Mathematical Induction

Question1. Conjecture a formula for the sum of the first *n* positive odd integers. Then prove your conjecture using mathematical induction.

Question 2. Use mathematical induction to show that

$$1 + 2 + 2^2 + \ldots\ldots + 2^n = 2^{n+1} - 1$$

For all non-negative integers n.

Question 3. Prove that $1^2 + 3^2 + 5^2 + \ldots\ldots + (2n+1)^2 = [(n+1)(2n+1)(2n+3)] / 3$, whenever n is a non-negative integer.

Question 4. Show that $n^3 + 2n$ is divisible by 3 for all n>=1 by induction.

Question 5. Show that $2^n * 2^n - 1$ is divisible by 3 for all n>=1 by induction.

Question 6. Prove by induction that for n>=1

$$1.1! + 2.2! + 3.3! + \ldots\ldots. + n.n! = (n+1)! - 1$$

Question 7. Show that

$$1 + 2 + 2^2 + 2^3 + \ldots. + 2^n = 2^{n+1} - 1$$

Question 8. Prove that 5 divides $n^5 - n$ whenever n is a non-negative integer.

Question 9. Prove that 2 divides $n^2 - n$ whenever n is a positive integer.

Question 10. Show that $1.2.3 + 2.3.4 + 3.4.5 + \ldots. + n(n+1)(n+2) = [n(n+1)(n+2)(n+3)] / 4$

Set Theory

Question 1. Determine whether these statements are true or false.
a) $\emptyset \in \{\emptyset\}$ **b)** $\emptyset \in \{\emptyset, \{\emptyset\}\}$
c) $\{\emptyset\} \in \{\emptyset\}$ **d)** $\{\emptyset\} \in \{\{\emptyset\}\}$
e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ **f)** $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Question 2. List the members of these sets.
a) $\{x \mid x$ is a real number such that $x2 = 1\}$
b) $\{x \mid x$ is a positive integer less than $12\}$
c) $\{x \mid x$ is the square of an integer and $x < 100\}$
d) $\{x \mid x$ is an integer such that $x2 = 2\}$

Question 3. a) Let *A* be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.

b) Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

Question 4. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find
a) $A \times B \times C$. **b)** $C \times B \times A$.
c) $C \times A \times B$. **d)** $B \times B \times B$.

Question 5.a) Find the power set of each of these sets, where *a* and *b* are distinct elements.
a) $\{a\}$ **b)** $\{a, b\}$ **c)** $\{\emptyset, \{\emptyset\}\}$

Question 5.b) What is the cardinality of each of these sets?
a) $\emptyset$ **b)** $\{\emptyset\}$
c) $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Question 6. Suppose that *A* is the set of sophomores at your school and *B* is the set of students in discrete mathematics at your school. Express each of these sets in terms of *A* and *B*.

a) the set of sophomores taking discrete mathematics in your school
b) the set of sophomores at your school who are not taking discrete mathematics
c) the set of students at your school who either are sophomores or are taking discrete mathematics
d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

Question 7 a) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$
7 b) Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$.
Find **a)** $A \cup B$. **b)** $A \cap B$. **c)** $A - B$. **d)** $B - A$.

Question 8. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.
a) $A \cap (B \cup C)$ **b)** $A \cap B \cap C$
c) $(A - B) \cup (A - C) \cup (B - C)$ **d)** $A \cap (B - C)$ **e)** $(A \cap B) \cup (A \cap C)$
f) $(A \cap B) \cup (A \cap C)$

Question 9. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.
a) $(A \cap B) \cup (C \cap D)$ **b)** $A \cup B \cup C \cup D$
c) $A - (B \cap C \cap D)$

Question 10. Let A, B, and C be sets. Show that
a) $(A \cup B) \subseteq (A \cup B \cup C)$.
b) $(A \cap B \cap C) \subseteq (A \cap B)$.
c) $(A - B) - C \subseteq A - C$.
d) $(A - C) \cap (C - B) = \emptyset$.
e) $(B - A) \cup (C - A) = (B \cup C) - A$.

Countability

Question 1. Use the product rule to show that the number of different subsets of a finite set S is $2|S|$.

Question 2. How many functions are there from a set with m elements to a set with n elements?

Question 3. There are 18 mathematics majors and 325 computer science majors at a college.
a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?
b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Question 4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each gender. How many different types of this shirt are made?

Question 5. A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Question 6. A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

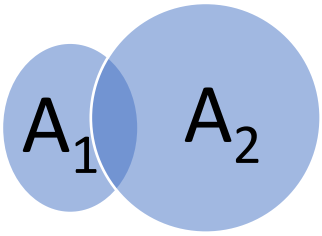
Question 7. **a)** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
b) How many must be selected to guarantee that at least three hearts are selected?

Question 8. Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.
Question 9. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

Question 10. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
a) How many balls must she select to be sure of having at least three balls of the same colour?
b) How many balls must she select to be sure of having at least three blue balls?

Principle of Inclusion-Exclusion

Question 1. Out of 200 students, 50 of them take the course Discrete Mathematics, 140 of them take the course Economics, and 24 of them take both the courses. Since both courses have schedules examinations for the following day, only students who are not in either one of these courses will be able to go to the party, the night before. We want to know how many students will be at the party. Examining the Venn Diagram in the figure, where A_1 is the set of students in the course Discrete Mathematics and A_2 is the set of students in the course Economics.



Question 2. In a discrete mathematics class, every student is a major in computer science or mathematics,
or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in this class?

Question 3. Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?

Question 4. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French,
and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Question 5. There are 345 students at a college who have taken a course in calculus, 212 who have taken a course in discrete mathematics, and 188 who have taken courses in both calculus and discrete mathematics. How many students
have taken a course in either calculus or discrete mathematics?

Question 6. A survey of households in the United States reveals that 96% have at least one television set, 98% have telephone service, and 95% have telephone service and at least one television set. What percentage of households in the United States have neither telephone service nor a television set?

Question 7. Find the number of elements in $A1 \cup A2 \cup A3$ if there are 100 elements in each set and if
a) the sets are pairwise disjoint.
b) there are 50 common elements in each pair of sets and no elements in all three sets.
c) there are 50 common elements in each pair of sets and 25 elements in all three sets.
d) the sets are equal.

Question 8. Find the number of elements in $A1 \cup A2 \cup A3$ if there are 100 elements in $A1$, 1000 in $A2$, and 10,000 in $A3$ if
a) $A1 \subseteq A2$ and $A2 \subseteq A3$.
b) the sets are pairwise disjoint.
c) there are two elements common to each pair of sets and one element in all three sets.