Section 16 Confidence Interval

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1 Confidence Interval

1.1 CI for μ

1. CI for μ, when $X_i \sim N(\mu, \sigma^2)$, σ^2 is known

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

2. CI for $\mu_1 - \mu_2$, when $X_i, i = 1, \ldots, n$ and $Y_j, j = 1, \ldots, m$ are from independent normal distribution $N(\mu_1, \sigma_1^2), N(\mu_2, \sigma_2^2)$, respectively, σ_1^2, σ_2^2 are known

$$Z = \frac{\bar{X}_n - \bar{Y}_n - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n + \sigma_2^2/m}} \sim N(0,1)$$

3. CI for μ, when $X_i \sim \ iid\ N(\mu, \sigma^2)$, σ^2 is unknown

$$T_n = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{(n-1)}$$

4. CI for $\mu_1 - \mu_2$, when $X_i, i = 1, \ldots, n$ and $Y_j, j = 1, \ldots, m$ are from independent normal distribution $N(\mu_1, \sigma_1^2), N(\mu_2, \sigma_2^2)$, respectively, σ_1^2, σ_2^2 are unknown

$$T = \frac{\frac{\bar{X}_n - \bar{Y}_n - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n + \sigma_2^2/m}}}{\sqrt{\left[\frac{(n-1)S_1^2}{\sigma_1^2} + \frac{(m-1)S_2^2}{\sigma_2^2}\right]/(n+m-2)}} \sim t_{(n+m-2)}$$

when $\sigma_1^2 = \sigma_2^2$:

$$T = \frac{\bar{X}_n - \bar{Y}_n - (\mu_1 - \mu_2)}{\sqrt{\frac{(n-1)S_1^2 + (m-1)S_2^2}{m+n-2}(\frac{1}{n} + \frac{1}{m})}} \sim t_{(n+m-2)}$$

5. CI for μ, when $X_i \sim \ iid\ N(\mu, \sigma^2)$, σ^2 is unknown, $n \to \infty$

$$T_n = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \xrightarrow{d} N(0,1)$$

6. CI for μ, when $X_i \sim \; iid \; (\mu, \sigma^2)$, σ^2 is unknown, $n \to \infty$

$$T_n = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \xrightarrow{d} N(0,1)$$

1.2 CI for σ^2

1. CI for σ^2, when $X_i \sim \; iid \; N(\mu, \sigma^2)$, μ is unknown

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

2. CI for $\frac{\sigma_1^2}{\sigma_2^2}$, when $X_i, i = 1, \ldots, n$ and $Y_j, j = 1, \ldots, m$ are from independent normal distribution $N(\mu_1, \sigma_1^2), N(\mu_2, \sigma_2^2)$, respectively

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{(n-1,m-1)}$$

2 Practice

2. A random sample $X_1, \cdots, X_n$ of size n is selected from a normal distribution with known mean μ and unknown variance σ^2. Two possible confidence intervals for σ^2 are shown below, where a_1, a_2, b_1 and b_2 are constants.

$$\left(a_1^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2,\ a_2^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2\right), \qquad \left(b_1^{-1} \sum_{i=1}^{n} (X_i - \mu)^2,\ b_2^{-1} \sum_{i=1}^{n} (X_i - \mu)^2\right).$$

For the case $n = 10$, find values of these constants which give intervals with confidence level 0.90. Compare the expected lengths of these intervals. Comment on your findings.

3. Let $X_1, \cdots, X_n$ be a random sample from the uniform distribution on the interval $[0, \theta]$ ($\theta > 0$). Find a confidence interval for θ.

4. There is a theory that people can postpone their death until after an important event. To test this theory, Phillips and King (1988, *Lancet*, pp.728–) collected data on deaths around the Jewish holiday Passover. Of 1919 deaths, 922 died the week before the holiday and 997 died the week after. Think of this as a binomial and test the null hypothesis that $\theta = 1/2$, where θ is the probability that a death occurs after the holiday. Also construct a confidence interval for θ.

3 Take-Home Practice

5. A sample of 11 observations from population $N(\mu, \sigma^2)$ yields the sample mean $\bar{X} = 8.68$ and the sample variance $S^2 = 1.21$. At 5% significance level, test the following hypotheses.

   (a) $H_0 : \mu = 8$ against $H_1 : \mu > 8$
   (b) $H_0 : \mu = 8$ against $H_1 : \mu < 8$
   (c) $H_0 : \mu = 8$ against $H_1 : \mu \neq 8$

   Repeat the above exercise with the additional assumption $\sigma^2 = 1.21$. Compare the results with those derived without this assumption and comment.

6. (a) Two independent random samples, of n_1 and n_2 observations, are drawn from normal distributions with the same variance σ^2. Let S_1^2 and S_2^2 be the sample variances of the first and the second sample, respectively. Show that $\hat{\sigma}^2 = \frac{1}{n_1+n_2-2}\{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2\}$ is an unbiased estimator for σ^2.

   (b) Two makes of car safety belts, A and B have breaking strengths which are normally distributed with the same variance. A sample of 140 belts of make A and a sample of 220 belts of make B were tested, the sample means, and the sums of squares about the means (i.e. $\sum_i (X_i - \bar{X})^2$), of the breaking strengths (in lbf units) were (2685, 19000) for make A, and (2680, 34000) for make B. Is there any significant evidence to support the hypothesis that belts of make A are stronger than belts of make B?

1. Let $N(0, 1)$, χ_k^2 and t_k denote, respectively, the standard normal, χ_k^2-distributed and t_k-distributed random variables. Find the unknown constants C and α in the equations below, either using the relevant tables in `StatisticalTables.pdf` or using R, and make a table itemizing these values.

$$P\{N(0, 1) > C\} = 0.975, \qquad P\{N(0, 1) < -2.3\} = \alpha, \qquad P\{-1.3 < N(0, 1) < 1.5\} = \alpha,$$
$$P\{\chi_{10}^2 > C\} = 0.975, \qquad P\{\chi_{14}^2 < C\} = 0.025, \qquad P\{13.5 < \chi_{17}^2 < 35.7\} = \alpha,$$
$$P\{t_{10} > C\} = 0.975, \qquad P\{t_{15} < -2.6\} = \alpha, \qquad P\{|t_{20}| < C\} = 0.95.$$

4 Advice

- Review course material carefully
- Review lecture notes carefully
- Review homework
- Do some practice
- Don't be anxious

Remark 1 *You can make your own plan for review.*

Remark 2 *You don't need to spend too much time on reviewing because I think all of you have a pretty good understanding of the course material. After going over all the concepts this course has covered, doing more (difficult) problems is more helpful to you.*

Solutions of Exercise 7

1. Using a statistical table, we may find out

   (a) For $P\{N(0,\ 1) > C\} = 0.975$, $P\{N(0,\ 1) < C\} = 0.025$, $C = -1.96$.

   (b) $\alpha = P\{N(0,\ 1) < -2.3\} = P\{N(0,\ 1) > 2.3\} = 0.0107$.

   (c) $\alpha = P\{-1.3 < N(0,\ 1) < 1.5\} = P\{N(0,\ 1) \geq -1.3\} - P\{N(0,\ 1) \geq 1.5\} = 0.5 + P\{-1.3 \leq N(0,\ 1) \leq 0\} - P\{N(0,\ 1) \geq 1.5\} = 0.5 + [0.5 - P\{N(0,\ 1) > 1.3\}] - P\{N(0,\ 1) \geq 1.5\} = 0.836$.

   (d) For $P\{\chi^2_{10} > C\} = 0.975$, $C = \chi^2_{0.975,\,10} = 3.247$.

   (e) For $P\{\chi^2_{14} < C\} = 0.025$, $P\{\chi^2_{14} \geq C\} = 0.975$, $C = 5.629$.

   (f) $\alpha = P\{13.5 < \chi^2_{17} < 35.7\} = P\{\chi^2_{17} \geq 13.5\} - P\{\chi^2_{17} \geq 35.7\} = 0.7 - 0.005 = 0.695$.

   (g) For $P\{t_{10} > C\} = 0.975$, $P\{t_{10} > -C\} = 0.025$, $-C = t_{0.025,\,10} = 2.228$. Hence $C = -2.228$.

   (h) $\alpha = P\{t_{15} < -2.6\} = P\{t_{15} > 2.6\} = 0.01$.

   (i) Since $P\{|t_{20}| < C\} = 0.95$, $P\{t_{20} > C\} = (1 - 0.95)/2 = 0.025$. Hence $C = t_{0.025,\,20} = 2.086$.

   Note. It is important to be familiar with the statistical tables, which contains a lot of information. In the exam, you will be provided with a copy of the table.

2. Note $\sum_{i=1}^{10}(X_i - \bar{X})^2/\sigma^2 \sim \chi^2_9$. Leaving out 5% at both ends of the distribution, we have

$$\begin{aligned} 0.9 &= P\{3.325 < \sum_{i=1}^{10}(X_i - \bar{X})^2/\sigma^2 < 16.92\} \\ &= P\{\sum_{i=1}^{10}(X_i - \bar{X})^2/16.92 < \sigma^2 < \sum_{i=1}^{10}(X_i - \bar{X})^2/3.325\}, \end{aligned}$$

   namely, $a_1 = 16.92$ and $a_2 = 3.325$.

   Similarly since $\sum_{i=1}^{10}(X_i - \mu)^2/\sigma^2 \sim \chi^2_{10}$, we have

$$\begin{aligned} 0.9 &= P\{3.94 < \sum_{i=1}^{10}(X_i - \bar{X})^2/\sigma^2 < 18.31\} \\ &= P\{\sum_{i=1}^{10}(X_i - \mu)^2/18.31 < \sigma^2 < \sum_{i=1}^{10}(X_i - \mu)^2/3.94\}, \end{aligned}$$

   namely, $b_1 = 18.31$ and $b_2 = 3.94$.

   The average lengths of the confidence intervals are

$$(a_2^{-1} - a_1^{-1})E\{\sum_{i=1}^{10}(X_i - \bar{X})^2\} = (a_2^{-1} - a_1^{-1}) \times 9\sigma^2 = 2.18\sigma^2,$$

$$(b_2^{-1} - b_1^{-1})E\{\sum_{i=1}^{10}(X_i - \mu)^2\} = (b_2^{-1} - b_1^{-1}) \times 10\sigma^2 = 1.99\sigma^2.$$

   The second interval is shorter since it makes use of given mean μ.

3. The MLE for θ is $X_{(n)}$, the sample maximum. It is easy to see that for $x \in [0, 1]$,

$$P\{X_{(n)}/\theta < x\} = P\{X_i/\theta < x \text{ for all } 1 \leq i \leq n\} = \left[P\{X_1/\theta < x\}\right]^n = x^n.$$

Hence $X_{(n)}/\theta$ is a pivot with probability $f(x) = nx^{n-1}$ for $0 \leq x \leq 1$. To find a $100(1-\alpha)\%$ confidence interval for θ, we need to find a and b such that

$$P\{a \leq X_{(n)}/\theta \leq b\} = 1 - \alpha.$$

Obviously there are many choices for a and b here. However we prefer the interval which has the shortest length. Therefore we look for an interval on which the probability density $f(x)$ is as large as possible. Hence we should let $b = 1$ and choose a according to α. This yields $a = \alpha^{1/n}$. The resulting confidence interval for θ is

$$[X_{(n)}/b,\ X_{(n)}/a] = [X_{(n)},\ X_{(n)}\alpha^{-1/n}].$$

4. Let $X_i = 1$ if the i-th person dies after Passover, or 0 otherwise, $i = 1, \cdots, 1919$. Then $\widehat{\theta} = \bar{X} = 997/1919 = 0.5195$. $\mathrm{SE}(\widehat{\theta}) = \sqrt{\widehat{\theta}(1-\widehat{\theta})/n} = \sqrt{0.5195(1-0.5195)/1919} = 0.0114$. To test $H_0 : \theta = 1/2$ against $H_1 : \theta \neq 1/2$, we use the Wald test with the test statistic $T = (\widehat{\theta} - 0.5)/\mathrm{SE}(\widehat{\theta})$. We reject H_0 at the 95% significance level if $|T| > z_{0.025} = 1.96$. Since $T = (0.5195 - 0.5)/0.0114 = 1.711$, we cannot reject the null hypothesis. We conclude that there is no significant evidence indicating that the death rates before and after the Passover are different. Note that the p-value is $P(|N(0,1)| > 1.711) = 0.087$.

An approximate 95% confidence interval for θ is $\widehat{\theta} \pm 1.96\mathrm{SE}(\widehat{\theta}) = (0.4972,\ 0.5419)$.

Remark. A more relevant setting for this problem: $H_0 : \theta = 0.5$ vs $H_1 : \theta > 0.5$. Then the p-value is then $P(N(0,1) > 1.711) = 0.0435$. We reject H_0 at the level 5%, but not reject at the level 1%.

5. When σ^2 is unknown, we use the test statistic $T = \sqrt{n}(\bar{X} - 8)/S$.

Under H_0, $T \sim t_{10}$. With $\alpha = 0.05\%$, we reject H_0 if

(a) $T > t_{0.05,10} = 1.81$, against $H_1 : \mu > 8$,
(b) $T < -t_{0.05,10} = -1.81$, against $H_1 : \mu < 8$, or
(c) $|T| > t_{0.025,10} = 2.23$, against $H_1 : \mu \neq 8$.

For the given sample, $T = 2.0503$. Hence we reject H_0 against the alternative $H_1 : \mu > 8$, but will not reject H_0 against the two other alternatives.

When σ^2 is known, we use the test statistic $T = \sqrt{n}(\bar{X} - 8)/\sigma$. Now under H_0, $T \sim N(0,1)$. With $\alpha = 0.05\%$, we reject H_0 if

(a) $T > Z_{0.05} = 1.64$, against $H_1 : \mu > 8$,
(b) $T < -Z_{0.05} = -1.64$, against $H_1 : \mu < 8$, or
(c) $|T| > Z_{0.025} = 1.96$, against $H_1 : \mu \neq 8$.

For the given sample, $T = 2.0503$. Hence we reject H_0 against the alternative $H_1 : \mu > 8$ or $H_1 : \mu \neq 8$, but will not reject H_0 against $H_1 : \mu < 8$.

With σ known, we should be able to do the inference better simply because we have more information about the population. More precisely for the given significance level, we require less extreme values to reject H_0. Put it another way, the p-value of the test is reduced with σ given. So the risk to reject H_0 is also reduced.

6. (a) Note $(n_i-1)S_i^2/\sigma^2 \sim \chi^2_{n_i-1}$. By the definition of χ^2-distributions, $E\{(n_i-1)S_i^2\} = \sigma^2(n_i-1)$, $i=1,2$. Hence

$$E\widehat{\sigma}^2 = \frac{1}{n_1+n_2-2}[E\{(n_1-1)S_1^2\} + E\{(n_2-1)S_2^2\}] = \frac{(n_1-1)\sigma^2 + (n_2-1)\sigma^2}{n_1+n_2-2} = \sigma^2.$$

(b) Denote $\bar{X} = 2685$ and $\bar{Y} = 2680$. Then $139S_x^2 = 19000,\ 219S_y^2 = 34000$. We test $H_0 : \mu_x = \mu_y$ against $H_1 : \mu_x > \mu_y$. Under H_0, $\bar{X} - \bar{Y} \sim N(0,\ \sigma^2(1/140 + 1/220)) = N(0,\ 0.01169\sigma^2)$, $(139S_x^2 + 219S_y^2)/\sigma^2 \sim \chi^2_{358}$. Hence

$$T = \frac{(\bar{X}-\bar{Y})/\sqrt{0.01169}}{\sqrt{(139S_x^2 + 219S_y^2)/358}} \sim t_{358}$$

under H_0. We reject H_0 if $T > t_{0.01,358} = 2.34$. Since we observe $T = 3.801$. We reject H_0, i.e. there is significant evidence to suggest that make A belt is stronger than make B belt.