Section 7 Expectation and Variance

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1 Function of a Continuous Random Variable

1. **[Linear Transformation]** Show that $Y = a + bX$ $(b \neq 0)$ also follows a normal distribution when $X \sim N(\mu, \sigma^2)$, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ x \in \mathbb{R}$$

.

Solution: $\Omega_Y = \mathbb{R}$

1. $b > 0$

$$\begin{aligned} F(Y \leq y) &= P(X \leq \frac{y-a}{b}) \\ &= \int_{-\infty}^{\frac{y-a}{b}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \end{aligned}$$

Therefore,

$$f_Y(y) = \frac{1}{b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\frac{y-a}{b}-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi} b\sigma} e^{-\frac{(y-a-b\mu)^2}{2b^2\sigma^2}}$$

2. $b < 0$

Method 1:

$$\begin{aligned} F(Y \leq y) &= P(X \geq \frac{y-a}{b}) \\ &= 1 - P(X \leq \frac{y-a}{b}) \\ &= 1 - \int_{-\infty}^{\frac{y-a}{b}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \end{aligned}$$

By taking derivative, we get

$$f_Y(y) = -\frac{1}{b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\frac{y-a}{b}-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}(-b\sigma)} e^{-\frac{(y-a-b\mu)^2}{2b^2\sigma^2}}$$

Method 2:

$$\begin{aligned} F(Y \leq y) &= P(X \geq \frac{y-a}{b}) \\ &= P(X \leq 2\mu + \frac{y-a}{-b}) \\ &= \int_{-\infty}^{2\mu + \frac{y-a}{-b}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \end{aligned}$$

Therefore,

$$f_Y(y) = -\frac{1}{b}\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(2\mu + \frac{y-a}{-b} - \mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}(-b\sigma)} e^{-\frac{(y-a-b\mu)^2}{2b^2\sigma^2}}$$

[**Take-Home Thinking Problem**] Why $P(X \geq \frac{y-a}{b}) = P(X \leq 2\mu + \frac{y-a}{-b})$?

Remark 1 *If* $X \sim N(\mu, \sigma^2)$, $Y = a + bX \sim N(a + b\mu, b^2\sigma^2)$, $b \neq 0$.

4. [**Probability Integral Transform**]: Suppose X has a continuous CDF $F_X(x)$ which is strictly monotonically increasing (which means it is inversible). Define $Y = F_X(X)$. Prove that Y follows a uniform distribution on $[0, 1]$.

Proof: The support of $Y = F_X(X)$ is the unit interval $[0, 1]$. Letting $y \in [0, 1]$, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P[F_X(X) \leq y]. \end{aligned}$$

Because $F_X(x)$ is strictly increasing, its inverse function, denoted as $F_X^{-1}(y)$, exists and is also strictly increasing. For any real-value x, we have

$$F_X^{-1}[F_X(x)] = x.$$

By using $Y = F_X(X)$ and applying the inverse function operation, we obtain

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P[F_X(X) \leq y] \\ &= P\{F_X^{-1}[F_X(X)] \leq F_X^{-1}(y)\} \\ &= P[X \leq F_X^{-1}(y)] \\ &= F_X[F_X^{-1}(y)] \\ &= y, \text{ for } y \in [0, 1]. \end{aligned}$$

It follows that the PDF of Y is given by

$$f_Y(y) = \begin{cases} 1, & \text{for } 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

This is a uniform distribution on $[0, 1]$, which is called the standard uniform distribution, denoted $U[0, 1]$.

5. [Thinking Problem] [Take-Home Practice] **[Functions of 2 random variables]** Suppose X_1, X_2 are independent and both follow $N(0,1)$. Show that $Y = X_1 + X_2 \sim N(0,2)$.

Hint: Let $Y = X_1 + X_2$.

$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y-x) f_{X_2}(x) dx$

Remark 2 *First, fix $X_2 = x$. Then, X_1 must be equal to $y - x$. Since X_2 could take any value on the real line, finally we do integration with respect to x.*

Remark 3 *Let X and Y be independent random variables that are normally distributed, then their sum is also normally distributed. i.e., if*

$$X \sim N(\mu_X, \sigma_X^2)$$
$$Y \sim N(\mu_Y, \sigma_Y^2)$$
$$Z = X + Y,$$

then

$$Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

2 Expectation & Variance

2.1 Mathematical Expectations

Suppose X is a rv with pmf or pdf $f_X(x)$. Then the expected value or mean of function $g(X)$ is defined as

$$
\begin{aligned}
E[g(X)] &= \int g(x)dF_X(x) \\
&= \begin{cases} \sum_x g(x)f_X(x), & \text{drv}, \\ \int_{-\infty}^{\infty} g(x)f_X(x)dx, & \text{crv}, \end{cases}
\end{aligned}
$$

where the summation is over all possible values of X for the discrete case.

Mean, variance, skewness, kurtosis.

$$
\begin{aligned}
\text{Mean } \mu_X &= E(X) \\
&= \begin{cases} \sum_x x f_X(x) & drv \\ \int_{-\infty}^{\infty} x f_X(x)dx & crv \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\text{Variance } \sigma_X^2 &= E[(X-\mu_X)^2] \\
&= \begin{cases} \sum_x (x-\mu_X)^2 f_X(x) & drv \\ \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx & crv \end{cases}
\end{aligned}
$$

$$\sigma_X^2 = E(X^2) - \mu_X^2 = E(X^2) - E^2X$$

$$\text{Skewness } S = \frac{E(X-\mu_X)^3}{\sigma_X^3}$$

$$\text{Kurtosis } K = \frac{E(X-\mu_X)^4}{\sigma_X^4}$$

E is a linear operator.

$$\begin{aligned} E(X+c) &= E(X)+c \\ E(X+Y) &= E(X)+E(Y) \\ E(aX+bY) &= aE(X)+bE(Y) \\ E(XY) &= E(X)E(Y) \text{ when X, Y are independent} \\ E(XY) &\neq E(X)E(Y) \text{ when X, Y are not independent} \end{aligned}$$

Var is not a linear operator.

$$\begin{aligned} Var(aX+c) &= a^2Var(X) \\ Var(X+Y) &= Var(X)+Var(Y) \text{ when X, Y are uncorrelated} \\ Var(aX+bY) &= a^2Var(X)+b^2Var(Y) \text{ when X, Y are uncorrelated} \\ Var(X+Y) &\neq Var(X)+Var(Y) \text{ when X, Y are correlated} \\ Var(XY) &\neq Var(X)Var(Y) \end{aligned}$$

Remark 4 *If X and Y are independent, they must be uncorrelated.*
If X and Y are uncorrelated, they are not necessarily independent.
If X and Y are correlated, they must be dependent.

2.2 Practice

1. Derive the expectation and variance of Binomial Distribution using the fact that if $Y \sim B(n,p)$, then Y could be regraded as a summation of independent Bernoulli random variables, i.e. $Y = X_1 + \ldots + X_n$ where X_i's are independent from each other and all follow $Ber(p)$.

Hint: $E(X_i) = p$, $Var(X_i) = p(1-p)$, $\forall i$

2. Show that if a CRV X's pdf $f(x)$ is symmetric (which means $f(\mu+c) = f(\mu-c)$), then its skewness equals 0.

$$\text{Skewness } S = \frac{E(X-\mu)^3}{\sigma^3}$$

3. Suppose $X \sim Unif(0, \frac{\pi}{2})$. Derive $E(\sin X)$

4. If $P(X = a) = p = 1 - P(X = b)$ (which means $\Omega_X = \{a, b\}$). Derive $E(X)$ and $Var(X)$.

5. The probability distribution of the discrete random variable X is

$$f(x) = \binom{3}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x}, \quad x = 0, 1, 2, 3$$

Find the mean and the variance of X. (Combination Number $C_n^x = \binom{n}{x}$)

Hint: $X \sim B(3, \frac{1}{4})$

6. Assume the length X in minutes of a particular type of telephone conversation is a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{5}e^{-x/5} & x > 0 \\ 0 & otherwise \end{cases}$$

(a) Determine the mean length $E(X)$ of this type of telephone conversation;

(b) Find the variance and standard deviation of X;

(c) Find $E[(X + 5)^2]$.

Hint: $X \sim Exp(\frac{1}{5})$

2.3 Take-Home Practice

1. Show that if $X \sim Poisson(\lambda)$, then $E(X) = \lambda, Var(X) = \lambda$

2. Derive the mean and the variance of Gamma Distribution (using the property of Gamma Distribution/Gamma Function), Chi-squared Distribution (using the fact that a Chi-squared Distribution equals the summation of independent squared normal distributions).

1. Show that if $X \sim Poisson(\lambda)$, then $E(X) = \lambda, Var(X) = \lambda$

$$P(X = k) = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda},\ k = 0, 1, \cdots,$$

$$\mathrm{E}X = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}\mathrm{e}^{-\lambda} = \lambda.$$

$$\begin{aligned}
\mathrm{E}X^2 &= \mathrm{E}[X(X-1)] + \mathrm{E}X \\
&= \sum_{k=0}^{\infty} k(k-1)\frac{\lambda^k}{k!}\mathrm{e}^{-\lambda} + \lambda \\
&= \lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}\mathrm{e}^{-\lambda} + \lambda = \lambda^2 + \lambda.
\end{aligned}$$