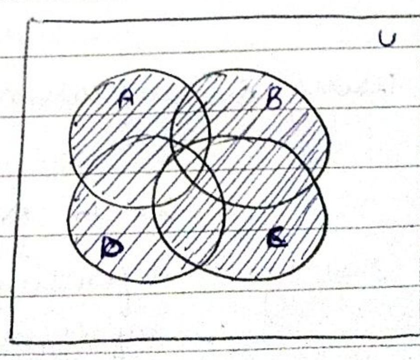
2) $(A \cap B) \cup (A \cap C)$



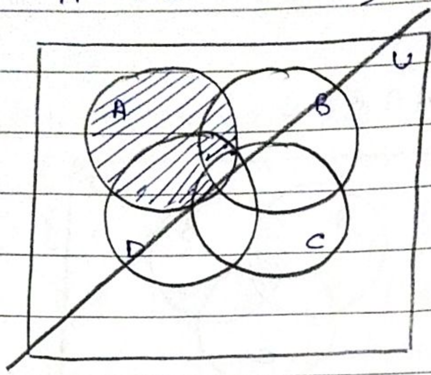
(a) $(A \cap B) \cup (C \cap B)$

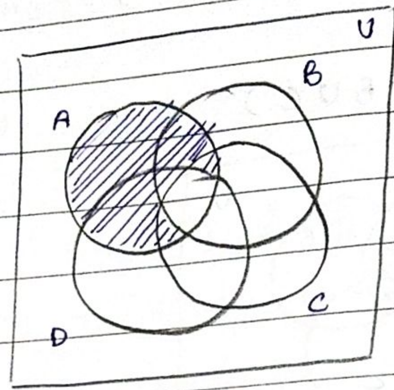


(b) $A \cup B \cup C \cup D$



(c) A − (B ∩ C ∩ D)





10) (a) Prove : $(A \cup B) \subseteq (A \cup B \cup C)$

Let $x \in A \cup B$

Acc^r definition of union

x belongs to union if it belongs to either A, B or b

$\Rightarrow$ $x \in A \vee x \in B$

using addition of prepositions : $\left(\frac{p}{\therefore p \vee q}\right)$

$x \in A \vee x \in B \vee x \in C$

By union $x \in A \cup B \cup C$

$\therefore$ by definition of subset $(A \cup B) \subseteq (A \cup B \cup C)$

(b) $(A \cap B \cap C) \subseteq (A \cap B)$

Let $x \in A \cap B \cap C$

using definition of intersection, x is in intersection ~~area~~ when it is in both sets $\Rightarrow$ $x \in A \wedge x \in B \wedge x \in C$

using simplification of propositions $\left(\frac{p \wedge q}{\therefore p}\right)$

$x \in A \wedge x \in B$

using definition of intersection $x \in A \cap B$

$\therefore$ by definition of subset $(A \cap B \cap C) \subseteq (A \cap B)$

$(A-B)-C \subseteq A-C$

Let $x \in (A-B)-C$

using definition of difference, x is in $A-B$ & x is not in C

$\Rightarrow x \in (A-B) \wedge \neg(x \in C)$

$x \in A \wedge \neg(x \in B) \wedge \neg(x \in C)$

using simplification of prepositions $\left(\frac{p \wedge q}{\therefore p}\right)$

$x \in A \wedge \neg(x \in C)$

Use definition of diff., x has to be in diff $A-C$

$\Rightarrow x \in A-C$

By definition of subset, $(A-B)-C \subseteq A-C$

$(A-C) \cap (C-B) = \phi$

Let $x \in (A-C) \cap (C-B)$

use definition of intersection, x is in both the sets

$x \in (A-C) \wedge x \in (C-B)$

using definition of difference, x is in A & not in C

$\Rightarrow x \in A \wedge \underbrace{\neg(x \in C) \wedge x \in C}_{False = F} \wedge \neg(x \in B)$

$x \in A \wedge F \wedge \neg(x \in B)$

By domination law, $= F$

$\therefore$ the set does not contain $x \in \phi$

By defin. of subset $(A-C) \cap (C-B) \subseteq \phi$

empty set is part of every set $\phi \subseteq (A-C) \cap (C-B)$

$\therefore$ as both are subset of each other

$(A-C) \cap (C-B) = \phi$

$(B-A) \cup (C-A) = (B \cup C) - A$

Let $x \in (B-A) \cup (C-A)$

By def. of union $x \in (B-A) \vee x \in (C-A)$

By def. of difference. $(x \in B \wedge \ldots$
$(x \in C \wedge (\neg x \in A))$

$(x \in B \vee x \in C) \wedge \neg(x \in A)$

By distributive law $(x \in B \vee x \in C) \wedge \neg(x \in A)$

By def. of union $(x \in B \cup C) \wedge \neg(x \in A)$

By def. of diff. $x \in (B \cup C) - A$

By def. of subset $(B-A) \cup (C-A) \subseteq (B \cup C) - A$

Let $x \in (B \cup C) - A$

$(x \in B \cup C) \wedge \neg(x \in A)$ — by def. of diff.

$(x \in B \vee x \in C) \wedge \neg(x \in A)$ — by def. of union

$(x \in B \wedge \neg(x \in A)) \vee (x \in C \wedge \neg(x \in A))$ - By distributive law

$x \in B - A \quad \vee \quad x \in C - A$

$\Rightarrow \quad x \in (B-A) \cup (C-A)$

$\Rightarrow \quad (B \cup C) - A \subseteq (B-A) \cup (C-A)$

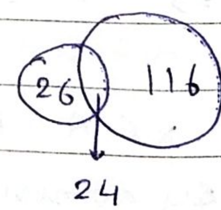
$\therefore \quad (B-A) \cup (C-A) = (B \cup C) - A$

Principle of Inclusion - Exclusion

1.) $U = 200$ $\quad A \cap B = 24$

$A = 50$ $\quad A \cup B = (A+B) - A \cap B$

$B = 140$



no: of students who will go to the party

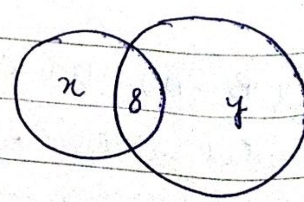
$= A \cup B - A \cap B$

$= 200 - 24$

$= 176$

2.) $A = 25$ $\quad A \cap B = 8$

$B = 13$



$A + B = A \cup B + A \cap B$

$x+8=A$ 191112220

$y+8=B$

$38 = A \cup B + 8$

$A \cup B = 30$

$A \cup B = 30$ ans.

$U = 1807$

$A = 453$ (computer science)

$B = 567$ (Maths)

$A \cap B = 299$

$A \cup B = A + B - A \cap B$

$\overline{A} \cap \overline{B} = ?$

$A \cup B = 453 + 567 - 299 = 721$

$(\overline{A} \cap \overline{B}) = (\overline{A \cup B}) = U - A \cup B$

$= 1807 - 721$

$= 1086$

) A (spanish) = 1232 $A \cap B = 103$

B (French) = 879 $B \cap C = 14$

C (Russia) = 114 $A \cap C = 23$

$A \cup B \cup C = 2092$

$A \cup B \cup C = A + B + C - (A \cap B + B \cap C + A \cap C) + (A \cap B \cap C)$

$2092 = 1232 + 879 + 114 - (103 + 14 + 23) + A \cap B \cap C$

$A \cap B \cap C = 7$

) A = calculus = 345

B = Discrete = 212

$A \cap B = 188$

$A \cup B = A + B - A \cap B$

$A \cup B = 345 + 212 - 188$

$A \cup B = 369$

6.) $U = 100$

$A = \text{TV set} = 96$

$B = \text{Telephone} = 98$

$A \cap B = 95$ $\quad \overline{A} \cap \overline{B} = ?$

$(\overline{A} \cap \overline{B}) = (\overline{A \cup B})$

$(A \cup B) = A + B - A \cap B$

$A \cup B = 96 + 98 - 95$

$A \cup B = 99$

$\overline{A \cup B} = 100 - 99 = 1$ $\quad \overline{A} \cap \overline{B} = 1$

7.) $|A| = 100$ $\quad A_1 \cup A_2 \cup A_3 = ?$

(a) 300

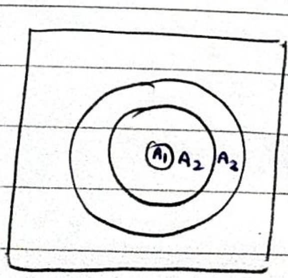
(b) 200

(c) 175

(d) 100

8.) $A_1 = 100$ $\quad A_2 = 1000$ $\quad A_3 = 10000$

(a) $A_1 \subseteq A_2$ & $A_2 \subseteq A_3$



$A_1 \cup A_2 \cup A_3 = 10{,}000$

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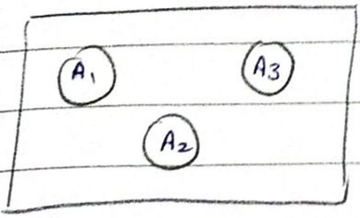
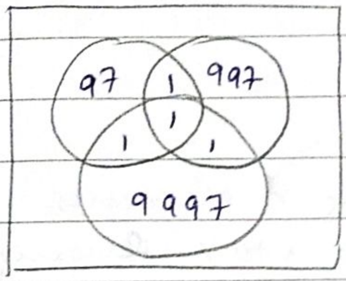
Pairwise disjoint set



$A_1 \cup A_2 \cup A_3 = 100 + 1000 + 10000$
$= 11100$

2 common element pairwise & 1 common in all 3:



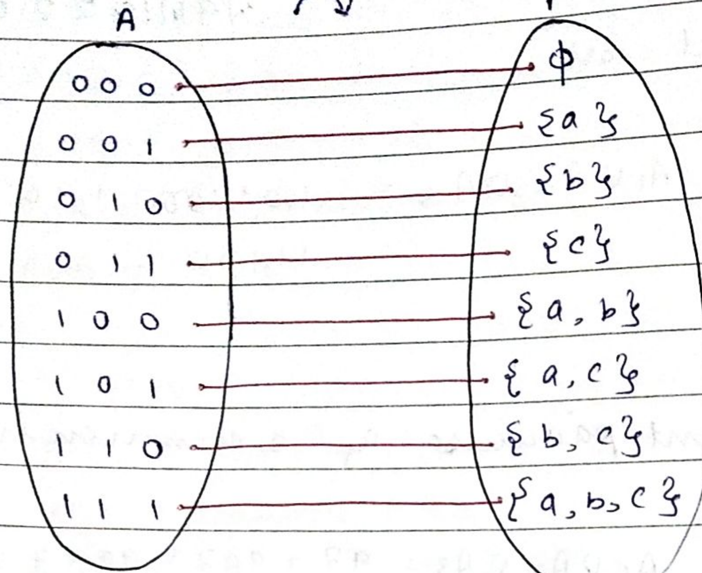
$A_1 \cup A_2 \cup A_3 = 97 + 997 + 9997$
$+ 1 + 1 + 1 + 1$
$= 11095$

Countability

We know that,
there are 2^n elements in the power set of a set with cardinality n.
Let us consider the one to one correspondance b/w subsets of S and bit strings (each element takes on a value of 0 or 1) of length $|S|$.
A subset of S is associated with a bit string with a 1 in the ith position of the ith element in the list in the subsets. By the multiplication rule, there are $2^{|S|}$ bit strings of length $|S|$.
$\therefore \; |P(S)| = 2^{|S|}$

A = bit string of length 3
P = Power set of S
r = a subset of S is associated with a 1 in the ith position of ith element is in the subsets.

$S = \{a, b, c\}$



Q2 Let X & Y be 2 sets having m & n elements respectively. In a function from X to Y every element of X must be mapped to an element of Y.
∴ each element of X has 'n' elements to be choosen from. ∴ total number of function will be $n \times n \times n \times \ldots\ldots$ m times
$= n^m$

Total number of $fn^c = n^m$

Q3 Given
Maths majors = 18
Computer science majors = 325

(a) We need to use the Product Rule, because the first event in picking a maths major & second even in picking a C.S major.

~~(18)(325) = 5850~~

$(18) \cdot (325) = 5850$

(b) We use the Sum Rule, because the event is picking a maths major or picking a computer major

$18 + 325 = 343$

given
colour $= 12$, gender $= 2$, size $= 3$

we need to use Product Rule, because the first event is picking the colour, the second event is picking the gender & the third event picking the size.

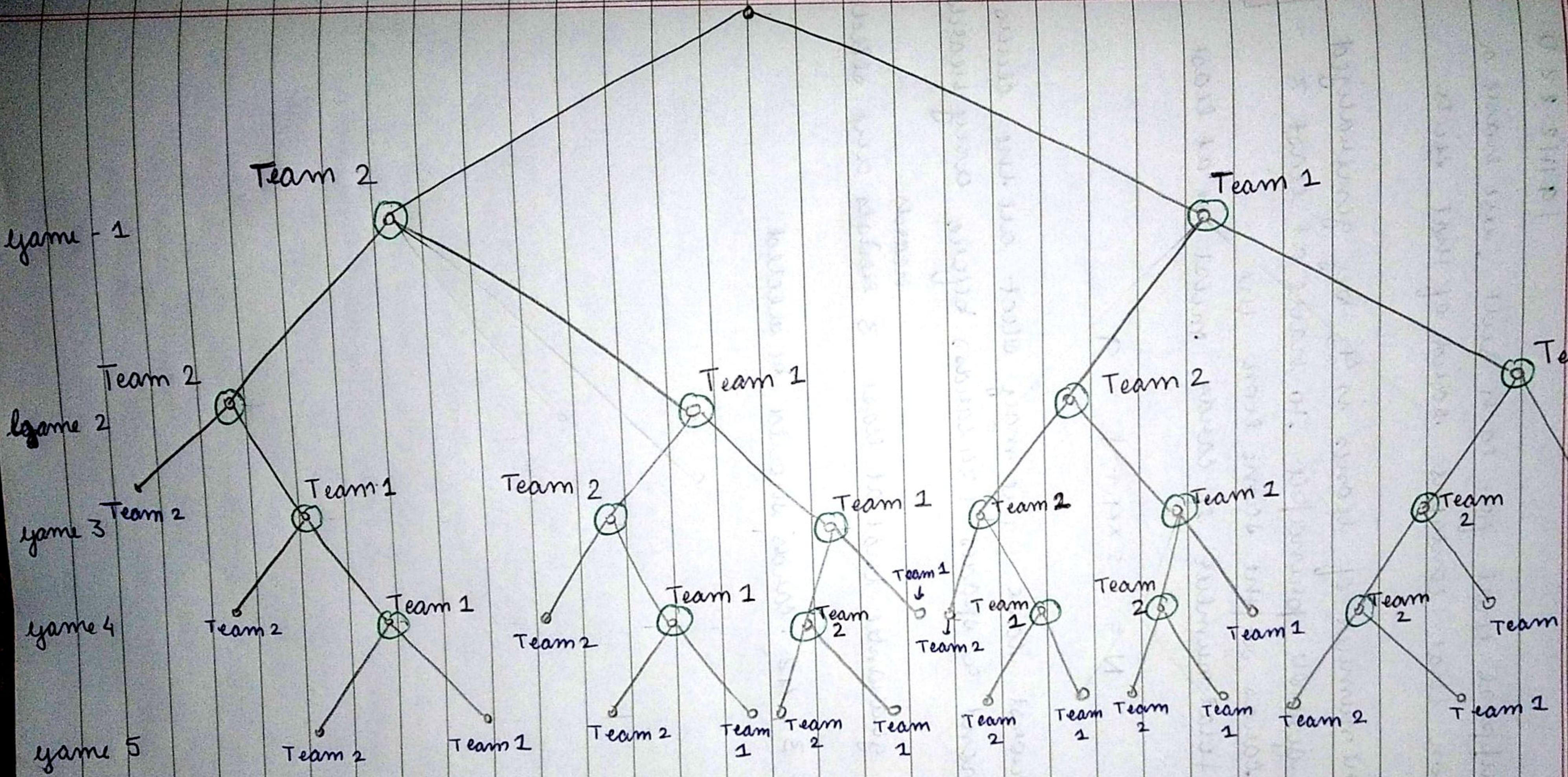
$12 \times 2 \times 3 = 72$

Student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by sum rule.

$23 + 15 + 19 = 57$

Assume on contrary that there is no integer in the set that divides another integer in the set.
If a is in the set such that $a \leq n$, then $2 \cdot a$ cannot be in the set. Thus, if there are k elements in the set not exceeding n, then there are k integer greater than $(n+1)$ but less than $2n$ which cannot be in the set. Thus, there are n elements in the set, contradicting the fact the s has $n+1$ elements. Here, there is an integer in the set that divides a diff. integer in set.

Best 3 game out of 3 playoffs



~~un~~ Winning Team in Black

classmate

Date

Page

Esha Jain

191112220

(a) suppose that for each suit, we have a box that contains card of that suits.

The number of boxes is 4, by generalized pigeonhole principle, to have at least 3 cards at the same box, the total number of cards must be at least

$= \left\lceil \frac{N}{4} \right\rceil$

$$N = 2 \times 4 + 1 = 9$$

b) The worst case, we may select all the clubs, diamonds & spades (34 cards) before any hearts.

So to guarantee that at least 3 ~~hearts~~ hearts are selected

$39 + 3 = 42$ cards should be selected.

Isha Jain

9 There are 30 students in class. Assume that each student has at least a first name & a last name.
Each last name has to begin with a letter of the alphabet.

objects = first letter of each last name
$\Rightarrow 30 \Rightarrow N$

holes = letters of Alphabets $= K = 36$

$$\left\lceil \frac{30}{26} \right\rceil = 2$$

So by pigeon hole principle there are at least 2 students who have a last name that begin with same letter.

10 There are 20 balls, half of them red & other half blue.

1(a) ~~Ans~~ Pigeonholes are colours, $\left\lceil \frac{x}{2} \right\rceil$ must be equal to 3, and the least positive integer to satisfy the equation is $\boxed{5}$.

(b) The first 10 choice may all be red balls, so the woman need to choose at least $\boxed{13}$ balls to ~~the~~ be sure that at least 3 of them are blue.