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(a) Let (1,1)(4,7) be two points. Assume an interpolation parameter $u=0$ at (1,1) and $u=1$ at (4,7). Find the coordinates of the 2D points between them using linear interpolation with an interpolation parameter of $u=0.3$.

$$p(u) = uP_2 + (1-u)P_1$$

$$x(u) = 4u + (1-u)(1) = 3u+1 \qquad x(0.3) = 3(0.3)+1 = 1.9$$
$$y(u) = 7u + (1-u)(1) = 6u+1 \qquad y(0.3) = 6(0.3)+1 = 2.8$$

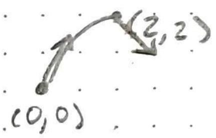
$$\boxed{(1.9, 2.8)}$$

(b) explain the differences between parametric and geometric continuity. Explain the advantages and challenges with piece wise interpolation.

The difference between parametric and geometric continuity is that parametric continuity has identical derivatives while geometric continuity has proportional derivatives.

Piecewise interpolation generates the curve in smaller lower degree polynomials for each piece, but guarenteeing absolute smoothness can serve to be somewhat of a challenge.

(c) Given the points (0,0) (2,2) and cooresponding tangents (1,1)(1,-1) respectively. Write the 4 constraint equations that are used to compute the Hermite interpolation coefficients for the x coordinate of the interpolation curve between the points.



$$P(0) = P_u = (0,0)$$
$$P(1) = P_{u+1} = (2,2)$$
$$p'(0) = dP_u = (1,1)$$
$$p'(1) = dP_{u+1} = (1,-1)$$

$$p(u) = [u^3 \ u^2 \ u^1 \ u^0] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$p'(u) = [3u^2 \ 2u \ 1 \ 0] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$p(u=0) = au^3 + bu^2 + cu + d = P_u$$
$$(0) + 0 + 0 + d = 0 \qquad d = 0$$

$$p(u=1) = au^3 + bu^2 + cu + d = P_{u+1}$$
$$= a + b + c + d = P_{u+1} = 2 \qquad a + b + 1 + 0 = 2$$
$$a + b = 1$$

$$p'(u=0) = 3au^2 + 2bu + c + 0 = P_{du}$$
$$3a(0) + 2b(0) + c = 1 \qquad c = 1$$

$$p'(u=1) = 3au^2 + 2bu + c + 0 = P_{du+1}$$
$$3a + 2b + 1 + 0 = 1 \qquad 3a + 2b + 1 + 0 = 1$$
$$3a + 2b = 0$$

$$\boxed{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}}$$

(d) Given the 2D control points (2,2)(4,2) with a tangent (1,1) at the point (2,2) and a tangent of (1,-1) at the point (4,2) compute the coordinate at the parameter $u=0.5$ using Hermite splines. Assume a parameter of $u=0$ at (2,2) and of $u=1$ at (4,2). Use matrix form for computations.

first, must calculate a, b, c, d, our constants.

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M_H \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

$x(u) = [u^3 \ u^2 \ u \ 1][-2 \ 3 \ 1 \ 2]^T$

$x(u) = -2u^3 + 3u^2 + u + 2$

$x(0.5) = -2(0.5)^3 + 3(0.5)^2 + (0.5) + 2 = 3$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M_H \begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

$y(u) = [u^3 \ u^2 \ u \ 1][0 \ -1 \ 1 \ 2]^T$

$y(u) = 0u^3 - u^2 + u + 2$ $\quad y(0.5) = -(0.5)^2 + (0.5) + 2 = 2.25$

$\boxed{\text{at } u=0.5, (3, 2.25)}$

(e) Repeat the previous ~~function~~ question but this time using the blending function

$P(u) = H_0(u)P_u + H_1(u)P_{u+1} + H_2(u)dP_u + H_3(u)dP_{u+1}$

$$[u^3 \ u^2 \ u \ 1]M_H = \begin{bmatrix} 2u^3 & -3u^2 & 0u & 1 \\ -2u^3 & +3u^2 & 0u & 0 \\ 1 \cdot u^3 & -2u^2 & 1u & 0 \\ 1 \cdot u^3 & -1u^2 & 0u & 0 \end{bmatrix}^T = \begin{bmatrix} H_0(u) \\ H_1(u) \\ H_2(u) \\ H_3(u) \end{bmatrix}^T = [0.5 \ 0.5 \ 0.125 \ -0.125]$$

$x(0.5) = 0.5(2) + 0.5(4) + 0.125(1) - 0.125(1) = 3$

$\boxed{3, 2.25}$

$y(0.5) = 0.5(2) + 0.5(2) + 0.125(1) - 0.125(-1) = 2.25$

(f) Given a set of 2D control points (1,1) (2,2) (4,2) (5,1) with a parameter at $u=0$ at (2,2) and parameter $u=1$ at (4,2) find the coordinate of the point at 0.5 when using Cardinal splines. Use the matrix form for the computations. Assume tension parameter is 0.5.

X-params:

$u=0$ $u=1$ P_4

$P(0) = P_u = 2$

$P(1) = P_{u+1} = 4$

$P'(0) = s(P_{u+1} - P_{u-1}) = 0.5(4-1) =$

$P'(1) = s(P_{u+2} - P_u) = 0.5(5-2)$

y-params:

$p(0) = P_u = 2$

$p(1) = P_{u+1} = 2$

$p'(0) = s(P_{u+1} - P_{u-1}) = 0.5(2-1)$

$p'(1) = s(P_{u+2} - P_u) = 0.5(1-2)$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M_c \begin{bmatrix} P_{u-1} \\ P_u \\ dP_{u+1} \\ dP_{u+2} \end{bmatrix} = \begin{bmatrix} -0.5 & 2-0.5 & 0.5-2 & 0.5 \\ 2(0.5) & 0.5-3 & 3-2(0.5) & -0.5 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.5 \\ 1.5 \\ 2 \end{bmatrix}$$

$x(u) = -1u^3 + 1.5u^2 + 1.5u + 2$

$x(0.5) = -(0.5^3) + 1.5(0.5^2) + 2 = 3$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M_c \begin{bmatrix} P_{k-1} \\ P_u \\ dP_{u+1} \\ dP_{u+2} \end{bmatrix} = \begin{bmatrix} -0.5 & 2-0.5 & 0.5-2 & 0.5 \\ 2(0.5) & 0.5-3 & 3-2(0.5) & -0.5 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0.5 \\ 2 \end{bmatrix}$$

$y(u) = 0u^3 - 0.5u^2 + 0.5u + 2$

$y(0.5) = -0.5(0.5^2) + 0.5(0.5) + 2 = 2.125$

$\boxed{(x,y) = (3, 2.125)}$

(g) repeat the above but using the blending function

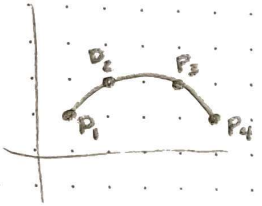
$p(u) = C_0(u)P_{u-1} + C_1(u)P_u + C_2(u)P_{u+1} + C_3(u)P_{u+2}$

$$x(u) = \begin{aligned}[t] &[-0.5(0.5^3) + 2(0.5)(0.5^2) - 0.5(0.5)](1) = -0.0625 \\ &[(2-0.5)(0.5^3) + (0.5-3)(0.5)^2 + 1](2) = 1.125 \\ &[(0.5-2)(0.5^3) + (3-2(0.5))(0.5)^2 + 0.5(0.5)](2) = 2.25 \\ &[(0.5(0.5^3) - 0.5(0.5^2)](1) = -0.3125 \\ &= 3 \end{aligned}$$

$$y(u) = \begin{aligned}[t] &[(-0.5)(0.5^3) + 2(0.5)(0.5^2) - (0.5)(0.5)](1) = -0.0625 \\ &[(2-0.5)(0.5^3) + (0.5-3)(0.5^2) + 1](2) = 1.125 \\ &[(0.5-2)(0.5^3) + (3-2(0.5))(0.5^2) + (0.5)(0.5)](2) = 1.125 \\ &[(0.5)(0.5^3) - (0.5)(0.5^2)](1) = -0.0625 \\ &= 2.125 \end{aligned}$$

$\boxed{(x,y) = (3, 2.125)}$

(h) Given a set of 2D control points (1,1)(2,2)(4,2)(5,1) with parameter u=0 at (1,1) and parameter u=1 at (5,1) find the coordinate of the point at u=0.5 when using cubic bezier curves. Use the matrix form for the computation.



$$x: \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M_B \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

$x(u) = -2u^3 + 3u^2 + 3u + 1$
$x(0.5) = -2(0.5^3) + 3(0.5^2) + 3(0.5) + 1 = 3$

$$y: \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M_B \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \\ 1 \end{bmatrix}$$

$y(u) = 0u^3 + (-3)u^2 + 3u + 1$
$y(0.5) = -3(0.5^2) + 3(0.5) + 1 = 1.75$

$\boxed{(x,y) = (3, 1.75)}$

(i) repeat the previous question but this time with blending function form

$P(u) = \sum_{k=0}^{n} P_k B_k^n(u) \quad B_k^n(u) = \binom{n}{k} u^k (1-u)^{n-k} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$x: (1)\left(\frac{3!}{0!\,(3-0)!}\right)(0.5)^0(1-0.5)^{3-0} = 0.125$

$(2)\left(\frac{3!}{1!\,(3-1)!}\right)(0.5)^1(1-0.5)^{3-1} = 0.75 \qquad 0.125 + 0.75 + 1.5 + 0.625 = 3$

$(4)\left(\frac{3!}{2!\,(3-2)!}\right)(0.5)^2(1-0.5)^{3-2} = 1.5$

$(5)\left(\frac{3!}{3!\,(3-3)!}\right)(0.5)^3(1-0.5)^{3-3} = 0.625$

$y: (1)\left(\frac{3!}{0!\,(3-0)!}\right)(0.5)^0(1-0.5)^{3-0} = 0.125$

$(2)\left(\frac{3!}{1!\,(3-1)!}\right)(0.5)^1(1-0.5)^{3-1} = 0.75 \qquad 0.125 + 0.75 + 0.75 + 0.125 = 1.75$

$(2)\left(\frac{3!}{2!\,(3-2)!}\right)(0.5)^2(1-0.5)^{3-2} = 0.75$

$(1)\left(\frac{3!}{3!\,(3-3)!}\right)(0.5)^3(1-0.5)^{3-3} = 0.125$

$\boxed{(x,y) = (3, 1.75)}$

(j) Assuming we want to add another cubic Bezier curve segment that will connect to the cubic Bezier curve segment in the previous question smoothly (with C1 continuity), compute the coordinates of the first control point in the second curve segment.

Two Bezier curve segments p, q may be connected with C1 continuity by setting:

$Q_0 = P_n$ so $Q_0 = (5, 1)$

$Q_1 = P_n + (P_n - P_{n-1})$ $Q_{x1} = 5 + (5-4) = 6$ $\boxed{Q_1 = (6, 0)}$

$Q_{y1} = 1 + (1-2) = 0$

(k) Explain the advantage of the Bezier curve blending functions.

The advantage of Bezier curve blending functions like Bernshten polynomial, allows us to have all our points lie within the convex hull of control points.

(l). Given a set of 2D control points (1, 1)(2, 2)(4, 2)(5, 1) and a knot vector [0, 1, 2, 3, 4, 5, 6] find the coordinate of the point at u=2 when using uniform quadratic B-splines.

$p(u) = \sum_{u=0}^{n} P_u B_u^d(u)$

$x(u) = (1)B_0^3(u) + (2)B_1^3(u) + (4)B_2^3(u) + (5)B_3^3(u)$

$y(u) = (1)B_0^3(u) + (2)B_1^3(u) + (2)B_2^3(u) + (1)B_3^3(u)$

$B_0^3 = \frac{2-0}{2-0} \cdot 0 + \frac{3-2}{3-1}(3-2) = 1/2$ $B_2^3 = \frac{2-2}{2-2} \cdot (3-2) + \frac{4-2}{4-2} \cdot (2-2) = 1/2$

$B_1^3 = \frac{2-1}{2-1} \cdot (3-2) + \frac{4-2}{4-2} \cdot (2-2) = 1/2$ $B_3^3 = \frac{2-3}{5-3} \cdot 0 + 0 = 0$

$x(2) = (1)(1/2) + (2)(1/2) + (4)(0) + (5)(0) = 1.5$

$y(2) = (1)(1/2) + (2)(1/2) + (2)(0) + (1)(0) = 1.5$ $\boxed{\text{at } u=2\ (1.5, 1.5)}$

(m) Given the uniform knot vector [0 1 2 3 4 5] for interpolating between 4 points using B-splines of degree ~~M~~ d=2 write the 4 B-spline blending functions $B_0^2(u)$, $B_1^2(u)$, $B_2^2(u)$, $B_3^2(u)$.

$$B_0^2(u) = \begin{cases} \frac{u-0}{1-0} = u & 0 \le u \le 1 \\ \frac{2-u}{2-1} = 2-u & 1 \le u < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$B_1^2(u) = \begin{cases} \frac{u-1}{2-1} = u-1 & 1 \le u < 2 \\ \frac{3-u}{3-2} = 3-u & 1 \le u < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$B_2^2(u) = \begin{cases} \frac{u-2}{3-2} = u-2 & 2 \le u < 3 \\ \frac{4-u}{4-3} = 4-u & 3 \le u < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$B_3^2(u) = \begin{cases} \frac{u-3}{4-3} = u-3 & 3 \le u < 4 \\ \frac{5-u}{5-4} = 5-u & 4 \le u < 5 \\ 0 & \text{otherwise} \end{cases}$$