

Online Learning and Unlearning

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Abstract

We formalize the problem of online learning-unlearning, where a model is updated sequentially in an online setting while accommodating unlearning requests between updates. After a data point is unlearned, all subsequent outputs must be statistically indistinguishable from those of a model trained without that point. We present two online learner-unlearner (OLU) algorithms, both built upon online gradient descent (OGD). The first, *passive OLU*, leverages OGD’s contractive property and injects noise when unlearning occurs, incurring no additional computation. The second, *active OLU*, uses an offline unlearning algorithm that shifts the model toward a solution excluding the deleted data. Under standard convexity and smoothness assumptions, both methods achieve regret bounds comparable to those of standard OGD, demonstrating that one can maintain competitive regret bounds while providing unlearning guarantees.

1 Introduction

Machine unlearning—the process of efficiently removing the influence of specific training data so that a model behaves as if it were retrained without that data—has recently attracted significant attention. Beyond mitigating privacy risks (e.g., membership inference attacks [1, 2]), unlearning algorithms are central to enforcing regulations such as the GDPR’s “right to be forgotten” and can even improve model performance by removing undesirable data points [3].

Most prior work on machine unlearning has focused on the offline setting [4, 5, 6, 7, 8], where data points are unlearned from an already trained model, either all at once or sequentially. However, as these methods operate *after training is complete*, they do not readily extend to settings where data arrives continuously. In practice, data often arrives incrementally, and the model is updated frequently with deletion requests interspersed among these updates. This gives rise to new challenges that offline methods do not address. This observation motivates our study of an *online learning and unlearning* framework.

In this work, we introduce an online learning and unlearning framework that continually updates the model with incoming data while simultaneously accommodating deletion requests. We formulate the online learning and unlearning problem and define an unlearning guarantee (inspired by the “delete-to-control” notion in Cohen et al. [9]) tailored to the online setting. Specifically, once a deletion request is processed, all *future outputs of the algorithm* must not reveal information about the deleted point. We formalise this in Definition 2.

To achieve this, we propose two algorithmic strategies: (1) *Passive Unlearning* leverages intrinsic properties of online algorithms which exhibit a Markovian Output property (Condition C1) and Contractiveness (Condition C2), to inject calibrated noise based on the time gap between learning and deletion. Notably Online Gradient Descent (OGD) exhibits both of these properties. This approach incurs no additional computational cost relative to standard online learning. (2) *Active Unlearning* exploits an auxiliary offline learning algorithm (e.g., one based on Empirical Risk Minimisation (ERM)) with well-established unlearning algorithms, actively shifting the online algorithm’s output towards that of the offline method. This strategy can reduce the amount of noise required and thus potentially improve regret bounds but comes with additional computational overhead.

For both strategies, we derive sub-linear regret bounds that closely match the guarantees of standard OGD. Roughly, for convex cost functions with quadratic growth properties, passive unlearning achieves an expected regret of $O(\sqrt{T} + k^2d)$ and for μ -strongly convex losses, the regret bounds of both passive unlearning and active unlearning are $O(\log T + k^2d)$ for k deletion requests on a d -dimensional parameter space and T learning steps.

To summarise, our contributions are three-fold:

- **Problem formulation** We initiate the *online learning and unlearning* setting and formalise an (α, ε) -online unlearning guarantee (Definition 2) for this framework.

- **Algorithms** We design two algorithms—passive unlearner and active unlearner—both provably satisfy the above unlearning guarantee.
- **Theoretical guarantees** We prove that our algorithms add only constant-factor computational overhead while attaining regret that nearly matches the best possible bound achievable without unlearning. A detailed comparison appears in Table 2.

The paper is organized as follows: Section 2 introduces the definitions of online convex optimization, machine unlearning, and online learning-unlearning; Sections 3 and 4 present the passive and active unlearners, with their theoretical guarantees; Section 5 concludes with performance insights.

2 Preliminaries and Problem Setup

In Section 2.1, we review online convex optimization (OCO) and formally define unlearning. Then, in Section 2.2, we introduce the *online learning and unlearning* framework.

2.1 Preliminaries

Online Convex Optimization Let \mathcal{K} denote a convex instance space and let \mathcal{F} be a class of convex cost functions mapping \mathcal{K} to \mathbb{R}_+ . OCO models an iterative game between a learner and an adversary over T time steps. At each time step $t = 1, \dots, T$, the learner \mathcal{A} selects a point $z_t \in \mathcal{K}$, while the adversary chooses a convex cost function $f_t \in \mathcal{F}$. The learner then incurs a cost $f_t(z_t)$. Formally, the learner’s update rule at each step is

$$z_t = g_t(f_{1:t}, z_{1:t-1}). \quad (1)$$

where g_t depends on all previously observed cost functions f_1, \dots, f_t and past outputs z_1, \dots, z_{t-1} . The performance of the learner is measured by its *regret*:

$$\text{Regret}_T(\mathcal{A}(f_{1:T})) = \sum_{t=1}^T f_t(z_t) - \min_{z \in \mathcal{K}} \sum_{t=1}^T f_t(z),$$

where a sublinear regret is desirable. Throughout, we assume an *oblivious* adversary that fixes f_1, \dots, f_T in advance, and that \mathcal{K} has a bounded diameter D . Further assumptions on f_t (e.g. Lipschitzness, smoothness, or strong convexity; see Definition 3) appear in later sections as needed.

Online Gradient Descent (OGD) is a canonical algorithm for this setting. At time t , given a cost function f_t at time t , OGD updates its output as:

$$z_t = \Pi_{\mathcal{K}} [z_{t-1} - \eta_t \nabla f_t(z_{t-1})]. \quad (2)$$

where $\Pi_{\mathcal{K}}$ is the projection operator onto the convex set \mathcal{K} . OGD achieves $O(\log T)$ regret for strongly convex and $O(\sqrt{T})$ regret for convex losses [10, 11].

Unlearning Machine unlearning aims to remove *post hoc* the influence of a specific training point on the learned model. Naturally, the gold standard is to retrain from scratch without that point and thus, a good unlearning procedure should produce a model close to the retrained model. Various works use different notions of statistical indistinguishability [4, 12, 13] to formalize this closeness, inspired by differential privacy [14]. We adopt the following definition of unlearning via Rényi divergence.

Definition 1. Let \mathcal{A} be a learning algorithm and \mathcal{R} an unlearning algorithm. For a dataset \mathcal{S} and a subset $\mathcal{S}^{\mathcal{U}} \subseteq \mathcal{S}$ of points to be removed, we say that \mathcal{R} is an (α, ε) -unlearner if

$$D_{\alpha}(\mathcal{R}(\mathcal{A}(\mathcal{S}), \mathcal{S}^{\mathcal{U}}) \parallel \mathcal{R}(\mathcal{A}(\mathcal{S} \setminus \mathcal{S}^{\mathcal{U}}), \emptyset)) \leq \varepsilon,$$

where $D_{\alpha}(\cdot \parallel \cdot)$ is the α -Rényi divergence.

In most unlearning approaches, the unlearning step can be decoupled into a deterministic component which adjusts the current output to approximately match the output that would have been obtained had the algorithm never seen the unlearned point, and a noise component which adds calibrated noise to the adjusted output to obfuscate the approximation error. We define the deterministic unlearning function as h and the perturbation function as ρ .

2.2 Online Learning-Unlearning

We now integrate unlearning into the OCO framework. In an *online learning-unlearning* game, the learner not only submits $z_t \in \mathcal{K}$ each round and incurs $f_t(z_t)$, but may also receive requests to unlearn specific cost functions encountered in the past. Perhaps closest to our work, is the turnstile model in the continual observation literature, which similarly accommodates both insertion and deletion [15].

Let k be the number of deletions, and let $\mathcal{U} = \{u[1], \dots, u[k]\}$ and $\mathcal{T} = \{\tau[1], \dots, \tau[k]\}$ denote respectively the *indices* of deleted functions and the *time steps* at which deletions occur (with $u[i] \leq \tau[i]$). Thus, at time $\tau[i]$, the learner must ensure that the future outputs are indistinguishable from those they would produce if all functions $f_{u[1]}, \dots, f_{u[i]}$ had never been observed. We call the learner in an online learning-unlearning game an *online learner-unlearner (OLU)*, denoted by $\mathcal{A}_{\mathcal{R}}$.

Online Learner-Unlearner An OLU can be constructed from a base online learner. Formally, let \mathcal{A} be a base online learner with update functions g_1, \dots, g_T . An *online learner-unlearner* $\mathcal{A}_{\mathcal{R}}$ implements update function $g_t^{\mathcal{A}_{\mathcal{R}}}$ at time t ,

$$g_t^{\mathcal{A}_{\mathcal{R}}} = \rho_t \circ h_t \circ g_t. \quad (3)$$

where h_1, \dots, h_T are the deterministic unlearning functions, and ρ_1, \dots, ρ_T are the perturbation functions.

For two types of online learning algorithms, OLU is constructed differently from Equation (3). For base learners that explicitly use all past cost functions at each step (e.g. FTL), unlearning (h_t and ρ_t) must be applied at every round, essentially treating each update as an independent offline unlearning problem. In contrast, incremental algorithms such as OGD, whose update depends only on the previous model and current cost function, unlearning is invoked only at deletion rounds; otherwise, OLU mirrors OGD exactly. However, because past data are encoded in the memory of the algorithm, unlearning can be more involved. In this paper, we focus on the latter case.

Certified OLU Let $\mathcal{S} = \{f_1, \dots, f_T\}$ be the cost functions chosen by the adversary, with $\mathcal{S}_t = \{f_1, \dots, f_t\}$ as the subset up to time t . The unlearned functions are $\mathcal{S}^{\mathcal{U}} = \{f_{u[1]}, \dots, f_{u[k]}\}$, where $\mathcal{S}_i^{\mathcal{U}} = \{f_{u[1]}, \dots, f_{u[i]}\}$ denotes the first i deletions. In the online setting, removing a previously used function can shift indices for future outputs. To manage this, we introduce a skip element \perp : an online learning algorithm does not update its output at time steps where it encounters \perp and ignores \perp in all future updates. The retraining dataset $\mathcal{S} \setminus \mathcal{S}^{\mathcal{U}}$ is obtained by replacing every occurrence of a point $f \in \mathcal{S}^{\mathcal{U}}$ in \mathcal{S} with \perp .

Definition 2. An OLU $\mathcal{A}_{\mathcal{R}}$ is an (α, ε) -OLU if, $\forall i = 1, \dots, k-1$,

$$D_{\alpha} \left([\mathcal{A}_{\mathcal{R}}(\mathcal{S}_{\tau[i+1]-1}, \mathcal{S}_i^{\mathcal{U}}, [\mathcal{T}]_{1:i})]_{\tau[i]:\tau[i+1]-1} \parallel [\mathcal{A}_{\mathcal{R}}(\mathcal{S}_{\tau[i+1]-1} \setminus \mathcal{S}_i^{\mathcal{U}}, \emptyset, [\mathcal{T}]_{1:i})]_{\tau[i]:\tau[i+1]-1} \right) \leq \varepsilon, \quad (4)$$

where $[\cdot]_{p:q}$ denotes the output sequence from time p to q .

In Definition 2, we require that for each interval $[\tau[i], \tau[i+1])$ the OLU's outputs be indistinguishable from those of retraining on the dataset with the first i points removed. Although this condition is stated separately for each sub-interval, it nonetheless guarantees that once a point is deleted it remains protected forever. For example, when the second point $f_{u[2]}$ is removed at $t = \tau[2]$, the interval-wise guarantees suffice to protect it for all future times:

- Interval $t \in [\tau[2], \tau[3])$: Definition 2 enforces indistinguishability from retraining on the dataset with the first two deleted points removed, thereby protecting $f_{u[2]}$.
- Interval $t \in [\tau[3], \tau[4])$: after deleting $f_{u[3]}$, Definition 2 enforces indistinguishability from retraining without the first three deleted points, which automatically continues to protect $f_{u[2]}$.
- All later intervals: Each subsequent interval enforces indistinguishability from retraining with all points deleted up to that time, preserving protection of every previously removed point.

OLU Regret Since the best-in-hindsight comparator may change after each deletion, we define regret in a manner reminiscent of dynamic regret [16, 17], allowing the comparator to change across time:

$$\text{Regret}_T(\mathcal{A}_{\mathcal{R}}(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T})) = \sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} [f_t(z_t) - f_t(z_i^*)], \quad z_i^* = \underset{z \in \mathcal{K}}{\text{argmin}} \left\{ \sum_{t=1}^T f_t(z) - \sum_{j=0}^i f_{u[j]}(z) \right\}. \quad (5)$$

Here, we define $\tau[k+1] = T$ and $\tau[0] = 0$. The term z_i^* is the best-in-hindsight estimator after the i^{th} and before the $(i+1)^{\text{th}}$ deletion, computed over $\{f_1, \dots, f_T\}$ with the first i deleted cost functions $\{f_{u[1]}, \dots, f_{u[i]}\}$ removed. This ensures both the learner and the comparator share the same history of cost functions, as in classical online learning setting.

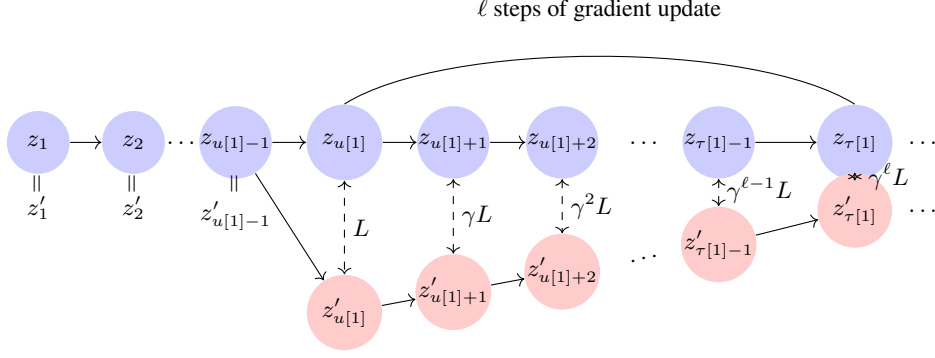


Figure 1: Visualization of the output sequence of algorithm \mathcal{A} up to the first deletion $\tau[1]$

$\gamma \in (0, 1]$ and $\Delta_t < \infty$, the update function g_t satisfies:

$$\textbf{Markovian Output:} \quad g_t(f_{1:t}, z_{1:t-1}) = g_t(f_t, z_{t-1}), \quad (\text{C1})$$

$$\gamma\text{-Contraction:} \quad \|g_t(f, z_1) - g_t(f, z_2)\|_2 \leq \gamma \|z_1 - z_2\|_2, \quad (\text{C2})$$

$$\Delta_{1:T}\text{-Sensitivity:} \quad \|g_t(f, x) - x\|_2 \leq \Delta_t. \quad (\text{C3})$$

Condition C1 ensures that update g_t depends only on the latest cost function and previous output, allowing it to be expressed as $g_t(f_t, z_{t-1})$ and simplifying subsequent conditions. The contraction property in Condition C2 has been central in privacy analyses of DP-SGD [20, 21], sampling [22], and generalisation bounds for SGD [23]. Recently, Chien et al. [13] utilised contraction in designing an *active* unlearning scheme for (noisy) SGD in the offline setting; to the best of our knowledge, our work is the first that applies this idea in an online framework of unlearning. Bounded sensitivity in Condition C3 requires that each single-step update does not lead to arbitrarily large changes in the model. This is a standard assumption that holds in many settings, as any continuous function defined on a bounded domain admits a finite sensitivity. Given these conditions, Theorem 1 proves that Algorithm 1 is an $(\alpha, \alpha\varepsilon)$ -OLU.

Theorem 1. *Let $\Delta_{1:T} < \infty$ and $\gamma \in (0, 1]$. If for all $t \in [T]$, the update function g_t of algorithm \mathcal{A} fulfills Conditions (C1), (C2) and (C3), then Algorithm 1 instantiated with \mathcal{A} is an $(\alpha, \alpha\varepsilon)$ -OLU.*

Proof Sketch. Consider a single deletion request at time $\tau[1]$ for the point that originally appears at time $u[1]$. Let $\ell = \tau[1] - u[1]$ denote the gap between the point's first inclusion and its requested removal. We compare two output sequences: (z_t) , the usual updates by the online learning algorithm \mathcal{A} on all cost functions, and (z'_t) , the updates when $f_{u[1]}$ is omitted. As illustrated in Figure 1, for $t < u[1]$, neither process has used $f_{u[1]}$, so $z_t = z'_t$. At $t = u[1]$, (z_t) takes one gradient step on $f_{u[1]}$ while (z'_t) does not, creating a maximum difference of $\Delta_{u[1]}$ (due to C3). For subsequent steps, both follow the same γ -contractive updates, shrinking their distance by a factor of $\gamma \leq 1$ each time. By $t = \tau[1]$, the distance is at most $\gamma^\ell \Delta_{u[1]}$. Injecting suitably calibrated Gaussian noise of scale proportional to $\gamma^\ell \Delta_{u[1]}$ makes the two processes statistically indistinguishable under Rényi divergence (see Lemma A), yielding the $(\alpha, \alpha\varepsilon)$ -OLU guarantee.

For subsequent deletions, we must track two sequences of random variables where each evolve according to a deterministic γ -contractive map with an added random noise term at specific time steps. Leveraging Lemma 4, an argument similar to the Privacy Amplification by Iteration in Feldman et al. [20], we ensure the guarantee holds after each deletion. See the full proof in Appendix B.1. \square

Finally, OGD (Equation 2) satisfies Conditions C1, C2, and C3 under standard smoothness and convexity assumptions. As a result, Algorithm 1 instantiated with OGD is an $(\alpha, \alpha\varepsilon)$ -OLU.

Corollary 1. *Assume each cost function f_t is β -smooth and convex. Then Algorithm 1 with OGD update step and learning rate $\eta \leq 2/\beta$ is an $(\alpha, \alpha\varepsilon)$ -OLU for $\gamma = 1$. If the cost functions are β -smooth and μ -strongly convex, then the same algorithm with $\eta \leq 1/(\beta + \mu)$ is an $(\alpha, \alpha\varepsilon)$ -OLU for $\gamma = \frac{\beta/\mu - 1}{\beta/\mu + 1}$.*

3.2 Regret guarantee

In this section, we analyze the regret of our passive unlearning algorithm with OGD as the base learner. First, we derive regret bounds under decreasing learning rates for strongly convex (Theorem 2) and convex (Theorem 3) cost functions, highlighting the role of the deletion index sets \mathcal{T}, \mathcal{U} . Next, Theorem 4 uses adaptive learning rates

to obtain regret bounds based on the decay of gradient norms. Finally, we show that a constant learning rate achieves a worst-case regret guarantee (Theorem 5) for any deletion schedule, provided the total number of steps and deletions are known. Table 1 summarizes the regret guarantees of Algorithm 1 under different learning rate schedules and assumptions.

Method	Learning Rate	Assumptions	Regret Guarantees
Decreasing	$\eta_t = \frac{1}{\mu t}$	Strongly convex (SC)	Theorem 2
	$\eta_t = \frac{D}{L\sqrt{t}}$	Convex + Quadratic Growth (QG)	Theorem 3
Adaptive	$\eta_t = \sqrt{\frac{D^2}{\sum_{i=1}^t \ \nabla f_i(z_i)\ _2^2}}$	Convex + Public Gradient Norms	Theorem 4
Constant	$\eta = \sqrt{\frac{2D^2}{TL^2(1 + \frac{1.2k^{2.2}d}{0.42\varepsilon})}}$	Knowledge of k, T ; QG + Convex/SC	Theorem 5

Table 1: Overview of assumptions and regret bounds using Algorithm 1 with different learning rates.

Decreasing learning rate In passive unlearning, the final regret increases with as the deleted point’s effect on subsequent outputs increases, since more noise must be added to obscure its impact. Two competing factors related to the time gap between the deletion request and the point’s initial occurrence dictate this effect: (i) the decreasing learning rate, which magnifies the impact of earlier points on future outputs, and (ii) the contractiveness of gradient descent, which reduces a point’s influence over time. Theorem 2 presents the regret of Algorithm 1 for strongly convex cost functions.

Theorem 2. Suppose \mathcal{U} (deletion indices) and \mathcal{T} (deletion times) are each of size k . If the cost functions are L -Lipschitz, β -smooth, and μ -strongly convex and $u[i] \geq \frac{1}{2} + \frac{\beta}{\mu}$, then for all $T \geq k$, Algorithm 1 with step size $\eta_t = 1/(\mu t)$ satisfies

$$\mathbb{E} \left[\text{Regret}(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T}) \right] = \frac{L^2}{\mu} \left(\log T + 2k^2 + \frac{\sqrt{3}dk^{1.7}}{\varepsilon} \mathcal{G}_1(\gamma, \mathcal{T}, \mathcal{U}) \right),$$

where $\gamma = \frac{\beta/\mu-1}{\beta/\mu+1}$ and $\mathcal{G}_1(\gamma, \mathcal{T}, \mathcal{U}) = \sqrt{\sum_{i=1}^k \tau[i]^2 \gamma^{4(\tau[i]-u[i])}}/u[i]^4$.

Here, the function \mathcal{G}_1 reflects the effect of deletion indices on the regret: $\tau[i]^2/u[i]^4$ captures the impact of the decreasing learning rate (larger when $u[i]$ is small), while $\gamma^{4(\tau[i]-u[i])}$ reflects contractiveness, which in the strongly convex case ($\gamma < 1$) is the dominant term in the expression. Therefore, each summand in \mathcal{G}_1 is of the order $O(1)$ for arbitrary \mathcal{T}, \mathcal{U} and we can get a loose regret bound of the order $O(\log T + k^2 + dk^{2.2}/\varepsilon)$ independent of the indices of deletion set.

Next, Theorem 3 addresses the case of convex but not necessarily strongly convex cost functions. Since the regret definition for the online learner-unlearner (Equation (5)) involves a changing comparator, we impose the Quadratic Growth (QG) assumption (Assumption 1)—a weaker assumption than strong convexity [24]—on each aggregate cost function $\sum_{t=1}^{\tau[i]} f_t^1$. The QG assumption on aggregate cost functions ensures that the comparator’s change after each unlearning is bounded.

Assumption 1 (Quadratic Growth). For any function F on \mathcal{K} , let $z^* = \arg\min_{z \in \mathcal{K}} F(z)$. Then F has quadratic growth with parameter κ if for all $z \in \mathcal{K}$,

$$F(z) - F(z^*) \geq \frac{\kappa}{2} \|z - z^*\|_2^2.$$

Theorem 3. Suppose \mathcal{U} and \mathcal{T} are each of size k . If f_1, \dots, f_T are L -Lipschitz, β -smooth convex cost functions such that for each i , $\sum_{t=1}^{\tau[i]} f_t$ satisfies Assumption 1 with parameter $\kappa(\tau[i] - \tau[i-1])$ and $u[i] \geq \frac{\beta^2 D^2}{4L^2}$, then for all $T \geq k$, Algorithm 1 with $\eta_t = \frac{D}{L\sqrt{t}}$ satisfies

$$\mathbb{E} \left[\text{Regret}(\mathcal{R}_{\mathcal{A}}(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T})) \right] \leq 3DL\sqrt{T} + \frac{2k^2 L^2}{\kappa} + \frac{3DLdk^{1.7}}{2\varepsilon} \mathcal{G}_2(\mathcal{U}, \mathcal{T}),$$

where $\mathcal{G}_2(\mathcal{T}, \mathcal{U}) = \sqrt{\sum_{i=1}^k \frac{\tau[i]}{u[i]^2}}$.

¹For the standard online learning problem, Theorem 3 in Chang et al. [24] implies that individual QG together with Assumption 2 yields an $O(\log T)$ regret bound, provided the learning rate decays rapidly ($\eta_t = O(1/t)$). When the loss functions are only convex, the OLU setting requires a slower decay of the learning rate ($\eta_t = O(1/\sqrt{t})$) to keep the unlearning overhead small. Therefore, individual QG no longer suffices to achieve logarithmic regret in the OLU setting.

As Theorem 3 shows, if the function is only convex, we have $\gamma = 1$, so the effect of the decreasing learning rate dominates over the contractiveness. Hence, deleting points that occur later (large $u[i]$) lead to smaller regret. In particular, if $\tau[i] = o(u[i]^2)$, $\mathcal{G}_2(\mathcal{T}, \mathcal{U})$ is constant and the regret remains $O(\sqrt{T})$, matching the bound achieved by OGD without unlearning.

The proof of Theorem 2 and 3 follows the standard regret analysis of OGD [25], with additional careful handling of the noise term. See Appendix B.2 for the full proof.

Adaptive learning rates If we assume that the gradient norms at each output is public information, one can use adaptive learning rates [26, 27]. Using this, Theorem 4 achieves a bound independent of \mathcal{T}, \mathcal{U} , but dependent on how quickly these norms decrease.

Theorem 4. *Let f_1, \dots, f_T be convex, L -Lipschitz and β -smooth and suppose \mathcal{U} and \mathcal{T} are each of size k . Define $p(t) = \sum_{i=1}^t \|\nabla f_i(z_i)\|_2^2$, if there exists some $u_0 \geq 1$ such that $p(u_0) \geq \beta^2/4$ and $u[i] \geq u_0$, then the expected regret of Algorithm 1 with adaptive learning rate $\eta_t = \frac{D}{\sqrt{p(t)}}$ is*

$$O \left(D^2\beta + D \sqrt{\sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} f_t(z_i^*)} + dk^2 L^2 D^2 \mathcal{G}_3(\mathcal{T}, \mathcal{U}, \mathcal{S}) \right),$$

where z_i^* is a best-in-hindsight solution after the k^{th} deletion, and $\mathcal{G}_3(\mathcal{T}, \mathcal{U}, \mathcal{S}) = \sqrt{\beta \sum_{i=1}^k \frac{p(\tau[i])}{p(u[i])^2}}$

The passive unlearner with an adaptive learning rate does not explicitly require $\tau[i] = O(u[i])$ and instead accounts for the algorithm's performance over time. Specifically, if the post-deletion gradients do not grow significantly i.e. $\mathcal{G}_3(\mathcal{T}, \mathcal{U}, \mathcal{S}) = O(\sqrt{T})$, the additional regret from unlearning stays $O(\sqrt{T})$. Furthermore, the second term, $D \sqrt{\sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} f_t(z_i^*)}$, depends on the best-in-hindsight estimator and is $O(\sqrt{T})$ in the worst case. However, it can yield a tighter bound when the best-in-hindsight estimator incurs a smaller loss [27]. Additionally, the adaptive method does not require prior knowledge of the Lipschitz constant of the cost functions.

Constant learning rate The preceding results depend on either \mathcal{T}, \mathcal{U} (in Theorem 2 and 3) or the gradient norms (in Theorem 4). Thus, unfavourable deletion sets or cost functions can lead to high regret well above the regret of their counterparts without unlearning.

In Theorem 5, we show that using a *constant* learning rate ensures $O(k^{1.1}\sqrt{T})$ regret uniformly over any deletion schedule for convex cost functions. While such a choice does not rely on the timing of deletions or gradient shrinkage, it does require knowledge of T and k beforehand and results in worse regret guarantee when the deletion schedule is favorable. In practice, this may be unreasonable or may necessitate a meta-strategy (e.g. doubling) to tune the constant learning rate.

Theorem 5. *For any \mathcal{U}, \mathcal{T} of size k , if the cost functions f_1, \dots, f_T are convex, L -Lipschitz, and β -smooth, and $\sum_{t=1}^{\tau[i]} f_t$ satisfies Assumption 1 with parameter $\kappa(\tau[i] - \tau[i-1])$, then there exists a constant step size η such that the expected regret of Algorithm 1 is*

$$L \left(D + k^{1.1} \sqrt{d/\varepsilon} \right) \sqrt{2T} + \frac{2L^2 k^2}{\kappa}.$$

4 Active Unlearning

The passive approach in Section 3 passively exploits OGD's properties without explicitly moving the current output toward the retrained solution. Analyzing deterministic OGD updates directly for unlearning can be difficult, and to the best of our knowledge, it has not been done before. Instead, we leverage the *descent-to-delete* method of Neel et al. [4], originally proposed for ERM, and integrate it into an active online learner-unlearner (OLU) (Algorithm 2).

Though designed for ERM rather than OGD, the descent-to-delete procedure can still bring our algorithm's output closer to the output that would arise from retraining on data excluding the deleted points, provided the current OGD output is not too far from the ERM minimizer. As shown in Theorem 6, this often allows the active unlearning algorithm to add less noise than the passive method, yielding improved regret bounds, as shown in Theorem 6.

Active OLU via Descent-to-Delete [4] Of the various ERM-based unlearning algorithms [4, 6, 8], we adapt the descent-to-delete approach of Neel et al. [4] thanks to its simplicity, computational efficiency, and certified unlearning guarantee.² In essence, this algorithm unlearns a set of points by running a few gradient-descent iterations

²Our framework also accommodates other ERM unlearning schemes, e.g. the Newton-based method of Sekhari et al. [6]; see Algorithm 3.

Algorithm	Assumptions	Regret	\mathcal{G} (Impact of deletion set)	Computation
Passive (Algorithm 1)	SC	$\log T + k^2 + \mathcal{G}$	$dk^{1.7} \sqrt{\sum_{i=1}^k \frac{\tau[i]^2 \gamma^4 (\tau[i] - u[i])}{u[i]^4}}$	1
	C + QG	$\sqrt{T} + k^2 + \mathcal{G}$	$dk^{1.7} \sqrt{\sum_{i=1}^k \frac{\tau[i]}{u[i]^2}}$	1
Active (Algorithm 2)	SC + 2	$\log T + k^2 + \mathcal{G}$	$\sum_{i=1}^k \gamma^{\Delta_\tau[i]} (\Delta_\tau[i])$	$\log \frac{1}{\gamma} \frac{k\mu D \tau[i]}{L}$
Discard-and-restart	SC	$k \log T + \mathcal{G}$	0	1
	C	$k\sqrt{T} + \mathcal{G}$	0	1
Online DP	SC	$dk \log^{2.5} T + \mathcal{G}$	0	$\log \tau[i]$
	C	$k\sqrt{dT \log^{2.5} T} + \mathcal{G}$	0	$\log \tau[i]$
Retraining	SC	$\log T + \mathcal{G}$	0	$\tau[i]$
	C	$\sqrt{T} + \mathcal{G}$	0	$\tau[i]$

Table 2: Comparison of regret and computation cost of different OLU: SC and C stands for strongly convex and convex setting, QG refers to Assumption 1 over aggregate cost functions, and 2 refers to Assumption 2. Computation cost is with respect to each unlearning step. $\Delta_\tau[i] = \tau[i] - \tau[i-1]$. All values are expressed in order terms, omitting constants and dependence on functional parameters like Lipschitzness for simplicity.

on the remaining data. Algorithm 2 combines OGD as the base learner with descent-to-delete as the deterministic unlearning function. The procedure alternates between two modes: during regular learning, it follows the OGD update rule. When a request arrives to delete a point $f_{u[i]}$, the algorithm first performs $\mathcal{I}_{1,i}$ gradient-descent steps on all previously seen cost functions, then runs \mathcal{I}_2 steps on all but the deleted points, and finally injects calibrated noise.

Compared with the original offline descent-to-delete algorithm, our online adaptation uses an additional $\mathcal{I}_{1,i}$ steps of gradient descent on *all* previously seen cost functions. In offline settings, the descent-to-delete procedure starts near the ERM solution of the retained set; but OGD outputs can be far from this solution, so the extra steps are required to bring the current model closer to the ERM solution. This shift reduces the noise needed to ensure unlearning and highlights an important caveat in adapting offline unlearning algorithms to online settings.

Because the unlearning procedure moves the model away from the pure OGD output toward an ERM solution, we require an additional assumption (Assumption 2) to control this shift. Intuitively, Assumption 2 ensures that each individual cost function’s minimizer is closely aligned with the overall ERM minimizer, so gradient descent over these functions sequentially pushes the model toward the global minimum.

Assumption 2. Let $\mathcal{T} = \{\tau[1], \dots, \tau[k]\}$, be the deletion times, $\tau[0] = 1$ and for each $1 \leq i \leq k$, define $z_i^* = \arg\min_z \sum_{t=1}^{\tau[i]} f_t(z)$. Then, for every i , there exists $a_i \in \mathcal{K}$ s.t. $\|a_i - z_i^*\| \leq \frac{1}{\tau[i]}$ and

$$\nabla f_t(a_i) = 0 \quad \text{for all } t \in (\tau[i-1], \tau[i]).$$

Under this assumption, the active OLU in Algorithm 2 achieves $O(\log T)$ regret *independently* of the deletion indices \mathcal{U} but dependent on the deletion time \mathcal{T} , as stated in Theorem 6.

Theorem 6. Let \mathcal{U} and \mathcal{T} each have size k . For all i , assume $\tau[i-1] \leq u[i] \leq \tau[i]$. Suppose each f_t is L -Lipschitz, μ -strongly convex, and β -smooth. If the number of gradient-descent steps on all previously seen points, $\mathcal{I}_{1,i}$ is at least $\log \frac{1}{\gamma} \frac{\mu D \tau[i]}{L}$ and $\mathcal{I}_2 \geq 2.2 \log \frac{1}{\gamma} k$, then Algorithm 2 is an $(\alpha, \alpha\epsilon)$ -OLU. Moreover, if Assumption 2 holds, the regret of Algorithm 2 is

$$O\left(\log T + k\left(LD^2 + \frac{LD}{\mu\epsilon}\right) + \mathcal{G}_2(\mathcal{T}, \gamma) + \frac{L^2 k^2}{\mu}\right),$$

where $\gamma = \frac{\beta/\mu-1}{\beta/\mu+1}$ and $\mathcal{G}_2 = \sum_{i=1}^k \gamma^{\tau[i]-\tau[i-1]} (\tau[i] - \tau[i-1])$.

5 Discussion and Open Problems

Table 2 compares our proposed unlearning algorithms to several baselines in terms of regret and per-deletion computational cost. Two naive baselines are *retraining from scratch*, which attains the best possible regret at $O(\tau[i])$ cost per deletion, and *discard-and-restart*, which reinitializes OGD after every deletion and trivially achieves $(\alpha, 0)$ -online unlearning (Definition 2). We also compare with the DP-online algorithm of Guha Thakurta and

Smith [18] as a baseline despite it incurring at least $O(\log t)$ computation per step for both learning and unlearning. We use the term \mathcal{G} in Table 2 to specifically show how the nature of deletion requests (i.e. \mathcal{T}, \mathcal{U}) affects the regret guarantees.

In the strongly convex setting with $O(1)$ computational overhead per unlearning, our passive OLU algorithm (Algorithm 1) nearly matches the regret of retraining, with the term \mathcal{G} decreasing exponentially in $\tau[i] - u[i]$ i.e. the time between learning and deletion. However, its performance can worsen in the convex setting if the adversary strategically selects earlier points for deletion i.e. small $u[i]$. In fact, unlike the strongly convex case, in the convex setting the additional term \mathcal{G} does not vanish exponentially fast. By allowing slightly more computation, $O(\log \tau[i])$ at each deletion and assuming OGD converges to the ERM solution, the *active* approach (Algorithm 2) can attain a regret bound that essentially matches retraining for strongly convex losses, with the term \mathcal{G} decreasing exponentially with $\tau[i] - \tau[i - 1]$ i.e. time between deletion requests. Both methods improve over the DP-based algorithm by a factor of d as well as a worse polylogarithmic dependence on T (since unlearning imposes privacy only on deleted data, unlike DP, which covers all points), and they also outperform the discard-and-restart baseline by a factor of k .

Although deriving a lower bound on regret in terms of T would be valuable, we defer it to future work, since such a bound requires restricting the computation complexity of the unlearning algorithms. Without such constraint, FTL-type algorithms can achieve exact unlearning with the same regret as in the standard online learning setting. [28] explores lower bounds on the space complexity of the exact unlearning in the offline setting, but it is orthogonal to establishing a regret lower bound.

Compared to offline unlearning algorithms [4, 6, 12, 13], our method requires weaker assumptions: while most offline approaches rely on strong convexity to establish unlearning guarantees, our passive OLU achieves this under merely convex losses (Corollary 1). However, we cannot fairly compare the accuracy of our algorithm to that of an offline method using online-to-batch conversion, because such conversions average over all past outputs, including those generated before the deleted point was removed, and therefore do not necessarily satisfy the unlearning guarantee.

To conclude, our unlearning methods offer preferable trade-offs between computational efficiency and regret guarantees in the online setting. Several open problems remain including setting lower bounds in this problem, designing more efficient active OLU algorithms that do not rely on strong convexity, and whether more unlearning friendly online learning algorithms can be designed.

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A Omitted Proofs for Section 2

Definition 3. A function $f : \mathcal{X} \rightarrow \mathcal{Y}$, is L -Lipschitz if the following hold for all $x, y \in \mathcal{X}$,

$$\|f(x) - f(y)\|_2 \leq L \|x - y\|_2.$$

f is called μ -strongly convex if for all $x, y \in \mathcal{X}$,

$$f(x) \geq f(y) + (\nabla f(y))^\top (x - y) + \frac{\mu}{2} \|x - y\|_2^2.$$

f is called β -smooth if for all $x, y \in \mathcal{X}$,

$$f(x) \leq f(y) + (\nabla f(y))^\top (x - y) + \frac{\beta}{2} \|x - y\|_2^2.$$

Definition 4. For any two random variables P, Q with corresponding distributions μ_P, μ_Q respectively, and for any positive value $\alpha > 0, \alpha \neq 1$, the Rényi divergence between these two distributions is defined as

$$D_\alpha(\mu_P \| \mu_Q) = \frac{1}{\alpha - 1} \log \int \mu_P(x)^\alpha \mu_Q(x)^{1-\alpha} dx.$$

For simplicity, we sometimes write $D_\alpha(P \| Q) = D_\alpha(\mu_P \| \mu_Q)$.

Proposition 1. For $k \in \mathbb{N}$, any (α, ε) -RDP online learning algorithm is an $(\alpha/k, k^{1.6}\varepsilon)$ -OLU for any deletion set U and deletion-time set \mathcal{T} of size k , if $\alpha \geq 2k$.

Proof. Let the set of cost functions up to $\tau[i]$, with and without the deleted points indexed by \mathcal{U} , be denoted by \mathcal{S}_i and \mathcal{S}'_i respectively, i.e.

$$\mathcal{S}_i = \{f_1, \dots, f_{\tau[i]}\}, \quad \mathcal{S}'_i = \{f_1, \dots, f_{\tau[i]}\} \setminus \{f_{u[j]}\}_{j=1}^i.$$

Then, for any $i \in \{1, \dots, k\}$, the number of points that \mathcal{S}_i and \mathcal{S}'_i differ at is upper bounded by k . As \mathcal{A} is an online learning algorithm that is (α, ε) -RDP, applying Lemma 2 on the dataset \mathcal{S}_i and \mathcal{S}'_i , we have

$$D_{\frac{\alpha}{k}}(\mathcal{A}(\mathcal{S}_i) \| \mathcal{A}(\mathcal{S}'_i)) \leq k^{1.6}\varepsilon.$$

Lemma 1 (Proposition 2 in Mironov [29]). If an algorithm \mathcal{A} is (α, ε) -RDP and if $\alpha \geq 2k$, then for any two dataset S, S' differing by at most k element,

$$D_{\frac{\alpha}{k}}(\mathcal{A}(S) \| \mathcal{A}(S')) \leq k^{1.6}\varepsilon$$

This concludes the proof. □

B Omitted Proofs for Section 3

B.1 Unlearning guarantee of passive unlearning

We denote Euclidean norm by $\|\cdot\|$ or $\|\cdot\|_2$.

Proposition 1. For $k \in \mathbb{N}$, any (α, ε) -RDP online learning algorithm is an $(\alpha/k, k^{1.6}\varepsilon)$ -OLU for any deletion set U and deletion-time set \mathcal{T} of size k , if $\alpha \geq 2k$.

Proof. Let the set of cost functions up to $\tau[i]$, with and without the deleted points in \mathcal{U} , be denoted by \mathcal{S}_i and \mathcal{S}'_i respectively, i.e.

$$\mathcal{S}_i = \{f_1, \dots, f_{\tau[i]}\}, \quad \mathcal{S}'_i = \{f_1, \dots, f_{\tau[i]}\} \setminus \{f_{u[j]}\}_{j=1}^i.$$

Then, for any $i \in \{1, \dots, k\}$, the number of points that \mathcal{S}_i and \mathcal{S}'_i differ at is upper bounded by k . As \mathcal{A} is an online learning algorithm that is $(k\alpha, \varepsilon)$ -RDP, applying Lemma 2 on the dataset \mathcal{S}_i and \mathcal{S}'_i , we have

$$D_\alpha(\mathcal{A}(\mathcal{S}_i) \| \mathcal{A}(\mathcal{S}'_i)) \leq k^{1.6}\varepsilon.$$

Lemma 2 (Proposition 2 in Mironov [29]). *If an algorithm \mathcal{A} is (α, ε) -RDP and if $\alpha \geq 2k$, then for any two dataset S, S' differing by at most k element,*

$$D_{\frac{\alpha}{k}}(\mathcal{A}(S) \parallel \mathcal{A}(S')) \leq k^{1.6} \varepsilon$$

This concludes the proof. \square

Definition 5 (shifted Rényi divergence). *Let μ, ν be two distributions. For parameters $e \geq 0$ and $\alpha \geq 1$, the e -shifted Rényi divergence between μ and ν is defined as*

$$D_{\alpha}^{(e)}(\mu \parallel \nu) = \inf_{\mu' : W_{\infty}(\mu, \mu') \leq e} D_{\alpha}(\mu' \parallel \nu),$$

where W_{∞} represents ∞ -Wasserstein distance.

Shifted Rényi divergence satisfies monotonicity, i.e. For $0 \leq e \leq e'$, $D_{\alpha}^{(e')}(\mu \parallel \nu) \leq D_{\alpha}^{(e)}(\mu \parallel \nu)$. For a distribution ζ and a vector x , we let $\zeta * x$ denote the distribution of $\eta + x$ where $\eta \sim \zeta$. We define

$$R_{\alpha}(\zeta, \alpha) = \sup_{x : \|x\| \leq a} D_{\alpha}(\zeta * x \parallel \zeta).$$

Definition 6 (Contractive Noise Iteration (CNI), Feldman et al. [20]). *Given an initial random state $X_0 \in \mathcal{Z}$, a sequence of contractive functions $\psi_t : \mathcal{Z} \rightarrow \mathcal{Z}$, and a sequence of noise distribution $\{\zeta_t\}$, we define the Contractive Noisy Iteration (CNI) by the following update rule:*

$$X_{t+1} = \psi_{t+1}(X_t) + \xi_{t+1},$$

where ξ_{t+1} is drawn independently from ζ_{t+1} . We denote the random variable output by this process after T steps as $CNI_T(X_0, \{\psi_t\}, \{\zeta_t\})$.

Theorem 1. *Let $\Delta_{1:T} < \infty$ and $\gamma \in (0, 1]$. If for all $t \in [T]$, the update function g_t of algorithm \mathcal{A} fulfills Conditions (C1), (C2) and (C3), then Algorithm 1 instantiated with \mathcal{A} is an $(\alpha, \alpha\varepsilon)$ -OLU.*

Proof. Let $\mathcal{S} = \{f_1, \dots, f_T\}$ be the set of cost functions given to the learner over time, $\mathcal{S}^{\mathcal{U}}$ be the set of deleted points with index in $\mathcal{U} = \{u[1], \dots, u[k]\}$. Let $\mathcal{T} = \{\tau[1], \dots, \tau[k]\}$ be the set of deletion times. Let $\mathcal{S}' = \{f'_1, \dots, f'_T\}$ with $f'_t = f_t$ at $t \notin \mathcal{T}$ and $f'_t = \perp$ at $t \in \mathcal{T}$.

We note that the output of Algorithm 1 at time t is a CNI with a sequence of update functions g_1, \dots, g_t , the noise distribution ζ_t is the Dirac delta distribution at 0 when there is no deletion request, $t \notin \mathcal{T}$, and $\zeta_t = \mathcal{N}(0, \sigma_i^2)$ for the $t = \tau[i]$.

The proof follows an application of Lemma 3, a more general version of Theorem 22 in [20]. Compared with Theorem 22 in their original paper, Lemma 3 leverages the fact that the contractive coefficient γ is sometimes strictly less than 1, allowing us to achieve the same guarantee in terms of Rényi divergence by adding less noise.

Lemma 3 (Privacy amplification by iteration). *Let X_T and X'_T denote the output of $CNI_T(X_0, \{\psi_t\}, \{\zeta_t\})$ and $CNI_T(X_0, \{\psi'_t\}, \{\zeta_t\})$, where ψ_t, ψ'_t have contractive coefficient γ . Let $s_t = \sup_x \|\psi_t(x) - \psi'_t(x)\|$. Let a_1, \dots, a_T be a sequence of reals and let $e_t = \sum_{i=1}^t \gamma^{t-i}(s_i - a_i)$ such that $e_t \geq 0$ for all t , then*

$$D_{\alpha}^{(e_T)}(X_T \parallel X'_T) \leq \sum_{t=1}^T R_{\alpha}(\zeta_t, a_t).$$

Let $\psi_t(z) = g_t(f_t, z)$ and $\psi'_t(z) = g_t(f'_t, z)$ represents the update function with the t th cost functions from \mathcal{S} and \mathcal{S}' respectively. Then,

$$s_t = \sup_{z \in \mathcal{K}} \|g(f_t, z) - g(f'_t, z)\| = \sup_{z \in \mathcal{K}} \|\psi_t(z) - \psi'_t(z)\|.$$

As all update functions g_t are Δ_t -bounded (Condition C3), we can compute the value of s_t for all $t \in \{1, \dots, T\}$,

$$s_t = \begin{cases} \Delta_t & t \in \{u[1], \dots, u[k]\} \\ 0 & \text{otherwise} \end{cases}$$

Next, we select the sequence a_1, \dots, a_T such that $R_{\alpha}(\zeta_t, a_t)$ are bounded and $e_t = \sum_{i=1}^t \gamma^{t-i}(s_i - a_i) \geq 0$ holds for all t . By definition of our algorithm (Algorithm 1), the noise is only added at steps $t \in \mathcal{T}$. Therefore,

we need to set $a_t = 0$ for all $t \notin \mathcal{T}$ to avoid unbounded $R_\alpha(\zeta_t, a_t)$ when ζ_t is a Dirac delta distribution at 0. Additionally, for $i \in \{1, \dots, k\}$, we set $a_{\tau[i]} = \gamma^{\tau[i]-u[i]} \Delta_{u[i]}$, i.e.

$$a_t = \begin{cases} \gamma^{\tau[i]-u[i]} \Delta_{u[i]} & \text{if } t = \tau[i], i \in \{1, \dots, k\} \\ 0 & \text{otherwise} \end{cases}. \quad (7)$$

This ensures $e_{\tau[i]} = 0$ for all $\tau[i] \in \mathcal{T}$ and $e_t \geq 0$ for all t .

For Algorithm 1 to satisfy the unlearning guarantee, it suffices to ensure the following indistinguishability condition holds at time $\tau[i]$. Specifically, for each $i \in \{1, \dots, k\}$, we require that the Rényi divergence between the outputs of the two CNIs at step $\tau[i]$ are bounded by ε according to Definition 2, i.e.,

$$D_\alpha(X_{\tau[i]} \| X'_{\tau[i]}) \leq \alpha \varepsilon.$$

To prove this, we apply Lemma 3 with the sequence a_t selected above in Equation (7): for all $\tau[i] \in \mathcal{T}$, where $i \leq k$,

$$\begin{aligned} D_\alpha(X_{\tau[i]} \| X'_{\tau[i]}) &\leq \sum_{j=1}^i R_\alpha(\zeta_{\tau[j]}, a_{\tau[j]}) \\ &\stackrel{(a)}{=} \sum_{j=1}^i \frac{\alpha(a_{\tau[j]}^2)}{2\sigma_j^2} \stackrel{(b)}{=} \sum_{j=1}^i \frac{\alpha \varepsilon \omega - 1}{j^\omega \omega} \stackrel{(c)}{\leq} \alpha \varepsilon \end{aligned} \quad (8)$$

where step (a) follows from Lemma A, step (b) follows by the definition of $\sigma_j^2 = \frac{j^\omega \omega (\gamma^{\tau[j]-u[j]} \Delta_{u[j]})^2}{2(\omega-1)\varepsilon}$ in our algorithm, and step (c) follows

$$\sum_{j=1}^i \frac{1}{j^\omega} = 1 + \sum_{j=2}^i \frac{1}{j^\omega} \leq 1 + \int_1^\infty \frac{1}{x^\omega} dx = 1 + \frac{1}{\omega-1} = \frac{\omega}{\omega-1}.$$

Lemma A (Corrolary 3 in Mironov [29]). *For any two Gaussian distributions of dimension d with the same variance $\sigma^2 I_d$ but different means μ_0, μ_1 , denoted by $\mathcal{N}(\mu_0, \sigma^2 I_d)$ and $\mathcal{N}(\mu_1, \sigma^2 I_d)$, the following holds,*

$$D_\alpha(\mathcal{N}(\mu_0, \sigma^2 I_d) \| \mathcal{N}(\mu_1, \sigma^2 I_d)) \leq \frac{\alpha \|\mu_0 - \mu_1\|^2}{2\sigma^2}.$$

Then the same guarantee in Equation (8) extends to the output sequence between $t \in (\tau[i], \tau[i+1])$ by post-processing property of Rényi divergence (Lemma B). Specifically, we consider a $(\tau[i+1] - 1 - \tau[i])$ -dimensional post-processing function $(I, \psi_{\tau[i]+1}, \psi_{\tau[i]+2} \circ \psi_{\tau[i]+1}, \dots, \psi_{\tau[i+1]-1} \circ \dots \circ \psi_{\tau[i]+1})$. Applying Lemma B, we have $D_\alpha(X_{\tau[i]:\tau[i+1]-1} \| X'_{\tau[i]:\tau[i+1]-1}) \leq \alpha \varepsilon$ as desired.

Lemma B (Mironov [29]). *For any Rényi parameter $\alpha \geq 1$, any (possibly random) function h , and any two random variable P, Q with corresponding distributions μ_P, μ_Q ,*

$$D_\alpha(h(P) \| h(Q)) \leq D_\alpha(P \| Q).$$

□

Proof of Lemma 3. The proof is by induction and similar to the original proof of Theorem 22 in [20].

Let X_t and X'_t denote the t 'th iteration of $CNI(X_0, \{\psi_t\}, \{\zeta_t\})$ and $CNI(X_0, \{\psi'_t\}, \{\zeta_t\})$ respectively. For each $t \leq T$, our goal is to show the following equation holds,

$$D_\alpha^{(e_t)}(X_t \| X'_t) \leq \sum_{i=1}^t R_\alpha(\zeta_i, a_i).$$

The base case follows by the definition that $e_0 = 0$ and $X_0 = X'_0$. For the induction step, let ξ_{t+1} denote the random variable drawn from ζ_{t+1} ,

$$\begin{aligned} D_\alpha^{(e_{t+1})}(X_{t+1} \| X'_{t+1}) &= D_\alpha^{(e_{t+1})}(\psi_{t+1}(X_t) + \xi_{t+1} \| \psi'_{t+1}(X'_t) + \xi_{t+1}) \\ &\stackrel{(a)}{\leq} D_\alpha^{(e_{t+1}+a_{t+1})}(\psi_{t+1}(X_t) \| \psi_{t+1}(X'_t)) + R_\alpha(\zeta_{t+1}, a_{t+1}) \\ &\stackrel{(b)}{\leq} D_\alpha^{(\gamma e_t + s_{t+1})}(\psi_{t+1}(X_t) \| \psi_{t+1}(X'_t)) + R_\alpha(\zeta_{t+1}, a_{t+1}) \\ &\stackrel{(c)}{\leq} D_\alpha^{(e_t)}(X_t \| X'_t) + R_\alpha(\zeta_{t+1}, a_{t+1}) \stackrel{(d)}{\leq} \sum_{i=1}^{t+1} R_\alpha(\zeta_i, a_i), \end{aligned} \quad (9)$$

where step (a) is due to Lemma C, step (b) is due to definition of $e_t = \sum_{i=1}^t \gamma^{t-i} (s_i - a_i)$, i.e. $e_t = \frac{e_{t+1} + a_{t+1} - s_{t+1}}{\gamma}$, step (c) follows from Lemma 4, a modification of privacy amplification of contraction (Lemma 21 in [20]), and step (d) follows by the induction hypothesis.

Lemma C (Shift-reduction lemma [20]). *Let $\mu * \zeta$ denote the distribution of $X + Y$ where $X \sim \mu, Y \sim \zeta$. Let μ, ν and ζ be distributions. Then, for any $\alpha \geq 0$,*

$$D_{\alpha}^{(e)}(\mu * \zeta \| \nu * \zeta) \leq D_{\alpha}^{(e+a)}(\mu \| \nu) + R_{\alpha}(\zeta, a),$$

where $R_{\alpha}(\zeta, a) = \sup_{x: \|x\| \leq a} D_{\alpha}(\zeta * x \| \zeta)$.

Lemma 4. *Suppose ψ, ψ' are contractive maps with coefficient γ and $\sup_x \|\psi(x) - \psi'(x)\| \leq s$. Then, for r.v. X and X' ,*

$$D_{\alpha}^{(\gamma e + s)}(\psi(X) \| \psi'(X)) \leq D_{\alpha}^{(e)}(X \| X').$$

□

Proof of Lemma 4. The proof follows from that of Lemma 21 in [20], by removing the step where the contractive coefficient γ is upper bounded by 1, and writing shift coefficient explicitly in terms of the contractive coefficient γ . We repeat their proof here for completeness.

By definition of $D_{\alpha}^{(e)}(\cdot \| \cdot)$, there exists a joint distribution (X, Y) such that

$$D_{\alpha}(Y \| X') = D_{\alpha}^{(e)}(X \| X') \text{ and } \mathbb{P}(\|X - Y\| \leq e) = 1.$$

By the post processing properties of Rényi divergence (Lemma B),

$$D_{\alpha}(\psi'(Y) \| \psi'(X')) \leq D_{\alpha}(Y \| X') = D_{\alpha}^{(e)}(X \| X').$$

Moreover,

$$\begin{aligned} \|\psi(X) - \psi'(Y)\| &\leq \|\psi(X) - \psi(Y)\| + \|\psi(Y) - \psi'(Y)\| \\ &\stackrel{(a)}{\leq} \gamma \|X - Y\| + s \leq \gamma e + s. \end{aligned} \tag{10}$$

where step (a) follows by the definition of contractive maps.

This shows that $(\psi(X), \psi'(Y))$ is a coupling establishing the claimed upper bound on $D_{\alpha}^{(\gamma e + s)}(\psi(X) \| \psi'(Y))$, and concludes the proof. □

Proof of Corollary 1. Our result follows from the contractive properties of projected gradient descent. For standard gradient descent without projection, their contractive parameters are provided in Lemma D.

Lemma D (Proposition 18 [20]; Lemma 2.2 [22]). *Gradient descent step is contractive if the loss function is smooth or strongly convex.*

- Suppose a function $\ell : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex, twice differentiable, and M -smooth. Then the function ψ defined as $\psi(w) = w - \eta \nabla \ell(w)$ is contractive with parameter $(1 - \eta M)$ for $\eta \leq 2/M$.

$$\|\psi(w) - \psi(w')\| \leq \beta \|w - w'\|$$

- Suppose ℓ is an m -strongly convex and M -smooth function for $0 < m \leq M < \infty$. For step size $\eta = \frac{2}{M+m}$, then ψ is contractive with parameter $\frac{M/m-1}{M/m+1}$.

Since the projection operation is contractive, applying it after each gradient update preserves the contractive nature of gradient descent. Therefore, each step of projected online gradient descent remains contractive with the same coefficient as in the case of gradient descent without projection. This completes the proof. □

B.2 Regret guarantee of passive unlearning

In this section, we present the proof of regret guarantees for the passive unlearning algorithm.

Theorem 2. *Suppose \mathcal{U} (deletion indices) and \mathcal{T} (deletion times) are each of size k . If the cost functions are L -Lipschitz, β -smooth, and μ -strongly convex and $u[i] \geq \frac{1}{2} + \frac{\beta}{\mu}$, then for all $T \geq k$, Algorithm 1 with step size $\eta_t = 1/(\mu t)$ satisfies*

$$\mathbb{E} \left[\text{Regret}_T(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T}) \right] = \frac{L^2}{\mu} \left(\log T + 2k^2 + \frac{\sqrt{3}dk^{1.7}}{\varepsilon} \mathcal{G}_1(\gamma, \mathcal{T}, \mathcal{U}) \right),$$

where $\gamma = \frac{\beta/\mu-1}{\beta/\mu+1}$ and $\mathcal{G}_1(\gamma, \mathcal{T}, \mathcal{U}) = \sqrt{\sum_{i=1}^k \tau[i]^2 \gamma^{4(\tau[i]-u[i])} / u[i]^4}$.

Proof. Recall that $\mathcal{T} = \{\tau[1], \dots, \tau[k]\}$ represents the set of deletion times, and $\mathcal{U} = \{u[1], \dots, u[k]\}$ represents the corresponding index of the points to be deleted. For a set of cost functions $\mathcal{S} = \{f_1, \dots, f_T\}$, for any $i \in \{1, \dots, k\}$, let z_i^* be the best-in-hindsight estimator after the i th deletion, i.e.

$$z_i^* = \arg \min_{z \in \mathcal{K}} \sum_{t=1}^T f_t(z) - \sum_{j=1}^i f_{u[j]}(z).$$

For the first part of the analysis, we consider the constant best-in-hindsight estimator

$$z^* = \arg \min_{z \in \mathcal{K}} \sum_{t=1}^{\tau[i]} f_t(z).$$

By the updating rule of Algorithm 1, for $t+1 \notin \mathcal{T}$,

$$\begin{aligned} \|z_{t+1} - z^*\|^2 &= \|\Pi_{\mathcal{K}}[z_t - \eta_t \nabla f_t(z_t) - z^*]\|^2 \\ &\leq \|z_t - z^*\|^2 + \eta_t^2 \|\nabla f_t(z_t)\|^2 - 2\eta_t (\nabla f_t(z_t))^\top (z_t - z^*). \end{aligned} \quad (11)$$

If there exists i such that $t+1 = \tau[i] \in \mathcal{T}$,

$$\begin{aligned} \|z_{t+1} - z^*\|^2 &= \|\Pi_{\mathcal{K}}[z_t - \eta_t \nabla f_t(z_t) + \xi_i - z^*]\|^2 \\ &\leq \|z_t - z^* + \xi_i\|^2 + \eta_t^2 \|\nabla f_t(z_t)\|^2 - 2\eta_t (\nabla f_t(z_t))^\top (z_t - z^* + \xi_i). \end{aligned} \quad (12)$$

Rearrange Equation (11) and Equation (12), we have

$$(\nabla f_t(z_t))^\top (z_t - z_i^*) \leq \begin{cases} \frac{\|z_t - z^*\|^2 - \|z_{t+1} - z^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} & t+1 \notin \mathcal{T} \\ -\nabla f_t(z_t)^\top \xi_i + \frac{\|z_t - z^* + \xi_i\|^2 - \|z_{t+1} - z^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} & t+1 = \tau[i] \in \mathcal{T} \end{cases} \quad (13)$$

As the loss functions are μ -strongly convexity and by the definition of strong convexity (Definition 3),

$$f_t(z_t) - f_t(z^*) \leq (\nabla f_t(z_t))^\top (z_t - z^*) - \frac{\mu}{2} \|z_t - z^*\|^2.$$

Summing over $t \in \{1, \dots, T\}$ and substituting Equation (13) into the equation,

$$\begin{aligned} \text{Regret}_T(\mathcal{R}_{\mathcal{A}}(\mathcal{S}, \emptyset, \mathcal{T})) &= \sum_{t=1}^T f_t(z_t) - f_t(z^*) \\ &\leq \sum_{t=1}^T (\nabla f_t(z_t))^\top (z_t - z^*) - \frac{\mu}{2} \|z_t - z^*\|^2 \\ &\stackrel{(a)}{=} \underbrace{\sum_{t=1}^T \frac{\|z_t - z^*\|^2 - \|z_{t+1} - z^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2}}_A - \frac{\mu}{2} \|z_t - z^*\|^2 \\ &\quad + \underbrace{\sum_{i=1}^k -(\nabla f_{\tau[i]-1}(z_{\tau[i]-1}))^\top \xi_i + \frac{2(z_{\tau[i]-1} - z^*)^\top \xi_i + \|\xi_i\|^2}{2\eta_{\tau[i]-1}}}_B \end{aligned}$$

where step (a) follows by substituting Equation (13).

This way, we then decompose the expected regret into two parts. In the following, we derive separate upper bounds for each.

$$\mathbb{E} \left[\text{Regret}_T(\mathcal{R}_{\mathcal{A}}(\mathcal{S}, \emptyset, \mathcal{T})) \right] = \mathbb{E}_{\xi_{1:k}} [A + B] = \mathbb{E}_{\xi_{1:k}} [A | \xi_{1:k}] + \mathbb{E}_{\xi_{1:k}} [B] \quad (14)$$

We first bound the term $\mathbb{E}[A | \xi_{1:k}]$. Given $\xi_{1:k}, z_1, \dots, z_T$ are deterministic, we get

$$\begin{aligned} \mathbb{E}[A | \xi_{1:k}] &= \sum_{t=1}^T \frac{\|z_t - z^*\|^2 - \|z_{t+1} - z^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} - \frac{\gamma}{2} \|z_t - z^*\|^2 \\ &= \mathbb{E} \left[\sum_{t=1}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} - \gamma \right) \|z_t - z^*\|^2 + \sum_{t=1}^T \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} \right] \\ &\stackrel{(a)}{=} \sum_{t=1}^T \frac{\|\nabla f_t(z_t)\|^2}{2\mu t} \stackrel{(b)}{\leq} \frac{L^2}{\mu} (1 + \log T) \end{aligned}$$

where step (a) is due to the definition of learning rate η_t , i.e. $\eta_t = \frac{1}{\mu t}$ and $\frac{1}{\eta_0} = 0$, and step (b) is due to the Lipschitzness of the cost function f_t and that $\sum_{t=1}^T \frac{1}{t} \leq 1 + \log T$.

Thus,

$$\mathbb{E}_{\xi_{1:k}} [A] = \mathbb{E}_{\xi_{1:k}} \mathbb{E} [A | \xi_{1:k}] \leq \frac{L^2}{\mu} (1 + \log T) \quad (15)$$

It remains to bound $\mathbb{E}_{\xi_{1:k}} [B]$,

$$\begin{aligned} \mathbb{E}_{\xi_{1:k}} [B] &= \mathbb{E} \left[\sum_{i=1}^k -(\nabla f_{\tau[i]-1}(z_{\tau[i]-1}))^\top \xi_i + \frac{2(z_{\tau[i]-1} - z_i^*)^\top \xi_i + \|\xi_i\|^2}{2\eta_{\tau[i]-1}} \right] \\ &\stackrel{(a)}{=} \mathbb{E} \left[\sum_{i=1}^k \frac{\|\xi_i\|^2}{2\eta_{\tau[i]-1}} \right] = \sum_{i=1}^k \frac{\mathbb{E} [\|\xi_i\|^2]}{2\eta_{\tau[i]-1}} \\ &\stackrel{(b)}{=} \sum_{i=1}^k \frac{d\sigma_i^2}{2\eta_{\tau[i]-1}} \stackrel{(c)}{=} \frac{d\omega L^2}{4\varepsilon\mu(\omega-1)} \sum_{i=1}^k \frac{(\tau[i]-1)}{u[i]^2} \gamma^{2(\tau[i]-u[i])} i^\omega \end{aligned} \quad (16)$$

where step (a) follows from the fact that ξ_i are independent and have zero mean. Step (b) holds because $\mathbb{E}(\|\xi_i\|^2) = d\sigma_i^2$ for $\xi_i \sim \mathcal{N}(0, \sigma_i^2 I_d)$. Step (c) follows by substituting $\sigma_i^2 = \frac{i^\omega \omega \alpha \gamma^{2(\tau[i]-u[i])} \eta_{u[i]}^2 L^2}{2\varepsilon(\omega-1)}$, $\eta_{u[i]} = \frac{1}{u[i]\mu}$, along with $\eta_{\tau[i]} = \frac{1}{\tau[i]\mu}$.

To simplify the term $\sum_{i=1}^k \frac{(\tau[i]-1)}{u[i]^2} \gamma^{2(\tau[i]-u[i])} i^\omega$, we apply Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{i=1}^k \frac{(\tau[i]-1)}{u[i]^2} \gamma^{2(\tau[i]-u[i])} i^\omega &\leq \sqrt{\sum_{i=1}^k \left(\frac{(\tau[i]-1)}{u[i]^2} \gamma^{2(\tau[i]-u[i])} \right)^2} \sqrt{\sum_{i=1}^k i^{2\omega}} \\ &\leq \sqrt{\sum_{i=1}^k \left(\frac{(\tau[i]-1)}{u[i]^2} \gamma^{2(\tau[i]-u[i])} \right)^2} \frac{k^{\omega+0.5}}{\sqrt{2\omega+1}} \\ &\leq \frac{k^{\omega+0.5}}{\sqrt{3}} \sqrt{\sum_{i=1}^k \left(\frac{(\tau[i]-1)}{u[i]^2} \gamma^{2(\tau[i]-u[i])} \right)^2} \end{aligned} \quad (17)$$

where the last inequality follows from $\omega > 1$.

Combining Equations (14), (16), (17) and (25), we have

$$\mathbb{E} \left[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \mathcal{S}^\mathcal{U}, \mathcal{T})) \right] \leq \frac{L^2}{\mu} (1 + \log T) + \frac{d\omega k^{\omega+0.5} L^2}{2\sqrt{3}\varepsilon\mu(\omega-1)} \sqrt{\sum_{i=1}^k \left(\frac{\tau[i]}{u[i]^2} \gamma^{2(\tau[i]-u[i])} \right)^2} \quad (18)$$

We note that up to this point in the proof, we have been computing the regret of our algorithm with respect to a constant best-in-hindsight estimator z^* . However, this differs from our regret definition in Equation (5).

Recall that $\mathcal{S}^\mathcal{U}$ is the set of points deleted by the algorithm with index in \mathcal{U} . In the following, we bound the difference between these two regret measures: $\mathbb{E}[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \emptyset, \mathcal{T}))]$, which corresponds to the regret with a constant comparator, and $\mathbb{E}[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \mathcal{S}^\mathcal{U}, \mathcal{T}))]$, which corresponds to our definition.

$$\begin{aligned} \left\| \mathbb{E} \left[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \emptyset, \mathcal{T})) \right] - \mathbb{E} \left[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \mathcal{S}^\mathcal{U}, \mathcal{T})) \right] \right\| &= \left\| \sum_{i=1}^k \sum_{t=\tau[i-1]}^{\tau[i]} f_t(z^*) - f_t(z_i^*) \right\| \\ &\leq \sum_{i=1}^k \sum_{t=\tau[i-1]}^{\tau[i]} L \|z^* - z_i^*\| \\ &\stackrel{(a)}{\leq} \sum_{i=1}^k \sum_{t=\tau[i-1]}^{\tau[i]} L \frac{2Li}{\mu T} = \frac{(1+k)kL^2}{\mu}, \end{aligned} \quad (19)$$

where step (a) follows Lemma 5 and the fact that $\tau[i] - \tau[i-1] \leq T$.

Lemma 5. For a set of functions f_1, \dots, f_T and for any set of index $\{u[i]\}_{i=1}^k$ such that $u[i] \leq T$, let $F(w) = \sum_{i=1}^T f_i(w)$ and $F_i(w) = \sum_{i=1}^T f_i(w) - \sum_{j=1}^{i-1} f_{u[j]}(w)$. Let $w^* = \operatorname{argmin}_w F(w)$ and $w_i^* = \operatorname{argmin}_w F_i(w)$. If each f_i 's are μ -strongly convex and L -Lipschitz, then

$$\|w^* - w_i^*\| \leq \frac{2(i-1)L}{\mu T}.$$

Combining Equations (18) and (19), and substituting $\omega = 1.2$ concludes the proof. \square

Proof of Lemma 5. For any $i \in \{1, \dots, k\}$ the definition of F ,

$$\begin{aligned} F(w_i^*) &= F_i(w_i^*) + \sum_{j=1}^{i-1} f_{u[j]}(w_i^*) \\ &\leq F_i(w^*) + \sum_{j=1}^{i-1} f_{u[j]}(w_i^*) \\ &= F(w^*) - \sum_{j=1}^{i-1} f_{u[j]}(w^*) + \sum_{j=1}^{i-1} f_{u[j]}(w_i^*) \\ &\leq F(w^*) + (i-1)L \|w^* - w_i^*\|, \end{aligned} \tag{20}$$

where the last inequality follows by the Lipschitzness of all component functions $f_{u[i]}$.

As each component functions f_i 's are μ -strongly convex, the sum of T such functions, F , is $T\mu$ -strongly convex. Then,

$$F(w_i^*) \geq F(w^*) + \frac{T\mu}{2} \|w_i^* - w^*\|^2. \tag{21}$$

Combining Equations (20) and (21) concludes the proof. \square

Corollary 7. Under the same assumption as Theorem 2, Algorithm 1 with step size $\eta_t = 1/(\mu t)$ satisfies

$$\mathbb{E} \left[\operatorname{Regret}_T(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T}) \right] \leq \frac{L^2}{\mu} (1 + k + k^2 + \log T) + \frac{d\omega k^{\omega+1} L^2}{2\sqrt{3}e \ln\left(\frac{1}{\gamma}\right) \varepsilon \mu (\omega - 1)}.$$

Proof. We can show that

$$\begin{aligned} \frac{\tau[i]}{u[i]^2} \gamma^{2(\tau[i]-u[i])} &= \frac{u[i] + (\tau[i] - u[i])}{u[i]^2} \gamma^{2(\tau[i]-u[i])} \\ &\leq (1 + (\tau[i] - u[i])) \gamma^{2(\tau[i]-u[i])} \leq 1 + \frac{1}{e \ln \frac{1}{\gamma}} \end{aligned} \tag{22}$$

Substituting into the regret bound of Theorem 2,

$$\mathbb{E} \left[\operatorname{Regret}_T(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T}) \right] \leq \frac{L^2}{\mu} (1 + k + k^2 + \log T) + \frac{d\omega k^{\omega+1} L^2}{2\sqrt{3}e \ln\left(\frac{1}{\gamma}\right) \varepsilon \mu (\omega - 1)}.$$

\square

Theorem 3. Suppose \mathcal{U} and \mathcal{T} are each of size k . If f_1, \dots, f_T are L -Lipschitz, β -smooth convex cost functions such that for each i , $\sum_{t=1}^{\tau[i]} f_t$ satisfies Assumption 1 with parameter $\kappa(\tau[i] - \tau[i-1])$ and $u[i] \geq \frac{\beta^2 D^2}{4L^2}$, then for all $T \geq k$, Algorithm 1 with $\eta_t = \frac{D}{L\sqrt{t}}$ satisfies

$$\mathbb{E} \left[\operatorname{Regret}_T(\mathcal{R}_{\mathcal{A}}(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T})) \right] \leq 3DL\sqrt{T} + \frac{2k^2 L^2}{\kappa} + \frac{3DLdk^{1.7}}{2\varepsilon} \mathcal{G}_2(\mathcal{U}, \mathcal{T}),$$

where $\mathcal{G}_2(\mathcal{T}, \mathcal{U}) = \sqrt{\sum_{i=1}^k \frac{\tau[i]}{u[i]^2}}$.

Proof. We consider the same notation of \mathcal{T} , \mathcal{U} , z_i^* and z^* as in the proof of Theorem 2.

Similar to Equations (11) to (13), we can derive the following equation from the updating rule of OGD,

$$(\nabla f_t(z_t))^\top (z_t - z^*) \leq \begin{cases} \frac{\|z_t - z^*\|^2 - \|z_{t+1} - z^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} & t+1 \notin \mathcal{T} \\ -\nabla f_t(z_t)^\top \xi_i + \frac{\|z_t - z^* + \xi_i\|^2 - \|z_{t+1} - z^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} & t+1 = \tau[i] \in \mathcal{T} \end{cases} \quad (23)$$

As the loss functions are convex, for any $i \in \{1, \dots, k\}$,

$$f_t(z_t) - f_t(z^*) \leq (\nabla f_t(z_t))^\top (z_t - z^*).$$

Summing over $t \in \{1, \dots, k\}$, we get the following regret bound with respect to a constant competitor,

$$\begin{aligned} \text{Regret}_T(\mathcal{R}_{\mathcal{A}}(S, \emptyset, \mathcal{T})) &= \sum_{t=1}^T f_t(z_t) - f_t(z^*) \leq \sum_{t=1}^T (\nabla f_t(z_t))^\top (z_t - z^*) \\ &\stackrel{(a)}{=} \underbrace{\sum_{t=1}^T \frac{\|z_t - z^*\|^2 - \|z_{t+1} - z^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2}}_A \\ &\quad + \underbrace{\sum_{i=1}^k -(\nabla f_{\tau[i]-1}(z_{\tau[i]-1}))^\top \xi_i + \frac{(z_{\tau[i]-1} - z^*)^\top \xi_i + \|\xi_i\|^2}{2\eta_{\tau[i]-1}}}_B \end{aligned}$$

where step (a) follows by substituting Equation (23).

This way, we decompose the expected regret into two parts. In the following, we derive separate upper bounds for each.

$$\begin{aligned} \mathbb{E} \left[\text{Regret}_T(\mathcal{R}_{\mathcal{A}}(S, \emptyset, \mathcal{T})) \right] &= \mathbb{E}_{\xi_{1:k}} [A + B] \\ &= \mathbb{E}_{\xi_{1:k}} [A] + \mathbb{E}_{\xi_{1:k}} [B] \end{aligned} \quad (24)$$

We first bound the term $\mathbb{E}[A|\xi_{1:k}]$. Given $\xi_{1:k}$, z_1, \dots, z_T are deterministic. Thus,

$$\begin{aligned} \mathbb{E}[A|\xi_{1:k}] &= \sum_{t=1}^T \frac{\|z_t - z^*\|^2 - \|z_{t+1} - z^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} \\ &= \mathbb{E} \left[\sum_{t=1}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \|z_t - z^*\|^2 + \sum_{t=1}^T \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} \right] \\ &\stackrel{(a)}{\leq} D^2 \sum_{t=1}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \frac{L^2}{2} \sum_{t=1}^T \eta_t \\ &\stackrel{(b)}{\leq} DL\sqrt{T} + \sum_{t=1}^T \frac{L^2 \eta_t}{2} \leq 3DL\sqrt{T} \end{aligned}$$

where step (a) is due to $\|z_t - z^*\| \leq D$ as D is the diameter of the parameter space \mathcal{K} and Lipschitzness of the cost functions, and step (b) follows by substituting in specified learning rate $\eta_t = \frac{D}{L\sqrt{t}}$, and the last inequality follows by $\sum_{t=1}^T \frac{1}{\sqrt{t}} \leq 2\sqrt{T}$.

Thus,

$$\mathbb{E}_{\xi_{1:k}} [A] = \mathbb{E}_{\xi_{1:k}} \mathbb{E}[A|\xi_{1:k}] \leq 3LD\sqrt{T}. \quad (25)$$

It remains to bound $\mathbb{E}_{\xi_{1:k}} [B]$,

$$\begin{aligned} \mathbb{E}_{\xi_{1:k}} [B] &= \mathbb{E} \left[\sum_{i=1}^k -(\nabla f_{\tau[i]-1}(z_{\tau[i]-1}))^\top \xi_i + \frac{2(z_{\tau[i]-1} - z^*)^\top \xi_i + \|\xi_i\|^2}{2\eta_{\tau[i]-1}} \right] \\ &\stackrel{(a)}{=} \mathbb{E} \left[\sum_{i=1}^k \frac{\|\xi_i\|^2}{2\eta_{\tau[i]-1}} \right] = \sum_{i=1}^k \frac{\mathbb{E}[\|\xi_i\|^2]}{2\eta_{\tau[i]-1}} \\ &\stackrel{(b)}{=} \sum_{i=1}^k \frac{d\sigma_i^2}{2\eta_{\tau[i]-1}} \stackrel{(c)}{\leq} \frac{DLd\omega}{4(\omega-1)\varepsilon} \sum_{i=1}^k \frac{i^\omega \sqrt{\tau[i]}}{u[i]} \stackrel{(d)}{\leq} \frac{DLd\omega k^{\omega+0.5}}{4\sqrt{3}(\omega-1)\varepsilon} \sqrt{\sum_{i=1}^k \frac{\tau[i]}{u[i]^2}} \end{aligned} \quad (26)$$

where step (a) is due to ξ_i 's independent from each other and have zero mean, step (b) follows by $\mathbb{E}[\|\xi_i\|^2] = d\sigma_i^2$ for $\xi_i \sim \mathcal{N}(0, \sigma_i^2 I_d)$. Step (c) follows by substituting in $\sigma_i^2 = \frac{i^\omega \omega L^2 \eta_{u[i]}^2}{2(\omega-1)\varepsilon}$, and step (d) follows an application of Cauchy-Schwarz inequality (similar to Equation (17)).

Combining Equations (14), (16) and (25), we have

$$\mathbb{E} \left[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \emptyset, \mathcal{T})) \right] \leq 3DL\sqrt{T} + \frac{DLd\omega k^{\omega+0.5}}{4\sqrt{3}(\omega-1)\varepsilon} \sqrt{\sum_{i=1}^k \frac{\tau[i]}{u[i]^2}} \quad (27)$$

Below, we bound the difference introduced by using a constant competitor: the distance between $\mathbb{E}[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \emptyset, \mathcal{T}))]$, which corresponds to the regret with a constant comparator, and $\mathbb{E}[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T}))]$, which corresponds to our regret definition Equation (5).

$$\begin{aligned} \left\| \mathbb{E} \left[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \emptyset, \mathcal{T})) \right] - \mathbb{E} \left[\text{Regret}_T(\mathcal{R}_A(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T})) \right] \right\| &= \left\| \sum_{i=1}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} f_t(z^*) - f_t(z_i^*) \right\| \\ &\stackrel{(a)}{\leq} \sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} L \|z^* - z_i^*\| \\ &\stackrel{(b)}{\leq} \sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} L \frac{2Li}{\kappa(\tau[i+1] - \tau[i])} \\ &\leq \frac{L^2(k^2 + k)}{\kappa} \end{aligned} \quad (28)$$

where the second last inequality follows by the Quadratic Growth assumption of the composite loss function, and the stability of ERM under QG condition Lemma 6.

Lemma 6 (Stability for ERM under QG condition). *Let $F = \sum_{j=1}^T f_j$ and for any k functions $\{i[1], \dots, i[k]\} \subset \{1, \dots, T\}$, $\hat{F} = F - \sum_{\ell=1}^k f_{i[\ell]}$. Let z^*, \hat{z}^* be the ERM solution of F, \hat{F} respectively. Assume the function F satisfies quadratic growth with parameter κ , i.e. $F(z) - F(z^*) \geq \frac{\kappa}{2} \|z - z^*\|^2$ for any z and each component function f_i are L -Lipschitz, then,*

$$\|z^* - \hat{z}^*\| \leq \frac{2kL}{\kappa}$$

Finally, we choose $\omega = 1.2$, which concludes the proof. \square

Proof of Lemma 6.

$$\begin{aligned} F(\hat{z}^*) &= \hat{F}(\hat{z}^*) + \sum_{\ell=1}^k f_{i[\ell]}(\hat{z}^*) \\ &\leq \hat{F}(z^*) + \sum_{\ell=1}^k f_{i[\ell]}(\hat{z}^*) \\ &= F(z^*) - \sum_{\ell=1}^k f_{i[\ell]}(z^*) + \sum_{\ell=1}^k f_{i[\ell]}(\hat{z}^*) \\ &\leq F(z^*) + kL \|z^* - \hat{z}^*\| \end{aligned} \quad (29)$$

where the last inequality follows the Lipschitzness of each f_i . By the Quadratic Growth assumption,

$$F(\hat{z}^*) - F(z^*) \geq \frac{\kappa}{2} \|z^* - \hat{z}^*\|^2 \quad (30)$$

Combining the two inequality complete the proof. \square

Theorem 4. *Let f_1, \dots, f_T be convex, L -Lipschitz and β -smooth and suppose \mathcal{U} and \mathcal{T} are each of size k . Define $p(t) = \sum_{i=1}^t \|\nabla f_i(z_i)\|_2^2$, if there exists some $u_0 \geq 1$ such that $p(u_0) \geq \beta^2/4$ and $u[i] \geq u_0$, then the expected regret of Algorithm 1 with adaptive learning rate $\eta_t = \frac{D}{\sqrt{p(t)}}$ is*

$$O \left(D^2\beta + D \sqrt{\sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} f_t(z_i^*)} + dk^2 L^2 D^2 \mathcal{G}_3(\mathcal{T}, \mathcal{U}, \mathcal{S}) \right),$$

where z_i^* is a best-in-hindsight solution after the k^{th} deletion, and $\mathcal{G}_3(\mathcal{T}, \mathcal{U}, \mathcal{S}) = \sqrt{\beta \sum_{i=1}^k \frac{p(\tau[i])}{p(u[i])^2}}$

Proof. For $i \in \{0, 1, \dots, k\}$, let

$$z_i^* = \underset{z}{\operatorname{argmin}} \sum_{t=1}^T f_t(z) - \sum_{j=1}^i f_{u[j]}(z)$$

be the best-in-hindsight estimator after i th deletion. Following a similar argument as in Equation (13),

$$(\nabla f_t(z_t))^\top (z_t - z_i^*) \leq \begin{cases} \frac{\|z_t - z_i^*\|^2 - \|z_{t+1} - z_i^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} & t+1 \notin \mathcal{T} \\ -\nabla f_t(z_t)^\top \xi_i + \frac{\|z_t - z_i^* + \xi_i\|^2 - \|z_{t+1} - z_i^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} & t+1 = \tau[i] \in \mathcal{T} \end{cases} \quad (31)$$

By convexity of the loss function, for each i , and its corresponding time steps $t \in [\tau[i-1], \tau[i]]$,

$$f_t(z_t) - f_t(z_i^*) \leq \nabla f_t(z_t)^\top (z_t - z_i^*)$$

Summing over $t \in \{1, \dots, T\}$ and substitute in Equation (31), we have

$$\begin{aligned} \operatorname{Regret}_T(\mathcal{A}_{\mathcal{R}}(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T})) &= \sum_{i=0}^k \sum_{t=\tau[i]}^{\tau[i+1]} f_t(z_t) - f_t(z_i^*) \\ &\leq \underbrace{\sum_{i=0}^k \sum_{t=\tau[i]}^{\tau[i+1]} \frac{\|z_t - z_i^*\|^2 - \|z_{t+1} - z_i^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2}}_A \\ &\quad + \underbrace{\sum_{i=1}^k -(\nabla f_{\tau[i]-1}(z_{\tau[i]-1}))^\top \xi_i + \frac{(z_{\tau[i]-1} - z_i^*)^\top \xi_i + \|\xi_i\|^2}{2\eta_t}}_B \end{aligned}$$

Then, the expected regret can be upper bounded as

$$\begin{aligned} \mathbb{E} \left[\operatorname{Regret}_T(\mathcal{A}_{\mathcal{R}}(\mathcal{S}, \mathcal{S}^{\mathcal{U}}, \mathcal{T})) \right] &= \mathbb{E}_{\xi_{1:k}} [A + B] \\ &= \mathbb{E}_{\xi_{1:k}} \mathbb{E} [A | \xi_{1:k}] + \mathbb{E}_{\xi_{1:k}} [B] \end{aligned} \quad (32)$$

We first bound the first term $\mathbb{E} [A | \xi_{1:k}]$. Given $\xi_{1:k}, z_1, \dots, z_T$ are deterministic. Thus,

$$\begin{aligned} \mathbb{E} [A] &= \mathbb{E} [A | \xi_{1:k}] = \mathbb{E} \left[\sum_{i=0}^k \sum_{t=\tau[i]}^{\tau[i+1]} \frac{\|z_t - z_i^*\|^2 - \|z_{t+1} - z_i^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} \right] \\ &\leq \sum_{i=0}^k \sum_{t=\tau[i]}^{\tau[i+1]} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) \frac{\|z_t - z_i^*\|^2}{2} + \sum_{t=1}^T \frac{\eta_t \|\nabla f_t(z_t)\|^2}{2} \\ &\leq \frac{D^2}{2\eta_T} + \sum_{t=1}^T \frac{D \|\nabla f_t(z_t)\|^2}{2\sqrt{\sum_{j=1}^t \|\nabla f_j(z_j)\|^2}} \\ &\stackrel{(a)}{\leq} \frac{D}{2} \sqrt{\sum_{t=1}^T \|\nabla f_t(z_t)\|^2} + D \sqrt{\sum_{t=1}^T \|\nabla f_t(z_t)\|^2} = \frac{3D}{2} \sqrt{\sum_{t=1}^T \|\nabla f_t(z_t)\|^2} \end{aligned}$$

where step (a) follows by Lemma 7.

Lemma 7 (Orabona [11]). *Let $a_0 \geq 0$ and $f : [0, \infty] \rightarrow [0, \infty]$ a nonincreasing function. Then,*

$$\sum_{t=1}^T a_t f \left(a_0 + \sum_{i=1}^t a_i \right) \leq \int_{a_0}^{\sum_{t=0}^T a_t} f(x) dx.$$

As the loss functions are β -smooth, we apply Lemma 8 to arrive an upper bound on part A.

Lemma 8 (Lemma 4.1 in Srebro et al. [27]). *If f is a β -smooth function, then the following holds,*

$$\begin{aligned} \|\nabla f(x)\|_*^2 &\leq 2\beta \left[f(x) - \inf_{y \in \mathbb{R}^d} f(y) \right]. \\ \mathbb{E}[A] &= \mathbb{E}[A|\xi_{1:k}] \leq \frac{3D}{2} \sqrt{\sum_{t=1}^T \|\nabla f_t(z_t)\|^2} \\ &\stackrel{(a)}{\leq} \frac{3D}{2} \sqrt{\beta \sum_{t=1}^T \left[f_t(z_t) - \inf_{y \in \mathbb{R}^d} f_t(y) \right]} \stackrel{(b)}{\leq} \frac{3D}{2} \sqrt{\beta \sum_{t=1}^T f_t(z_t)} \end{aligned} \quad (33)$$

where step (a) follows Lemma 8, and step (b) follows by non-negativity of the loss functions.

It remains to bound $\mathbb{E}_{\xi_{1:k}}[B]$,

$$\begin{aligned} \mathbb{E}_{\xi_{1:k}}[B] &= \mathbb{E} \left[\sum_{i=1}^k -(\nabla f_{\tau[i]-1}(z_{\tau[i]-1}))^\top \xi_i + \frac{(z_{\tau[i]-1} - z^*)^\top \xi_i + \|\xi_i\|^2}{2\eta_{\tau[i]-1}} \right] \\ &\stackrel{(a)}{=} \mathbb{E} \left[\sum_{i=1}^k \frac{\|\xi_i\|^2}{2\eta_{\tau[i]-1}} \right] \stackrel{(b)}{=} \sum_{i=1}^k \frac{d\omega\omega L^2 \eta_{u[i]}^2}{4(\omega-1)\varepsilon\eta_{\tau[i]}} \\ &\stackrel{(c)}{\leq} \frac{dk\omega+0.5\omega L^2}{4\sqrt{3}(\omega-1)\varepsilon} \sqrt{\sum_{i=1}^k \frac{\eta_{u[i]}^4}{\eta_{\tau[i]}^2}} \stackrel{(d)}{=} \frac{dk\omega+0.5\omega L^2 D}{4\sqrt{3}(\omega-1)\varepsilon} \sqrt{\sum_{i=1}^k \frac{\sum_{j=1}^{\tau[i]} \|\nabla f_j(z_j)\|^2}{\left(\sum_{j=1}^{u[i]} \|\nabla f_j(z_j)\|^2\right)^2}} \end{aligned} \quad (34)$$

where step (a) follows from the fact that ξ_i are independent and have zero mean. Step (b) follows by $\mathbb{E}[\|\xi\|^2] = d\sigma_i^2$ for $\xi_i \sim \mathcal{N}(0, \sigma_i^2)$ where $\sigma_i^2 = \frac{i\omega\omega L^2 \eta_{u[i]}^2}{2(\omega-1)\varepsilon}$. Step (c) follows by Cauchy-Schwarz inequality and step (d) follows by substituting in $\eta_t = \frac{D}{\sqrt{\sum_{i=1}^t \|f_i(z_i)\|^2}}$.

Combining Equations (32) to (34), we have

$$\sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} f_t(z_t) - f_t(z_i^*) \leq \mathbb{E}[B] + \frac{3D}{2} \sqrt{\beta \sum_{t=1}^T f_t(z_t)} \quad (35)$$

Then, we apply Lemma 9 with $x = \sum_{t=1}^T f_t(z_t)$ and $a = \frac{9D^2\beta}{4}$, $b = 0$ and $c = \sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} f_t(z_i^*) + \mathbb{E}[B]$

Lemma 9. *Let $a, c > 0$, $b \geq 0$, and $x \geq 0$ such that $x - \sqrt{ax+b} \leq c$. Then $x \leq a + c + 2\sqrt{b+ac}$.*

Therefore, the regret is upper bounded by

$$\begin{aligned} \sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} (f_t(z_t) - f_t(z_i^*)) &\leq \frac{9D^2\beta}{4} + \mathbb{E}[B] + 3D \sqrt{\beta \sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} f_t(z_i^*) + \mathbb{E}[B]} \\ &\leq \frac{9D^2\beta}{4} + 3D \sqrt{\sum_{i=0}^k \sum_{t=\tau[i]+1}^{\tau[i+1]} f_t(z_i^*)} \\ &\quad + \frac{dk\omega+0.5\omega L^2 D^2}{4(\omega-1)\varepsilon} \sqrt{\sum_{i=1}^k \frac{\beta \sum_{j=1}^{\tau[i]} \|\nabla f_j(z_j)\|^2}{\left(\sum_{j=1}^{u[i]} \|\nabla f_j(z_j)\|^2\right)^2}}. \end{aligned} \quad (36)$$

Taking $\omega = 1.5$ concludes the proof. \square

Proof of Lemma 9. Starting with $x - c \leq \sqrt{ax+b}$, we can square both sides and get

$$x^2 - (2c+a)x + c^2 \leq b.$$

Completing the square,

$$\left(x - \frac{2c+a}{2}\right)^2 \leq b + ac + \frac{a^2}{4}$$

Taking square root and rearrange terms, we get

$$\begin{aligned} x &\leq \frac{2c+a}{2} + \sqrt{b + ac + \frac{a^2}{4}} \\ &\leq \frac{2c+a}{2} + \sqrt{b + ac} + \sqrt{\frac{a^2}{4}} \\ &= a + c + \sqrt{b + ac} \end{aligned} \tag{37}$$

This concludes the proof. \square

Theorem 5. For any \mathcal{U}, \mathcal{T} of size k , if the cost functions f_1, \dots, f_T are convex, L -Lipschitz, and β -smooth, and $\sum_{t=1}^{\tau[i]} f_t$ satisfies Assumption 1 with parameter $\kappa(\tau[i] - \tau[i-1])$, then there exists a constant step size η such that the expected regret of Algorithm 1 is

$$L \left(D + k^{1.1} \sqrt{d/\varepsilon} \right) \sqrt{2T} + \frac{2L^2 k^2}{\kappa}.$$

Proof. Similar to the previous proof, from the update rule of OGD, we have

$$(\nabla f_t(z_t))^\top (z_t - z_i^*) = \begin{cases} \frac{\|z_t - z_i^*\|^2 - \|z_{t+1} - z_i^*\|^2}{2\eta} + \frac{\eta \|\nabla f_t(z_t)\|^2}{2} & t+1 \notin \mathcal{T} \\ -\nabla f_t(z_t)^\top \xi_i + \frac{\|z_t - z_i^* + \xi_i\|^2 - \|z_{t+1} - z_i^*\|^2}{2\eta} + \frac{\eta \|\nabla f_t(z_t)\|^2}{2} & t+1 = \tau[i] \in \mathcal{T} \end{cases} \tag{38}$$

Therefore, using the property of convex functions, we can upper bound the expected regret by

$$\begin{aligned} \mathbb{E} \left[\sum_t^T f_t(z_t) - f_t(z^*) \right] &\leq \mathbb{E} \left[\sum_t^T (\nabla f_t(z_t))^\top (z_t - z^*) \right] \\ &\stackrel{(a)}{\leq} \frac{1}{2} \left[\sum_{t=1}^T \frac{\|z_t - z^*\|^2 - \|z_{t+1} - z^*\|^2}{\eta} + \eta \|\nabla f_t(z_t)\|^2 + \sum_{i=1}^k \eta \mathbb{E} \|\xi_i\|^2 \right] \\ &\stackrel{(c)}{\leq} \frac{1}{2} \left[\sum_{t=1}^T \frac{2D^2}{\eta} + \eta \left(L^2 + \sum_{i=1}^k \frac{L^2 d j^\omega \omega}{2(\omega-1)\varepsilon} \right) \right] \\ &\stackrel{(d)}{\leq} \frac{1}{2} \left[\frac{2D^2}{\eta} + \eta T \left(L^2 + \frac{\omega k^{\omega+1} L^2 d}{2(\omega^2-1)\varepsilon} \right) \right] \end{aligned} \tag{39}$$

where step (a) follows Equation (38), step (c) follows by Lipschitzness of the cost functions and the boundedness of the parameter space, and step (d) follows by the fact that

$$\sum_{j=1}^k j^\omega \leq \int_0^k x^\omega dx \leq \frac{k^{\omega+1}}{\omega+1}.$$

Setting $\eta = \sqrt{\frac{2D^2}{T \left(L^2 + \frac{\omega k^{\omega+1} L^2 d}{2(\omega^2-1)\varepsilon} \right)}}$, we arrive at the upper bound

$$\mathbb{E} \left[\sum_t^T f_t(z_t) - f_t(z^*) \right] \leq \sqrt{T \left(2D^2 L^2 + \frac{\omega k^{\omega+1} L^2 d}{2(\omega^2-1)\varepsilon} \right)} \leq \left(DL + \frac{L}{2} \sqrt{\frac{\omega k^{\omega+1} d}{(\omega^2-1)\varepsilon}} \right) \sqrt{2T}.$$

\square

C Omitted Proofs for Section 4

Algorithm 2 First-order active online learner and unlearner

Require: Cost functions f_1, \dots, f_T that are L -Lipschitz, learning rates η_1, \dots, η_T , a deletion time set \mathcal{T} , a deletion index set \mathcal{U} , and privacy parameter ε , the auxiliary unlearner \mathcal{U}_{aux} and its Lipschitz parameter L_R .

- 1: Initialize $z_1 \in \mathcal{K}$.
 - 2: **for** Time step $t = 2, \dots, T$ **do**
 - 3: Set $z_t = \Pi_{\mathcal{K}} [z_{t-1} - \eta_t \nabla f_{t-1}(z_{t-1})]$
 - 4: **if** there exists $\tau[i] \in \mathcal{T}$ such that $t = \tau[i]$ **then**
 - 5: Set $\hat{z}_{\tau[i]} = \Pi_{\mathcal{K}} [z_{\tau[i]-1} - \eta_t \nabla f_{\tau[i]-1}(z_{t-1})]$
 - 6: Starting from $\hat{z}_{\tau[i]}$, $\mathcal{I}_{1,i}$ steps of gradient descent with learning rate $\frac{1}{\beta+\mu}$ on all points till now $f_{1:\tau[i]}$ and output $z'_{\tau[i]}$
 - 7: Starting from $z'_{\tau[i]}$, perform \mathcal{I}_2 steps of gradient descent with learning rate $\frac{1}{\beta+\mu}$ on all remaining data points $f_{i:\tau[i]} \setminus \{f_{u[1]} : f_{u_i}\}$ and output $z''_{\tau[i]}$.
 - 8: Set $z_{\tau[i]} = z''_{\tau[i]} + \xi_i$, where $\xi_i \sim \mathcal{N}(0, \sigma_i^2)$ for
 - $$\sigma_i = \gamma^{\mathcal{I}_2} \sqrt{\frac{i\omega\omega}{2(\omega-1)\varepsilon} \frac{L(6i + L\gamma^{\tau[i]-u[i]}\eta_{u[i]})}{\tau[i]\mu}}. \quad (40)$$
 - 9: **end if**
 - 10: Output z_t
 - 11: **end for**
-

Theorem 6. Let \mathcal{U} and \mathcal{T} each have size k . For all i , assume $\tau[i-1] \leq u[i] \leq \tau[i]$. Suppose each f_i is L -Lipschitz, μ -strongly convex, and β -smooth. If the number of gradient-descent steps on all previously seen points, $\mathcal{I}_{1,i}$ is at least $\log \frac{1}{\gamma} \frac{\mu D \tau[i]}{L}$ and $\mathcal{I}_2 \geq 2.2 \log \frac{1}{\gamma} k$, then Algorithm 2 is an $(\alpha, \alpha\varepsilon)$ -OLU. Moreover, if Assumption 2 holds, the regret of Algorithm 2 is

$$O\left(\log T + k\left(LD^2 + \frac{Ld}{\mu\varepsilon}\right) + \mathcal{G}_2(\mathcal{T}, \gamma) + \frac{L^2 k^2}{\mu}\right),$$

where $\gamma = \frac{\beta/\mu-1}{\beta/\mu+1}$ and $\mathcal{G}_2 = \sum_{i=1}^k \gamma^{\tau[i]-\tau[i-1]} (\tau[i] - \tau[i-1])$.

Unlearning guarantee in Theorem 6. For any i , we express Algorithm 2's output at time $\tau[i]$ as two CNIs (Definition 6). Let $\mathcal{S}_{\tau[i]} = \{f_1, \dots, f_{\tau[i]}\}$, and define $\mathcal{S}'_{\tau[i]} = \{f'_1, \dots, f'_{\tau[i]}\}$ as the same set with functions at $u[j]$, $j \leq i$ replaced with \perp .

Then, ψ_j denote the standard online gradient descent steps between $\tau[j-1]$ and $\tau[j]$ using points in $\mathcal{S}_{\tau[j]}$. Formally, define the updating functions $g_t(z) = \Pi_{\mathcal{K}} [z - \eta_t \nabla f_t(z_t)]$ and $g'_t(z) = \Pi_{\mathcal{K}} [z - \eta_t \nabla f'_t(z_t)]$ with respect to cost functions \mathcal{F} and \mathcal{F}' respectively, then ψ_j and ψ'_j are defined as follows,

$$\psi_j = g_{t_j} \circ \dots \circ g_{\tau[j-1]}, \quad \psi'_j = g'_{t_j} \circ \dots \circ g'_{\tau[j-1]}.$$

We denote the deterministic unlearning procedure (Step 6 and 7 in Algorithm 2) as the function \mathcal{U}_{aux} that takes as input the current output $\hat{z}_{\tau[i]}$, past cost functions $\mathcal{S}_{\tau[i]} = \{f_1, \dots, f_{\tau[i]}\}$ and the deleted functions $\mathcal{S}^{\mathcal{U}}_i = \{f_{u[1]}, \dots, f_{u[i]}\}$. Then, the CNIs are

$$\left\{z_0, \{\mathcal{U}_{\text{aux}}(\psi_j, \mathcal{S}_{\tau[i]}, \mathcal{S}^{\mathcal{U}}_i)\}_{j=1}^i, \{\zeta_j\}_{j=1}^i\right\}, \quad \left\{z_0, \{\mathcal{U}_{\text{aux}}(\psi'_j, \mathcal{S}_{\tau[i]} \setminus \mathcal{S}^{\mathcal{U}}_i, \emptyset)\}_{j=1}^i, \{\zeta_j\}_{j=1}^i\right\}$$

Next, we apply Lemma 3 to upper bound $D_{\alpha}(z_{\tau[i]} \| z'_{\tau[i]})$. To do this, we need to derive a sequence of $a_1, \dots, a_{\tau[i]}$ such that $e_t = \sum_{j=1}^t \gamma^{t-j} (s_j - a_j) \geq 0$ for all $t \in \{1, \dots, \tau[i]\}$, and $e_{\tau[i]} = 0$, where s_j is defined as follows,

$$s_j = \max_{z \in \mathcal{K}} \|\mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]}, \mathcal{S}^{\mathcal{U}}_j) - \mathcal{U}_{\text{aux}}(\psi'_j(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}^{\mathcal{U}}_j, \emptyset)\|_2 \quad (41)$$

As we assume $u[i] \in (\tau[i-1], \tau[i])$ for all i ,

$$\begin{aligned}
s_j &= \max_{z \in \mathcal{K}} \left\| \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]}, \mathcal{S}_j^{\mathcal{U}}) - \mathcal{U}_{\text{aux}}(\psi_j'(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) \right\|_2 \\
&= \max_{z \in \mathcal{K}} \left\| \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]}, \mathcal{S}_j^{\mathcal{U}}) - \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) + \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) - \mathcal{U}_{\text{aux}}(\psi_j'(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) \right\|_2 \\
&= \max_{z \in \mathcal{K}} \left\| \mathcal{U}_{\text{aux}}(z, \mathcal{S}_{\tau[j]}, \mathcal{S}_j^{\mathcal{U}}) - \mathcal{U}_{\text{aux}}(z, \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) \right\|_2 + L_R(\psi_j(z) - \psi_j(z')) \\
&= \max_{z \in \mathcal{K}} \left\| \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]}, \mathcal{S}_j^{\mathcal{U}}) - \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) \right\|_2 + L_R \gamma^{\tau[i]-u[i]} \eta_{u[i]} L
\end{aligned} \tag{42}$$

where L_R is the Lipschitzness coefficient of the unlearning algorithm \mathcal{U}_{aux} .

Setting $a_i = s_i$ and $\sigma_j^2 = \frac{\omega_j^\omega a_j^2}{2(\omega-1)\varepsilon}$, we have the following guarantee: for all $i \in [k]$,

$$D_\alpha \left(z_{\tau[i]} \| z'_{\tau[i]} \right) \stackrel{(a)}{\leq} \sum_{j=1}^i R(\zeta_{\tau[j]}, a_j) = \sum_{j=1}^i \frac{\alpha \varepsilon (\omega - 1)}{\omega_j^\omega} \leq \alpha \varepsilon.$$

where (a) follows Lemma 3.

It remains to compute L_R and $\max_{z \in \mathcal{K}} \left\| \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]}, \mathcal{S}_j^{\mathcal{U}}) - \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) \right\|_2$ for \mathcal{U}_{aux} .

Due to the contractiveness of gradient descent under strongly convex and smooth cost functions (Lemma E), our unlearning algorithm \mathcal{U}_{aux} , which consists of $\mathcal{I}_1 + \mathcal{I}_2$ contractive steps with contraction coefficient γ , is Lipschitz continuous with constant $L_R = \gamma^{\mathcal{I}_1 + \mathcal{I}_2}$.

Lemma E (Convergence of gradient descent [30]). *If the loss function ℓ is μ -strongly convex and β -smooth, then the output w_t of T -step gradient descent on S with learning rate $\eta = \frac{2}{\mu + \beta}$ and initialization w_0 satisfies*

$$\|w_t - w^*\|_2 \leq \gamma^T \|w_0 - w^*\|_2,$$

where $\gamma = \frac{\beta/\mu-1}{\beta/\mu+1}$ and $w^* = \arg \min_w \sum_{x \in S} \ell(w, x)$ is the minimizer of the loss function ℓ on the dataset S .

Next, we derive an upper bound on $\max_{z \in \mathcal{K}} \left\| \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]}, \mathcal{S}_j^{\mathcal{U}}) - \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) \right\|_2$. Consider the first \mathcal{I}_1 gradient descent (GD) steps of the unlearning algorithm on the set $\mathcal{S}_{\tau[j]}$, denoted by the function F_0 , followed by \mathcal{I}_2 GD steps on the set $\mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}$, denoted by F_1 . The unlearning auxiliary function $\mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]}, \mathcal{S}_j^{\mathcal{U}})$ performs these two phases: \mathcal{I}_1 GD steps on $\mathcal{S}_{\tau[j]}$ (via F_0) and \mathcal{I}_2 GD steps on $\mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}$ (via F_1).

Similarly, $\mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset)$ performs $\mathcal{I}_1 + \mathcal{I}_2$ GD steps entirely on the set $\mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}$, denoted by F_0 and \tilde{F}_1 , respectively.

Then,

$$\begin{aligned}
&\max_{z \in \mathcal{K}} \left\| \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]}, \mathcal{S}_j^{\mathcal{U}}) - \mathcal{U}_{\text{aux}}(\psi_j(z), \mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}, \emptyset) \right\|_2 \\
&= \max_z \left\| F_0 \circ F_1(z) - F_0 \circ \tilde{F}_1(z) \right\|_2 \\
&\stackrel{(a)}{\leq} \gamma^{\mathcal{I}_2} \left\| F_1(z) - \tilde{F}_1(z) \right\| \\
&= \gamma^{\mathcal{I}_2} \left\| \left(F_1(z) - \text{ERM}_1 + \text{ERM}_1 - \text{ERM}_0 + \text{ERM}_0 - \tilde{F}_1(z) \right) \right\| \\
&\stackrel{(c)}{\leq} \frac{2(j+1)\gamma^{\mathcal{I}_2} L}{\tau[i]\mu}
\end{aligned} \tag{43}$$

where step (a) follows by the contractiveness of the GD, or the convergence bound of GD Lemma E, ERM_1 and ERM_0 represents the ERM of all points $\mathcal{S}_{\tau[j]}$ and $\mathcal{S}_{\tau[j]} \setminus \mathcal{S}_j^{\mathcal{U}}$ respectively. Step (c) follows by the contractiveness of GD (Lemma E), stability of ERM for strongly convex functions (Lemma 5) and the lower bound on \mathcal{I}_1 by $\log(L/\mu D \tau[j])/\log \gamma$. \square

Regret guarantee in Theorem 6. Consider the sequence of z_t output by active online learner and unlearner \mathcal{A} , i.e.

$$\begin{aligned}
z_{t+1} &= \Pi_{\mathcal{K}} [z_t - \eta_t \nabla f_t(z_t)] & t+1 \notin \mathcal{T} \\
z_{t+1} &= \mathcal{U}_{\text{aux}}(\Pi_{\mathcal{K}} [z_t - \eta_t \nabla f_t(z_t)], f_{1:t}, f_{u[1]:u[i]}) + \xi_i & t+1 = \tau[i] \in \mathcal{T}
\end{aligned} \tag{44}$$

Similar as in the previous proof, let

$$\begin{aligned}
z_{i,0}^* &= \operatorname{argmin}_{z \in \mathcal{K}} \sum_{t=1}^{\tau[i]} f_t(z) - \sum_{j=1}^i f_{u[j]}(z) \\
\|z_{t+1} - z^*\|^2 &\leq \|z_t - z^*\|^2 + \eta_t^2 \|\nabla f_t(z_t)\|^2 + 2\eta_t (\nabla f_t(z_t))^\top (z_t - z^*), & t+1 \notin \mathcal{T} \\
\|z_{t+1} - z^*\|^2 &\leq \|z_{i,0}^* - z^* + d_i + \xi_i\|^2, & t+1 = \tau[i] \in \mathcal{T}
\end{aligned} \tag{45}$$

where

$$\|\Pi_{\mathcal{K}} [\mathcal{U}_{\text{aux}}(z_{\tau[i]-1} - \eta_{\tau[i]-1} \nabla f_{\tau[i]-1}(z_{\tau[i]-1}))], f_{1:\tau[i]}, f_{u[1]:u[i]}] - z_{0,i}^*\| \leq \frac{2\gamma^{\mathcal{I}_2} DL}{\tau[i]\mu} =: d_i$$

for step 5 and 6 in Algorithm 2 following Lemma E.

Let $t_0 = 0, t_{k+1} = T$. By strong convexity of the cost function,

$$\begin{aligned}
\operatorname{Regret}_T(\mathcal{A}, \mathcal{T}, \mathcal{U}) &= \sum_{t=1}^T f_t(z_t) - f_t(z^*) = \sum_{i=0}^k \sum_{t=t_i+1}^{\tau[i+1]} f_t(z_t) - f_t(z^*) \\
&\leq \underbrace{\sum_{i=0}^k \sum_{t=t_i+1}^{\tau[i+1]-1} (\nabla f_t(z_t))^\top (z_t - z^*) - \frac{\mu}{2} \|z_t - z^*\|^2}_A + \underbrace{\sum_{i=1}^k L \|z_{\tau[i]} - z^*\|}_B,
\end{aligned} \tag{46}$$

We first bound part B

$$\begin{aligned}
B &= \sum_{i=1}^k L \|z_{i,0}^* + \xi_i + d_i - z^*\| \\
&\leq L \sum_{i=1}^k \|\xi_i\| + d_i + \|z_{i,0}^* - z^*\|
\end{aligned} \tag{47}$$

By Jensen's inequality and stability of ERM (Lemma 5),

$$\mathbb{E}[B] \leq \sum_{i=1}^k \sigma_i + d_i + \frac{L(T-i-\tau[i])}{T\mu} = O\left(\sum_{i=1}^k \frac{Ld}{\tau[i]\mu\varepsilon} + \frac{kL}{\mu}\right),$$

where σ_i is the standard deviation of ξ_i defined in Equation (59), and the last equality follows by $\mathcal{I}_2 \geq (2.2 \log k) / \log \frac{1}{\gamma}$.

It remains to bound 2 times part A, we first note that when $t = \tau[i] - 1$, the unlearning takes place,

$$\begin{aligned}
2A &= \sum_{i=0}^k \sum_{t=t_i+1}^{\tau[i+1]-1} (\nabla f_t(z_t))^\top (z_t - z^*) - \frac{\mu}{2} \|z_t - z^*\|^2 \\
&= \underbrace{\sum_{i=0}^k \sum_{t=t_i+1}^{\tau[i+1]-2} \frac{\|z_t - z^*\|^2 - \|z_{t+1} - z^*\|^2}{\eta_t} - \mu \|z_t - z^*\|^2}_C + \underbrace{\sum_{i=0}^k \sum_{t=t_i+1}^{\tau[i+1]-2} \eta_t \|\nabla f_t(z_t)\|^2}_D \\
&\quad + \underbrace{\sum_{i=1}^k (\nabla f_{t_i-1}(z_{\tau[i]-1}))^\top (z_{\tau[i]-1} - z^*)}_E
\end{aligned} \tag{48}$$

Part D is upper bounded by $O(\log T)$,

$$D \leq \sum_{t=1}^T \eta_t L^2 = \sum_{t=1}^T \frac{L^2}{t\mu} = O(\log T). \tag{49}$$

Part E is upper bounded by

$$E \leq 2kLD \tag{50}$$

Next, we bound part C,

$$\begin{aligned}
C &= \sum_{i=0}^k \|z_{\tau[i]+1} - z^*\|^2 \left(\frac{1}{\eta_{\tau[i]+1}} - \mu \right) - \frac{1}{\eta_{\tau[i+1]-1}} \|z_{\tau[i+1]-1} - z^*\|^2 \\
&\leq \sum_{i=0}^k \|z_{\tau[i]+1} - z^*\|^2 \tau[i]\mu - t_{i+1}\mu \|z_{\tau[i+1]-1} - z^*\|^2 + \mu \sum_{i=1}^k \|z_{\tau[i]-1} - z^*\|^2 \\
&\stackrel{(a)}{\leq} \sum_{i=1}^k \tau[i]\mu \left(\|z_{\tau[i]+1} - z^*\|^2 - \|z_{\tau[i]-1} - z^*\|^2 \right) + \mu k D^2
\end{aligned} \tag{51}$$

It remains to upper bound the term $\|z_{\tau[i]+1} - z^*\|^2 - \|z_{\tau[i]-1} - z^*\|^2$. We use the following inequality (Lemma 10),

Lemma 10. For any $a, b, c \in \mathbb{R}^d$,

$$\|a - b\| - \|c - b\|^2 \leq \|a - c\|^2 + 2\|a - c\| \|c - b\| \tag{52}$$

As the unlearning algorithm moves $z_{\tau[i]}$ close to ERM solution without the deleted points $z_{i,0}^*$, and that $z_{\tau[i]+1}$ is obtained by doing one gradient descent step on $z_{\tau[i]}$, we have

$$\begin{aligned}
\|z_{\tau[i-1]} - z_{i-1,0}^*\| &\leq \gamma^{\mathcal{I}_2} \left(\gamma^{\mathcal{I}_1} D + \frac{L}{\mu\tau[i-1]} \right) = \frac{\gamma^{\mathcal{I}_2} L}{\mu\tau[i]} \\
\|z_{\tau[i]+1} - z_{i,0}^*\| &= \|z_{\tau[i]} - \eta_{\tau[i]} \nabla f_{\tau[i]}(z_{\tau[i]}) - z^*\| \leq \|z_{\tau[i]} - z_{i,0}^*\| + \frac{L}{\tau[i]\mu} \leq \frac{2\gamma^{\mathcal{I}_2} L}{\mu\tau[i]}
\end{aligned} \tag{53}$$

Consider the composition of update functions of OGD with learning rate $\eta_t = \frac{1}{\mu t}$ on the functions $f_{\tau[i-1]}, \dots, f_{\tau[i]+1}$ as GD, then GD is $\gamma^{\tau[i]+1-\tau[i-1]}$ -contractive. By Assumption 2, a_i is a fixed point of GD and is close to the ERM solution $z_{i,0}^*$, i.e. $GD(z_{i,0}^*) = z_{i,0}^*$ and $\|z_{i,0}^* - a_i\| \leq \frac{1}{\tau[i]}$. Then, we can upper bound $\|z_{\tau[i]+1} - z_{\tau[i]-1}\|$ by

$$\begin{aligned}
\|z_{\tau[i]+1} - z_{\tau[i]-1}\| &\leq \left\| z_{i,0}^* + \frac{\gamma^{\mathcal{I}_2} L}{\mu\tau[i]} - GD \left(z_{i-1,0}^* + \frac{\gamma^{\mathcal{I}_2} L}{\mu\tau[i-1]} \right) \right\| \\
&\leq \left\| a_i + \frac{\mu + \gamma^{\mathcal{I}_2} L}{\mu\tau[i]} - GD \left(z_{i-1,0}^* + \frac{\gamma^{\mathcal{I}_2} L}{\mu\tau[i-1]} \right) \right\| \\
&= \left\| GD(a_i) + \frac{\gamma^{\mathcal{I}_2} L}{\mu\tau[i]} - GD \left(z_{i-1,0}^* + \frac{\gamma^{\mathcal{I}_2} L}{\mu\tau[i-1]} \right) \right\| \\
&\leq \gamma^{\tau[i]-\tau[i-1]} \left\| a - z_{i-1,0}^* + \frac{\gamma^{\mathcal{I}_2} L}{\mu\tau[i-1]} \right\| + \frac{\mu + \gamma^{\mathcal{I}_2} L}{\mu t_i} \\
&\leq \gamma^{\tau[i]-\tau[i-1]} \left\| z_{i,0}^* - z_{i-1,0}^* + \frac{\mu + \gamma^{\mathcal{I}_2} L}{\mu\tau[i-1]} \right\| + \frac{\mu + \gamma^{\mathcal{I}_2} L}{\mu t_i} \\
&= \frac{\gamma^{\tau[i]-\tau[i-1]} (\tau[i] - \tau[i-1]) L}{\mu\tau[i]} + \frac{2(\mu + \gamma^{\mathcal{I}_2} L)}{\mu\tau[i]}
\end{aligned} \tag{54}$$

Substituting this inequality into Equation (51), we get an upper bound

$$C \leq \sum_{i=1}^k \gamma^{\tau[i]-\tau[i-1]} (\tau[i] - \tau[i-1]) L + k\mu(2\gamma^{\mathcal{I}_2} L + D^2) \leq \frac{kL}{e \ln \frac{1}{\gamma}} + k\mu(2\gamma^{\mathcal{I}_2} L + D^2) \tag{55}$$

as $\gamma^{\tau[i]-\tau[i-1]} (\tau[i] - \tau[i-1]) \leq \frac{1}{e \ln \frac{1}{\gamma}}$.

Combining the upper bounds on part A, B, C, D, E, (Equations (47) to (51)) we can upper bound the regret by

$$O \left(\log T + kLD^2 + \frac{Lkd}{\mu\varepsilon} + \frac{kL}{e \ln \frac{1}{\gamma}} \right) \tag{56}$$

The above regret bound is measured against a fixed competitor, as in previous proofs. Next, we adjust the analysis to account for a dynamic competitor, reflecting changes in the best-in-hindsight estimator after each

deletion.

$$\begin{aligned}
\left\| \mathbb{E} \left[\text{Regret}_T(\mathcal{R}_A(S, \emptyset, \mathcal{T})) \right] - \mathbb{E} \left[\text{Regret}_T(\mathcal{R}_A(S, S_U, \mathcal{T})) \right] \right\| &= \left\| \sum_{i=1}^k \sum_{t=\tau[i-1]}^{\tau[i]} f_t(z^*) - f_t(z_i^*) \right\| \\
&\leq \sum_{i=1}^k \sum_{t=\tau[i-1]}^{\tau[i]} L \|z^* - z_i^*\| \\
&\stackrel{(a)}{\leq} \sum_{i=1}^k \sum_{t=\tau[i-1]}^{\tau[i]} L \frac{2Li}{\mu T} = \frac{(1+k)kL^2}{\mu},
\end{aligned} \tag{57}$$

This complete the proof. \square

Proof of Lemma 10.

$$\begin{aligned}
\|a - b\|^2 - \|c - b\|^2 &= \|a\|^2 + \|b\|^2 - 2a^\top b - \|c\|^2 - \|b\|^2 + 2c^\top b \\
&= \|a\|^2 - 2a^\top b - \|c\|^2 + 2c^\top b \\
&= (a - c)^\top (a + c - 2b) = (a - c)^\top (a - c + 2(c - b)) \\
&= \|a - c\|^2 + 2(a - c)^\top (c - b) \leq \|a - c\|^2 + 2\|a - c\| \|c - b\|
\end{aligned} \tag{58}$$

\square

Algorithm 3 Second-order active online learner and unlearner

Require: Cost functions f_1, \dots, f_T that are L -Lipschitz, learning rates η_1, \dots, η_T , a deletion time set \mathcal{T} , a deletion index set \mathcal{U} , and privacy parameter ε , the auxiliary unlearner \mathcal{U}_{aux} and its Lipschitz parameter L_R and error functions e_2, e_1 .

- 1: Initialize $z_1 \in \mathcal{K}$.
- 2: **for** Time step $t = 2, \dots, T$ **do**
- 3: Set $z_t = z_{t-1} - \eta_t \nabla f_{t-1}(z_{t-1})$
- 4: **if** there exists $\tau[i] \in \mathcal{T}$ such that $t = \tau[i]$ **then**
- 5: Starting from $\hat{z}_{\tau[i]}$, \mathcal{I}_1 steps of gradient descent with learning rate $\frac{1}{\beta + \mu}$ on all points till now $f_{1:\tau[i]}$ and output $z'_{\tau[i]}$
- 6: Set $z'_{\tau[i]} = z_{\tau[i]} - \frac{1}{\tau[i] - i} \hat{H}^{-1} \sum_{j=1}^i \nabla f_{u_j}(\hat{z}_{\tau[i]})$, where

$$\hat{H}^{-1} = \frac{1}{\tau[i] - i} \left(\sum_{j=1}^{\tau[i]} \nabla^2 f_j(\hat{z}_{\tau[i]}) - \sum_{j=1}^i \nabla^2 f_{u_j}(\hat{z}_{\tau[i]}) \right)$$

- 7: Set $z_{\tau[i]} = z'_{\tau[i]} + \xi_i$, where $\xi_i \sim \mathcal{N}(0, \sigma_i^2)$ for

$$\sigma_i = \sqrt{\frac{\alpha j^\omega \omega}{2(\omega - 1)\varepsilon} \frac{L \left(2 + \frac{k\beta}{\mu(\tau[i] - k)} \left(\frac{M}{\mu} - 1 \right) \right)}{\mu(\tau[i] - i)}}. \tag{59}$$

- 8: **end if**
 - 9: Output z_t
 - 10: **end for**
-