ECON130264.01: Section 1 Math Review

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1 Differentiation

1.1 Chain rule

Let g be a real-valued function on $[a, b]$ which is differentiable at $c \in [a, b]$; and suppose that f is a real-valued function defined on an interval I containing the range of g and suppose further that $g(c)$ is an interior point of I. If f is differentiable at $g(c)$, then

- $(f \circ g)(x)$ is differentiable at $x = c$,
- $(f \circ g)'(c) = f'(g(c))g'(c)$

Intuitively, if a variable, y, depends on a second variable, u, which in turn depends on a third variable, x, that is $y = y(u(x))$, then the rate of change of y with respect to x can be computed as the rate of change of y with respect to u multiplied by the rate of change of u with respect to x. Schematically,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

1.1.1 Chain rule for function of several variables

- Consider the function $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$, and $g(t)$ and $h(t)$ are differentiable with respect to t, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

  Remark 1 *The function z here could be regarded as a function of one variable only w.r.t. t.*

- Suppose that each argument of $z = f(u, v)$ is a two-variable function such that $u = h(x, y)$ and $v = g(x, y)$, and that these functions are all differentiable. Then the chain rule would look like:

$$\begin{aligned}\frac{dz}{dx} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\ \frac{dz}{dy} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.\end{aligned}$$

1.1.2 Derivative of inverse function

Suppose $y = f(x)$ (*) is invertible, then

$$x = f^{-1}(y).$$

Taking derivative w.r.t y on both sides of (*), we could derive

$$1 = \frac{df(x)}{dx} \cdot \frac{dx}{dy},$$
$$\Longrightarrow \frac{dx}{dy} = 1/(\frac{df(x)}{dx}),$$

i.e.

$$\frac{df^{-1}(y)}{dy} = \frac{1}{f'(x)}.$$

1.1.3 Examples

Compute derivative of the following function $f(x)$ with respect to (w.r.t) x.

1. $f(x) = (x^2+1)^3$
   $f'(x) = 3(x^2+1)(2x)$
2. $f(x) = \sin(x^2)$
   $f'(x) = \cos(x^2)(2x)$
3. $f(x) = \arctan(\sin x)$
   $f'(x) = \frac{1}{1+\sin^2(x)} \cos(x)$
4. Given $u = x^2 + 2y$ where $x = r\sin(t)$ and $y = \sin^2(t)$, determine the value of $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial t}$ using the chain rule.
   $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} = 2x\sin(t) = 2r\sin^2(t)$
   $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} = 2xr\cos(t) + 2 * 2\sin(t)\cos(t)$
5. Given $y = f(x) = x^2$ for $x > 0$, calculate the derivative of $f^{-1}(y)$
   $\frac{df^{-1}(y)}{dy} = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$
   or you could compute directly by $x = f^{-1}(y) = \sqrt{y}$

2 Integration

2.1 Double integral

$$\int\int_D f(x,y)dxdy$$

2.1.1 Examples

1. Consider this region: $D = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 1\}$. Calculate $\int\int_D xy^2dxdy$
   $\int\int_D xy^2dxdy = \int_0^2 xdx \int_0^1 y^2dy = 2/3$
2. Consider this region: $D = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq x/2\}$. Calculate $\int\int_D xy^2dxdy$
   $\int\int_D xy^2dxdy = \int_0^2 x(\int_0^{x/2} y^2dy)dx = \int_0^2 x(x^3/24)dx = 4/15$

2.1.2 Take-Home practice

1. Consider this region: $D = \{(x, y) : x \geq 0, y \leq 1, y \geq x^2\}$. Calculate $\int\int_D (x + y)dxdy$
   Hint: $D = \{(x, y) : 1 \geq x \geq 0, 1 \geq y \geq x^2\}$

2. Consider this region: $D = \{(x, y) : 0 \leq x \leq y \leq 1\}$. Calculate $\int\int_D 8xydxdy$
   Hint: $D = \{(x, y) : 1 \geq x \geq 0, 1 \geq y \geq x\}$

2.2 Integration by parts

If $u = f(x)$, $v = g(x)$, and the differentials $du = f\prime(x)dx$ and $dv = g\prime(x)dx$, then

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

Equivalently,

$$\int udv = uv - \int vdu.$$

Proof. $d(uv) = udv + udu \Longrightarrow \int d(uv) = \int udv + \int vdu$ ■

2.2.1 Examples

Compute the following integrals.

1. $\int x\sin(x)dx$
   $= \int xd(-\cos(x)) = x(-\cos(x)) - \int(-cos(x))dx = -x\cos(x) + \sin(x) + C$

   Remark 2 *Think about how to compute* $\int x^2\sin(x)dx$, $\int x^3\sin(x)dx$.

2. $\int x^2e^xdx$
   $= x^2e^x - \int e^x(2x)dx = x^2e^x - 2(\int xe^xdx) = x^2e^x - 2(xe^x - e^x) + C = x^2e^x - 2xe^x + 2e^x + C$

   Remark 3 *Think about how to compute* $\int x^3e^xdx$.

3. $\int \ln xdx$
   $= x\ln(x) - \int xd\ln(x) = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - x + C$

Remark 4 *A rule of thumb: Six types of functions in order (inverse trigonometric, logarithmic, power, exponential, trigonometric, fraction). When two types of functions appear in the target function at the same time, put the one on the right into the differential.*