Section 6 Functions of a Random Variable

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1 Gamma Function

The gamma function for **complex numbers with a positive real part** is defined as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx\ .$$

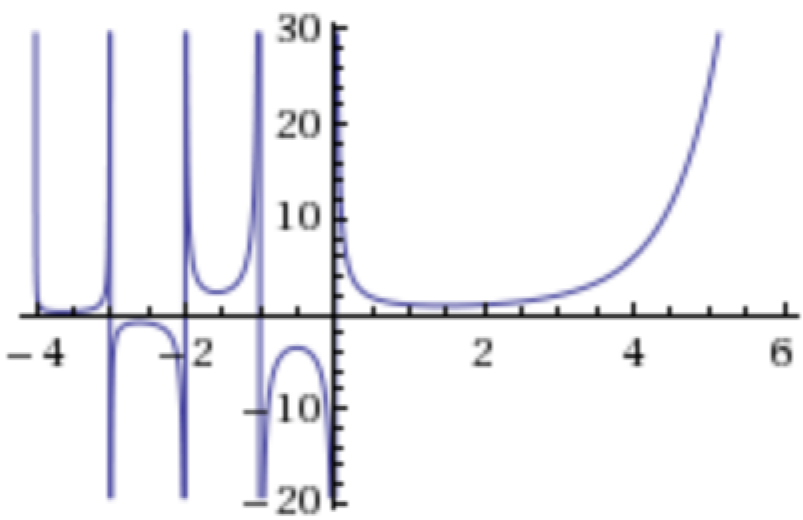

Figure 1: Gamma Function on Real Line

Try to prove:

1. $\Gamma(z+1) = z\Gamma(z)$
2. $\Gamma(1) = 1$
3. $\Gamma(n) = (n-1)!$
4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
5. $\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$

where z is positive real number, and n is positive integer.

1.1 Property of Gamma Function

Using integration by parts, one sees that:

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty x^z e^{-x}\,dx \\ &= \Big[-x^z e^{-x}\Big]_0^\infty + \int_0^\infty z x^{z-1} e^{-x}\,dx \\ &= \lim_{x\to\infty} \left(-x^z e^{-x}\right) - \left(-0^z e^{-0}\right) + z\int_0^\infty x^{z-1} e^{-x}\,dx \\ &= z\Gamma(z)\end{aligned}$$

We can calculate $\Gamma(1)$:

$$\begin{aligned}\Gamma(1) &= \int_0^\infty x^{1-1} e^{-x}\,dx \\ &= \Big[-e^{-x}\Big]_0^\infty \\ &= \lim_{x\to\infty} \left(-e^{-x}\right) - \left(-e^{-0}\right) \\ &= 0-(-1) \\ &= 1.\end{aligned}$$

Given that $\Gamma(1) = 1$ and $\Gamma(n+1) = n\Gamma(n)$,

$$\Gamma(n) = 1\cdot 2\cdot 3\cdots(n-1) = (n-1)!$$

Perhaps the best-known value of the gamma function at a non-integer argument is

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x} dx = \int_0^\infty t^{-1} e^{-t^2} 2t dt = \int_{-\infty}^\infty e^{-t^2} dt = \sqrt{\pi},$$

Given that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(z+1) = z\Gamma(z)$, for non-negative integer values of n we have:

$$\Gamma\left(\tfrac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

1.2 Reading Material on Beta Function (if you are interested)

In mathematics, the beta function is a special function that is closely related to the gamma function and to binomial coefficients. It is defined by the integral

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

for complex number inputs x, y with positive real part.

The beta function is symmetric, meaning that

$$\mathrm{B}(x,y) = \mathrm{B}(y,x)$$

A key property of the beta function is its close relationship to the gamma function:

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

To derive this relation, write the product of two factorials as

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_{u=0}^{\infty} e^{-u}u^{x-1}\,du \cdot \int_{v=0}^{\infty} e^{-v}v^{y-1}\,dv \\ &= \int_{v=0}^{\infty}\int_{u=0}^{\infty} e^{-u-v}u^{x-1}v^{y-1}\,du\,dv.\end{aligned}$$

Changing variables by $u = zt$ and $v = z(1-t)$ produces

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_{z=0}^{\infty}\int_{t=0}^{1} e^{-z}(zt)^{x-1}(z(1-t))^{y-1}z\,dt\,dz \\ &= \int_{z=0}^{\infty} e^{-z}z^{x+y-1}\,dz \cdot \int_{t=0}^{1} t^{x-1}(1-t)^{y-1}\,dt \\ &= \Gamma(x+y)\cdot \mathrm{B}(x,y).\end{aligned}$$

Dividing both sides by $\Gamma(x+y)$ gives the desired result.

The probability density function (PDF) of the beta distribution, for $0 \le x \le 1$, and shape parameters $\alpha, \beta > 0$, is

$$f(x;\alpha,\beta) = \frac{1}{\mathrm{B}(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

2 Gamma Distribution

$$f(x) = \frac{x^{r-1}e^{-\lambda x}\lambda^r}{\Gamma(r)} \quad \text{for } x > 0 \quad \lambda, r > 0,$$

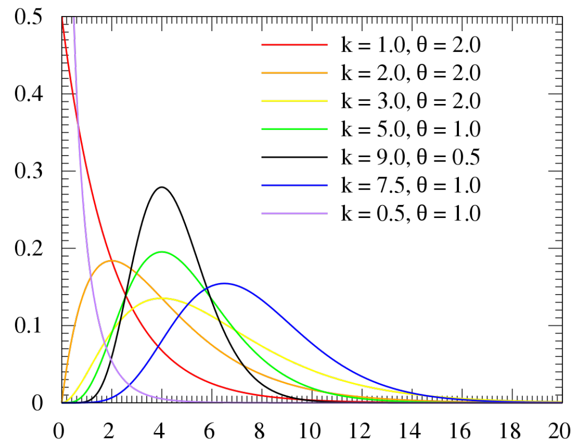

Figure 2: Gamma Distribution ($k = r, \theta = \frac{1}{\lambda}$)

Remark 1 $\int_0^\infty x^{r-1}e^{-\lambda x}dx = \frac{\Gamma(r)}{\lambda^r}$

Remark 2 *Chi-square distribution and exponential distribution are special cases of Gamma distribution.*

Remark 3 ***[Functions of n Random Variables]*** *If $X_i \sim \Gamma(r_i, \lambda)$ for $i = 1, 2, ..., N$ (i.e., all distributions have the same scale parameter λ), then*

$$\sum_{i=1}^{N} X_i \sim \Gamma\left(\sum_{i=1}^{N} r_i, \lambda\right)$$

provided all X_i are independent.

Think about what is the distribution of summation of independent chi-squared distributions?

Some facts:

- summation of independent Bernoulli distributions with the same p : Binomial distribution
- summation of independent Chi-squared distributions: Chi-squared distribution (Gamma distribution)
- summation of independent Normal distribution: Normal distribution

3 Function of a Discrete Random Variable

$f_X(x) = P(X = x)$ is known for a D.R.V. X.

To derive the pmf of $Y = r(X)$:

1. Find Ω_Y
2. $P(Y = y) = P(r(X) = y) = P(X \in A)$, where $A = \{x : x \in \Omega_X, r(X) = y\}$
3. Verify $f_Y(y) = P(Y = y)$ is a pmf

Remark 4 *Why not $P(X = r^{-1}(y))$?*

Ans: The function r is not necessarily inversible, which means many different x's might map to the same y.

3.1 Practice

1.

Example 80 (3.22). Suppose X follows the probability distribution:

X	-2	-1	0	1	2
$f_X(x)$	0.2	0.1	0.1	0.3	0.3

Find the PMF of $Y = X^2 + X$.

2. **[Functions of 2 Random Variables]** X_1, X_2 are independent and both follow $Ber(p)$. Show that $Y = X_1 + X_2$ follows $B(2, p)$.

Remark 5 *Think about how to calculate functions of n random variables.*

4 Function of a Continuous Random Variable

$f_X(x)$ is known for a C.R.V. X.

To derive the pdf of $Y = r(X)$:

1. Find $\Omega_Y = \{y : f_Y(y) > 0\}$
2. $F_Y(y) = P(Y \leq y) = P(r(X) \leq y) = P(X \in A)$, where $A = \{x : x \in \Omega_X, r(X) \leq y\}$
3. $f_Y(y) = \frac{dF_Y(y)}{dy}$
4. Verify $f_Y(y)$ is a pdf

4.1 Practice

1. [**Linear Transformation**] Show that $Y = a + bX$ also follows a normal distribution when $X \sim N(\mu, \sigma^2)$, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ x \in \mathbb{R}$$

.

2. [**Chi-squared Distribution**] A random variable X follows a standard normal distribution, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ x \in \mathbb{R}$$

Find the PDF $f_Y(y)$ of $Y = X^2$.

3. [**Lognormal Distribution**] Suppose a CRV X follows a normal distribution with parameters (μ, σ^2), i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ x \in \mathbb{R}$$

Find the PDF of $Y = e^X$. (Y follows a lognormal distribution.)

4. [**Probability Integral Transform**]: Suppose X has a continuous CDF $F_X(x)$ which is strictly monotonically increasing (which means it is inversible). Define $Y = F_X(X)$. Prove that Y follows a uniform distribution on $[0, 1]$.

5. [Thinking Problem] [Take-Home Practice] [**Functions of 2 random variables**] Suppose X_1, X_2 are independent and both follow $N(0, 1)$. Show that $Y = X_1 + X_2\ \sim N(0, 2)$.

Hint: Let $Y = X_1 + X_2$.

$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y - x) f_{X_2}(x) dx$

1. **[Linear Transformation]** Show that $Y = a + bX$ $(b \neq 0)$ also follows a normal distribution when $X \sim N(\mu, \sigma^2)$, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ x \in \mathbb{R}$$

.

Solution: $\Omega_Y = \mathbb{R}$

1. $b > 0$

$$\begin{aligned} F(Y \leq y) &= P(X \leq \frac{y-a}{b}) \\ &= \int_{-\infty}^{\frac{y-a}{b}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \end{aligned}$$

Therefore,

$$f_Y(y) = \frac{1}{b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\frac{y-a}{b}-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi} b\sigma} e^{-\frac{(y-a-b\mu)^2}{2b^2\sigma^2}}$$

2. $b < 0$

$$\begin{aligned} F(Y \leq y) &= P(X \geq \frac{y-a}{b}) \\ &= P(X \leq 2\mu + \frac{y-a}{-b}) \\ &= \int_{-\infty}^{2\mu + \frac{y-a}{-b}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \end{aligned}$$

Therefore,

$$f_Y(y) = -\frac{1}{b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(2\mu + \frac{y-a}{-b} - \mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}(-b\sigma)} e^{-\frac{(y-a-b\mu)^2}{2b^2\sigma^2}}$$

Remark 6 *If* $X \sim N(\mu, \sigma^2)$, $Y = a + bX \sim N(a + b\mu, b^2\sigma^2)$, $b \neq 0$.

2. **[Chi-squared Distribution]** A random variable X follows a standard normal distribution, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \ x \in \mathbb{R}$$

Find the PDF $f_Y(y)$ of $Y = X^2$.

Solution: Given $Y = X^2$ always takes nonnegative values, we can let $y \geq 0$ and obtain

$$\begin{aligned} P(Y &\leq y) = P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

Therefore, by the chain rule of differentiation, we obtain

$$\begin{aligned} f_Y(y) &= F_X'(\sqrt{y}) \frac{1}{2\sqrt{y}} + F_X'(-\sqrt{y}) \frac{1}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y/2}, \quad y \geq 0. \end{aligned}$$

It follows that

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y/2}, & y \geq 0, \\ 0, & y < 0. \end{cases}$$

The random variable X is called a standard normal variable, denoted as $N(0,1)$, and $Y = X^2$ is called, a chi-square random variable with degree of freedom 1, denoted as χ_1^2.

3. **[Lognormal Distribution]** Suppose a CRV X follows a normal distribution with parameters (μ, σ^2), i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ x \in \mathbb{R}$$

Find the PDF of $Y = e^X$. (Y follows a lognormal distribution.)

Solution: Given $Y = e^X$ is always positive, we let $y > 0$. Then

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(e^X \leq y) \\ &= P(X \leq \ln y) \\ &= F_X(\ln y). \end{aligned}$$

By the chain rule of differentiation, we obtain

$$\begin{aligned} f_Y(y) &= F'_X(\ln y)\frac{1}{y} \\ &= f_X(\ln y)\frac{1}{y} \\ &= \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{y} e^{-(\ln y-\mu)^2/2\sigma^2}, \qquad y > 0. \end{aligned}$$

It follows that

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{y} e^{-(\ln y-\mu)^2/2\sigma^2}, & \text{if } y > 0, \\ 0, & \text{if } y \leq 0. \end{cases}$$

4. **[Probability Integral Transform]**: Suppose X has a continuous CDF $F_X(x)$ which is strictly monotonically increasing (which means it is inversible). Define $Y = F_X(X)$. Prove that Y follows a uniform distribution on $[0,1]$.

Proof: The support of $Y = F_X(X)$ is the unit interval $[0,1]$. Letting $y \in [0,1]$, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P[F_X(X) \leq y]. \end{aligned}$$

Because $F_X(x)$ is strictly increasing, its inverse function, denoted as $F_X^{-1}(y)$, exists and is also strictly increasing. For any real-value x, we have

$$F_X^{-1}[F_X(x)] = x.$$

By using $Y = F_X(X)$ and applying the inverse function operation, we obtain

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P[F_X(X) \leq y] \\ &= P\{F_X^{-1}[F_X(X)] \leq F_X^{-1}(y)\} \\ &= P[X \leq F_X^{-1}(y)] \\ &= F_X[F_X^{-1}(y)] \\ &= y, \text{ for } y \in [0,1]. \end{aligned}$$

It follows that the PDF of Y is given by

$$f_Y(y) = \begin{cases} 1, & \text{for } 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

This is a uniform distribution on $[0,1]$, which is called the standard uniform distribution, denoted $U[0,1]$.

5. [Thinking Problem] [Take-Home Practice] **[Functions of 2 random variables]** Suppose X_1, X_2 are independent and both follow $N(0,1)$. Show that $Y = X_1 + X_2 \ \sim N(0,2)$.

Hint: Let $Y = X_1 + X_2$.

$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y-x) f_{X_2}(x) dx$

Remark 7 *Let X and Y be independent random variables that are normally distributed, then their sum is also normally distributed. i.e., if*

$$X \sim N(\mu_X, \sigma_X^2)$$
$$Y \sim N(\mu_Y, \sigma_Y^2)$$
$$Z = X + Y,$$

then

$$Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$