Section 5 Discrete & Continuous Random Variable

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1 Random Variable

1.1 Discrete Random Variable

[**Prob (Mass) Function**] of a discrete R.V. X is defined as

$$f_X(x) = P(X = x), \ \forall x \in \mathbb{R}$$

Properties of pmf:

(i) $0 \leq P(X = x) \leq 1$ for all $x \in \mathbb{R}$.

(ii) $\sum_x P(X = x) = 1$.

Relationship between pmf and cdf:

pmf→cdf: $F_X(x) = P(X \leq x) = \sum_{y \leq x} P(X = y)$;

cdf→pmf: $P(X = x_i) = \begin{cases} F(x_i) & i = 1 \\ F(x_i) - F(x_{i-1}) & i > 1. \end{cases}$

Remark 1 *pmf and cdf are in one-to-one correspondence.*

1.1.1 Practice

Derive the cdf of X if

$$P(X = x) = \begin{cases} 0.2 & X = 0 \\ 0.5 & X = 2 \\ 0.3 & X = 8 \\ 0 & otherwise \end{cases}$$

1.2 Discrete Distribution Examples

1. Discrete Uniform Distribution

$$P(X = x) = \frac{1}{k},\ x = 1, 2, \ldots, k$$

2. Bernoulli Distribution

$$P(X = x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$$

Remark 2 *Toss a coin and the prob of getting a head* $(X = 1)$ *is* p.

Ber(p)*:* $X = 1$ *head up;* $X = 0$ *tail up.*

3. Binomial Distribution

$$P(X = x) = C_n^x p^x (1 - p)^{n-x},\ x = 0, 1, \ldots, n$$

Remark 3 *Toss a coin* n *times and the prob of getting a head each time is* p.

$B(n, p)$*: Number of getting a head.* $X = 0, 1, \ldots, n$

Remark 4 *If* $Y_1, Y_2, \ldots, Y_n$ *are independent and all follow a Bernoulli distribution* $Ber(p)$*, then* $Y_1 + Y_2 + \ldots + Y_n$ *follows* $B(n, p)$.

4. Poisson Distribution

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!},\ \lambda > 0,\ x = 0, 1, 2, \ldots$$

1.2.1 Practice

1. Verify that $\sum_x P(X = x) = 1$ when X follows Binomial distribution or Poisson distribution.

2. Calculate the probability of getting at least 2 heads in the experiment of tossing a fair coin 10 times.

3. Calculate $P(2 \leq X \leq 4)$ when X follows Poisson distribution when $\lambda = 1$.

1.2.2 Take-Home Practice

Derive the cdf of the distributions mentioned above.

1.3 Continuous Random Variable

[Prob Density Function] Suppose the (cumulative) distribution function $F_X(x)$ of a continuous R.V. X is absolutely continuous, then there exists a prob density function $f_X(x)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du, \ \forall x \in (-\infty, \infty).$$

Remark 5 *Normally, for a C.R.V. X, $P(X = x) = 0$ for any x.*

Therefore, $P(a < X < b) = P(a \le X \le b)$ for a C.R.V X.

Remark 6 *An easy way to understand pdf: The magnitude of $f_X(x)$ at each x indicates the probability that X takes values in a small neighborhood of each point x, i.e.*

$$P(X \in [x, x + dx]) = f_X(x)dx$$

Properties of pdf:

(i) $f_X(x) \ge 0$ for all $x \in \mathbb{R}$;

(ii) $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

Remark 7 *Can $f_X(x) > 1$? Yes.*

Relationship between pdf and cdf:

pdf→cdf: $F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)du$;

cdf→pdf: $f_X(x) = \frac{dF_X(x)}{dx}$.

Remark 8 *Is $f_X(x)$ unique for a given $F_X(x)$? No!*

Is $F_X(x)$ unique for a given $f_X(x)$? Yes!

Again, we emphasize that the fact that $P(X = x) = 0$ for any given x for a C.R.V. allows us to change the value of the pdf of a continuous random variable X at a single point without altering the distribution of X. For instance, the pdf

$$f_X(x) = \lambda e^{-\lambda x}, \ x > 0$$

can be written as

$$f_X(x) = \lambda e^{-\lambda x}, \ x \ge 0$$

without changing $F_X(x)$ for any x. We observe these two functions differ only at $x = 0$ and $P(X = 0) = 0$. ***More generally, if two pdf's of continuous random variables differ only on a set having probability zero, the two corresponding probability distribution functions are exactly the same.*** *Unlike the continuous random variables, the pmf of a discrete random variable cannot be changed at any point, since a change in such a pmf alters the distribution of probability.*

1.3.1 Practice

1. For each of the following, determine the value of c that makes $f_X(x)$ a pdf.

(a) $f_X(x) = c(x+1)^2$, $-1 < x < 1$

(b) $f_X(x) = ce^{-2x}$, $0 < x < \infty$

(c) $f_X(x) = c\cos x, 0 < x < \frac{\pi}{2}$

Remark 9 *When the type of $f_X(x)$ (like exponential, triangular, or composite) is given, then $f_X(x)$ can be determined (because we only need to calculate the constant c).*

Think about how to derive the c if $f_X(x) = ce^{-x^2/2},\ x \in \mathbb{R}$.

1.1 Is it possible to construct a pdf $f_X(x)$ from any nonnegative $g(x)$ with a finite integral, i.e. $0 < \int_{\mathbb{R}} g(x)dx < \infty$?

1.2 Is it possible to construct a symmetric pdf $f_X(x)$ from any nonnegative $g(x)$ with a finite integral, i.e. $0 < \int_{\mathbb{R}} g(x)dx < \infty$?

2.
Find $f_Y(y)$, the pdf of Y, where

$$F_Y(y) = P(Y \le y) = \begin{cases} 1 - \frac{1}{y^2}, & 1 \le y < \infty \\ 0 & otherwise. \end{cases}$$

3. Suppose a C.R.V. X has a pdf $f_X(x) = \begin{cases} |x| & for\ -1 < x < 1 \\ 0 & otherwise \end{cases}$,

Find $P(X < \frac{1}{2})$.

1.4 Gaussian Intergral

The Gaussian integral, also known as the Euler–Poisson integral, is the integral of the Gaussian function $f(x) = e^{-x^2}$over the entire real line. Named after the German mathematician Carl Friedrich Gauss, the integral is

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Proof:

$$\begin{aligned}
\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\, dr\, d\theta \\
&= 2\pi \int_0^{\infty} re^{-r^2}\, dr \\
&= 2\pi \int_{-\infty}^{0} \tfrac{1}{2} e^s\, ds \qquad\qquad s = -r^2 \\
&= \pi \int_{-\infty}^{0} e^s\, ds \\
&= \pi \left(e^0 - e^{-\infty}\right) \\
&= \pi,
\end{aligned}$$

In addition,

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dxdy = \left(\int_{\mathbb{R}} e^{-x^2} dx\right)\left(\int_{\mathbb{R}} e^{-y^2} dy\right) = \left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2$$

1.4.1 Practice

Use the result of Gaussian Intergral to calculate

$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx$

$\int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)}\, dx$

$\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\, dx$

1.5 Continuous Distribution Examples

1. Uniform Distribution

$$f_X(x) = 1,\ 0 \le x \le 1$$

2. Generalized Uniform Distribution

$$f_X(x) = \frac{1}{b-a},\ a \le x \le b$$

3. Exponential Distribution

$$f_X(x) = \lambda e^{-\lambda x},\ x > 0$$

or

$$f_X(x) = \frac{1}{\beta} e^{-x/\beta},\ x > 0$$

4. Standard Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ x \in \mathbb{R}$$

5. Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ x \in \mathbb{R}$$

1.5.1 Practice

Derive the cdf of Uniform distribution, generalized uniform distribution, and exponential distribution.

Remark 10 *The cdf of normal distribution does not have analytical form, which means we cannot write it down explicitly.*