CSE464 - Digital Image Processing - Homework 1

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Q1

We are asked to show that the Laplacian operator is rotation invariant. When we rotate the x and y coordinates, we get:

$$x = x'cos\theta - y'sin\theta$$

$$y = x'sin\theta + y'cos\theta$$

and we need to prove:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

Taking partial derivative with respect to x gives us:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial x}$$

The second degree derivative is:

$$\frac{\partial^2 f}{\partial x^2} = (\frac{\partial^2 f}{\partial x'^2} \cdot \frac{\partial^2 x'}{\partial x^2}) + (\frac{\partial^2 f}{\partial x'\partial y'} \cdot \frac{\partial y'}{\partial x} \cdot \frac{\partial x'}{\partial x}) + (\frac{\partial^2 f}{\partial y'\partial x'} \cdot \frac{\partial x'}{\partial x} \cdot \frac{\partial y'}{\partial x}) + (\frac{\partial^2 f}{\partial y'^2} \cdot \frac{\partial^2 y'}{\partial x^2})$$

Which can also be written as:

$$\frac{\partial^2 f}{\partial x^2} = \underbrace{(\frac{\partial^2 f}{\partial x'^2} \cdot cos^2\theta)}_{\text{i}} + \underbrace{(\frac{\partial^2 f}{\partial x'\partial y'} \cdot sin\theta \cdot cos\theta)}_{\text{ii}} + \underbrace{(\frac{\partial^2 f}{\partial y'\partial x'} \cdot cos\theta \cdot sin\theta)}_{\text{iii}} + \underbrace{(\frac{\partial^2 f}{\partial y'^2} \cdot cos^2\theta)}_{\text{iv}}$$

Taking partial derivative with respect to y gives us:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial y} + \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial y}$$

The second degree derivative is:

$$\frac{\partial^2 f}{\partial y^2} = (\frac{\partial^2 f}{\partial y'^2} \cdot \frac{\partial^2 y'}{\partial y}) + (\frac{\partial^2 f}{\partial y'\partial x'} \cdot \frac{\partial y'}{\partial y} \cdot \frac{\partial x'}{\partial y}) + (\frac{\partial^2 f}{\partial y'\partial x'} \cdot \frac{\partial x'}{\partial y} \cdot \frac{\partial y'}{\partial y}) + (\frac{\partial^2 f}{\partial x'^2} \cdot \frac{\partial^2 x'}{\partial y^2})$$

Which can also be written as:

$$\frac{\partial^2 f}{\partial y^2} = \underbrace{(\frac{\partial^2 f}{\partial y'^2} \cdot sin^2\theta)}_{\text{I}} - \underbrace{(\frac{\partial^2 f}{\partial x' \partial y'} \cdot cos\theta \cdot sin\theta)}_{\text{II}} - \underbrace{(\frac{\partial^2 f}{\partial y' \partial x'} \cdot sin\theta \cdot cos\theta)}_{\text{III}} + \underbrace{(\frac{\partial^2 f}{\partial x'^2} \cdot sin^2\theta)}_{\text{IV}}$$

In the initial expression, we had $\frac{\partial^2 f}{\partial x'^2}$ and $\frac{\partial^2 f}{\partial y'^2}$. Since we have found each of these expression separately, now we need to sum up them to get our result.
→ ii and II are negatives of each other.
→ iii and III are negatives of each other.

Then let's just add the remaining terms:

$$\underbrace{(\frac{\partial^2 f}{\partial x'^2} \cdot cos^2\theta)}_{\text{i}} + \underbrace{(\frac{\partial^2 f}{\partial x'^2} \cdot sin^2\theta)}_{\text{IV}} + \underbrace{(\frac{\partial^2 f}{\partial y'^2} \cdot sin^2\theta)}_{\text{I}} + \underbrace{(\frac{\partial^2 f}{\partial y'^2} \cdot cos^2\theta)}_{\text{iv}}$$

And since $cos^2\theta + sin^2\theta = 1$, then $i + IV$ is $\frac{\partial^2 f}{\partial x^2}$ and because of the same rule, $I + iv$ is $\frac{\partial^2 f}{\partial y^2}$.
By that, we get the initial coordinates we have after we do the rotation. Thus, The Laplacian Operator is rotation invariant.

Q2

To prove that Cityblock Distance can be a metric, there are three axioms we need to show. This question asks us to prove the third axiom (triangular inequality).

Cityblock distance is defined as:

The distance between points P(p1, p2) and Q(q1, q2) is $d(p,q) = |p1 - q1| + |p2 - q2|$

And triangular inequality says that, $d(P,Q) + d(Q,R) \geq d(P,R)$

$d(P,Q) = |p1 - q1| + |p2 - q2|$ and $d(Q,R) = |q1 - r1| + |q2 - r2|$

Then $d(P,Q) + d(Q,R) = \underbrace{|p1 - q1|}_{i} + \underbrace{|p2 - q2|}_{ii} + \underbrace{|q1 - r1|}_{iii} + \underbrace{|q2 - r2|}_{iv}$

$i + iii = |p1 - q1| + |q1 - r1| = |p1 - q1 + q1 - r1| = \underbrace{|p1 - r1|}_{I}$

$ii + iv = |p2 - q2| + |q2 - r2| = |p2 - q2 + q2 - r2| = \underbrace{|p2 - r2|}_{II}$

$I + II = |p1 - r1| + |p2 - r2| = d(p,r)$ and this distance is indeed $\geq d(p,r)$

Q3

We have to align two images with known matching coordinates.
Following points in the image A:

$$[2, 2]$$
$$[-1, 4]$$
$$[-4, 4]$$

matches with following points in the image B:

$$[1, 2]$$
$$[2, 1]$$
$$[3, 1]$$

respectively.

We need to get a transformation matrix using these coordinates.
Let our transformation matrix be:

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

For a single point:

$$P.T = P'$$

$$= \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cdot T = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$x \cdot t_{11} + y \cdot t_{12} + 1 \cdot t_{13} + 0 \cdot t_{21} + 0 \cdot t22 + 0 \cdot t_{23} = x'$$
$$0 \cdot t_{11} + 0 \cdot t_{12} + 0 \cdot t_{13} + x \cdot t_{21} + y \cdot t22 + 1 \cdot t_{23} = y'$$

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \end{bmatrix} \begin{bmatrix} t11 \\ t12 \\ t13 \\ t21 \\ t22 \\ t23 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

For all points:

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} t11 \\ t12 \\ t13 \\ t21 \\ t22 \\ t23 \end{bmatrix} = \begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \\ x_3' \\ y_3' \end{bmatrix}$$

If we put the points we have in our question, we get:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} t11 \\ t12 \\ t13 \\ t21 \\ t22 \\ t23 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 4 \\ -4 \\ 4 \end{bmatrix}$$

This gives us the following set of equations:

$$t_{11} + 2t_{12} + t_{13} = 2$$

$$t_{21} + 2t_{22} + t_{23} = 2$$

$$2t_{11} + t_{12} + t_{13} = -1$$

$$2t_{21} + t_{22} + t_{23} = 4$$

$$3t_{11} + t_{12} + t_{13} = -4$$

$$3t_{21} + t_{22} + t_{23} = 4$$

And then we construct the equation matrix to solve it with elimination:

$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 \\ 3 & 1 & 1 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 3 & 1 & 1 & 4 \end{array}\right]$$

Find the pivot in the 1st column in the 1st row and eliminate the 1st column:

$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & -5 & -2 & 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & 3 & 1 & 1 & 4 \end{array}\right]$$

Make the pivot in the 2nd column by dividing the 3rd row by -3 and swap the 3rd and the 2nd rows:

$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1/3 & 1 & 2 & 1 & 5/3 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & -5 & -2 & 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & 3 & 1 & 1 & 4 \end{array}\right]$$

Eliminate the 2nd column:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1/3 & 0 & 0 & 0 & -4/3 \\ 0 & 1 & 1/3 & 1 & 2 & 1 & 5/3 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & 0 & -1/3 & 0 & 0 & 0 & -5/3 \\ 0 & 0 & 0 & 3 & 1 & 1 & 4 \end{array}\right]$$

Make the pivot in the 3rd column by dividing the 5th row bt -1/3 and swap the 5th and the 3rd rows:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1/3 & 0 & 0 & 0 & -4/3 \\ 0 & 1 & 1/3 & 1 & 2 & 1 & 5/3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 & 1 & 4 \end{array}\right]$$

Eliminate the 3rd column:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 & 1 & 4 \end{array}\right]$$

Find the pivot in the 4th column and swap the 5th and the 4th rows:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & 0 & 0 & 3 & 1 & 1 & 4 \end{array}\right]$$

Eliminate the 4th column:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -5 & -2 & -2 \end{array}\right]$$

Make the pivot in the 5th column by dividing the 5th row by -3:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & -5 & -2 & -2 \end{array}\right]$$

Eliminate the 5th column:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 1/3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/3 & -2 \end{array}\right]$$

Make the pivot in the 6th column by dividing the 6th row by -1/3:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 1/3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Eliminate the 6th column:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

This gives us the transformation matrix T as:

$$\begin{bmatrix} -3 & 0 & 5 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

Q4

In this question, we are asked to prove the distribution of a structuring element over union. We have given two structural element C and D. operator $\oplus$ is defined as: $X \oplus B = \bigcup_{x \in X} B_x$.

So if we dilate C, we get: $\underbrace{C \oplus B = \bigcup_{b1 \in C} B_{b1}}_{I}$.

And if we dilate D, we get: $\underbrace{D \oplus B = \bigcup_{b2 \in D} B_{b2}}_{II}$.

We need to prove that adding up the dilated structures is the same operation as adding the structures then dilating the result.

$$I \cup II = (C \cup D) \oplus \check{B}$$

$$= I \cup II = \bigcup_{b1 \in C} \check{B}_{b1} \cup \bigcup_{b2 \in D} \check{B}_{b2}$$

$$= \bigcup_{b \in C \cup D} \{\check{B}_b \cap \check{B}_{b1}\} \cup \bigcup_{b \in C \cup D} \{\check{B}_b \cap \check{B}_{b2}\}$$

Then we can combine the two and get:

$$= \bigcup_{b \in C \cup D} \{\check{B}_b \cap \check{B}_{b1} \cup \check{B}_b \cap \check{B}_{b2}\}$$

$$= \bigcup_{b \in C \cup D} \check{B}_b \cap \{\check{B}_{b1} \cup \check{B}_{b2}\}$$

$$= \bigcup_{b \in C \cup D} \check{B}_b = (C \cup D) \oplus \check{B}$$

This verifies our statement.