# Section 9 Bivariate Distribution

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## 1 Mean and Variance

### 1.1 Mean

$X$ and $Y$ are two random variables.

Since $E$ is a linear operator,

$$E(aX + bY) = aEX + bEY$$
$$E(aX - bY) = aEX - bEY$$
$$E(ag(X) + bh(Y)) = aEg(X) + bEh(Y)$$

**What about the nonlinear case?**

**If $X$ and $Y$ are independent**, then

$$E(XY) = EXEY$$
$$E(g(X)h(Y)) = Eg(X)Eh(Y)$$

Proof:

$$\begin{aligned}
E(XY) &= \int_{x,y} xyf_{XY}(x,y)dxdy \\
&= \int_{x,y} xyf_X(x)f_Y(y)dxdy \\
&= \int_x xf_X(x)dx \int_y yf_Y(y)dy \\
&= EXEY
\end{aligned}$$

The proof for $E(g(X)h(Y)) = Eg(X)Eh(Y)$ is similar.

**If $X$ and $Y$ are not independent**, then

$$E(XY) \neq EXEY$$
$$E(g(X)h(Y)) \neq Eg(X)Eh(Y)$$

**Remark 1** *Think about how to use $E(g(X)h(Y)) = Eg(X)Eh(Y)$ to find the mgf of binomial distribution using the mgf of Bernoulli distribution.*

### 1.2 Variance

Var is not a linear operator.

But **when $X$ and $Y$ are independent**, we have

$$\begin{aligned} Var(X+Y) &= Var(X) + Var(Y) \\ Var(aX+bY) &= a^2Var(X) + b^2Var(Y) \\ Var(X-Y) &= Var(X) + Var(Y) \\ Var(aX-bY) &= a^2Var(X) + b^2Var(Y) \end{aligned}$$

Proof:

$$\begin{aligned} Var(X+Y) &= E(X+Y-E(X+Y))^2 \\ &= E((X-EX)+(Y-EY))^2 \\ &= E(X-EX)^2 + 2E[(X-EX)(Y-EY)] + E(Y-EY)^2 \\ &= VarX + 2E(X-EX)E(Y-EY) + VarY \\ &= VarX + VarY \end{aligned}$$

The proof for other conclusions is similar.

**Remark 2** *Think about why $Var(X-Y) = Var(X) + Var(Y)$ when $X$ and $Y$ are independent?*

**Remark 3** *Whether $X$ and $Y$ are independent or not, $VarXY \neq (VarX)(VarY)$*

**1. CDF: Joint Distribution Function:**

$$\begin{aligned} F_{XY}(x,y) &= P(X \le x, Y \le y) \\ &= P(X \le x \cap Y \le y) \end{aligned}$$

for any pair $(x, y)$ in the $xy-$plane.

**2. pmf/pdf**

**DRV:** Let $X$ and $Y$ be two drv, then their **joint probability mass function (pmf)** is defined as

$$f_{XY}(x,y) = P(X = x \ \cap \ Y = y) = P(X = x, Y = y)$$

for any point $(x, y)$ in the $xy$-plane.

(i) $0 \le f_{XY}(x, y) \le 1$ for all $(x, y)$.

(ii) $\sum_y \sum_x f_{XY}(x, y) = 1$.

**CRV:** Two rv's $X$ and $Y$ are said to have a continuous joint distribution if there exists a nonnegative function $f_{XY}(x, y)$ such that for any subset $A$ on the $xy$-plane,

$$P[(X,Y) \in A] = \int\int_{(x,y)\in A} f_{XY}(x,y)dxdy.$$

The function $f_{XY}(x, y)$ is called the **joint probability density function (pdf).**

(i) $f_{XY}(x, y) \ge 0$;

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)dxdy = 1$.

**Examples:**

1. If the joint probability distribution of $X$ and $Y$ is given by $f(x,y) = \frac{x+y}{42}$, for $x = 1,2,3,4$; $y = 0,1,2$

find $P(X \geq Y)$ and $P(X+Y > 4)$

2. If the joint probability distribution of $X$ and $Y$ is given by

$$f(x,y) = \begin{cases} 6x & 0 < x < 1,\ 0 < y < 1-x \\ 0 & elsewhere \end{cases}$$

Find (a) $P(X+Y=1)$;

(b) $P(X+Y<1/2)$

## 3. Marginal distributions

**DRV:** Suppose $X$ and $Y$ have a joint discrete distribution with joint pmf $f_{XY}(x,y)$, then the marginal pmf of $X$ and $Y$ are defined as

$$f_X(x) = P(X = x) = \sum_y f_{XY}(x,y), \text{ where } -\infty < x < \infty,$$
$$f_Y(y) = P(Y = y) = \sum_x f_{XY}(x,y), \text{ where } -\infty < y < \infty.$$

(i) $0 \le f_X(x) \le 1$, $0 \le f_Y(y) \le 1$;

(ii) $\sum_x f_X(x) = 1$, $\sum_y f_Y(y) = 1$.

**CRV:** Suppose $X$ and $Y$ have a joint continuous distribution with joint pdf $f_{XY}(x,y)$, then the marginal pdf's of $X$ and $Y$ are defined as

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy, \text{ where } -\infty < x < \infty,$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx, \text{ where } -\infty < y < \infty.$$

(i) $f_X(x) \ge 0$, $f_Y(y) \ge 0$;

(ii) $\int f_X(x)dx = 1$, $\int f_Y(y)dy = 1$.

**Exampls:**

1. Suppose that $X$ and $Y$ have the following joint probability distribution:

$$f(x,y) = \begin{cases} \frac{3x-y}{9} & 1 < x < 3,\ 1 < y < 2 \\ 0 & elsewhere \end{cases}$$

(a) Find the marginal distribution of X.

(b) Find the marginal distribution of Y.

2. We have a following bivariate pdf of $X$ and $Y$ :

$$f(x,y) = \{ \begin{array}{ll} 2, & 0 < x < y < 1 \\ 0, & elsewhere \end{array}$$

(a) Find the marginal distribution of X.

(b) Find the marginal distribution of Y.

**4. Conditional Distributions:** Let $X$ and $Y$ have a joint discrete(continuous) distribution with joint probability function $f_{XY}(x,y)$ and marginal probability functions $f_X(x)$ and $f_Y(y)$. Then the conditional probability function of $X$ given $Y = y$ is defined as

$$f_{X|Y}(x|y)$$
$$= \begin{cases} f_{XY}(x,y)/f_Y(y), & f_Y(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

**5. Independence:** Two r.v. $X$ and $Y$ are independent iff

$$F_{XY}(x,y) = F_X(x)F_Y(y) \text{ for all } (x,y) \in R^2.$$

where $F_{XY}, F_X, F_Y$ are the joint and marginal cdf's.

**Theorem1:** Two discrete(continuous) rv $(X,Y)$ are independent iff

$$f_{XY}(x,y) = f_X(x)f_Y(y) \text{ for all } (x,y),$$

where $f_{Xy}, f_X, f_Y$ are the joint and marginal pmf(pdf)'s.

**Theorem2:** The two random variables $X$ and $Y$ are independent if and only if the joint pmf/pdf can be written as

$$f_{XY}(x,y) = g(x) \cdot h(y),$$

for all $-\infty < x < \infty$, and $-\infty < y < \infty$.

**Theorem3:** Suppose two rv $X$ and $Y$ are independent, then

$$f_{X|Y}(x|y) = f_X(x) \text{ for all } x \text{ and } y$$

and

$$f_{Y|X}(y|x) = f_Y(y) \text{ for all } x \text{ and } y.$$

**Examples:**

1. Suppose $(X, Y)$ has a joint pdf

$$f_{XY}(x, y) = \begin{cases} c(x+y) & 0 < x < y < 1 \\ 0 & elsewhere \end{cases}$$

(a) find the value of $c$;

(b) find marginal density $f_Y(y)$ of $Y$;

(c) check if $X$ and $Y$ are independent.

2. Suppose $(X, Y)$ has a joint pdf

$$f_{XY}(x, y) = \begin{cases} 6x & 0 < x < 1,\ 0 < y < 1 - x \\ 0 & elsewhere \end{cases}$$

(a) Find the conditional density function $f_{Y|X}(y|x)$;

(b) Are $X$ and $Y$ independent? Give your reasoning.