ECON130264.01: Section 3 Conditional Probability

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1 Rules & Definitions

1. Rule of Total Probabilities:

If $\{C_i\}_{i=1}^n$ m.e. & c.e., then

$$P(A) = \sum_{i=1}^{n} P(A \cap C_i)$$

2. Conditional Probability

If $P(B) > 0$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

3. Product Rule

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ P(A \cap B \cap C) &= P(A|B \cap C)P(B|C)P(C) \\ P(\cap_{i=1}^n A_i) &= P(A_1| \cap_{i=2}^n A_i)P(A_2| \cap_{i=3}^n A_i) \cdots P(A_n) \end{aligned}$$

4. Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

If $\{A_i\}_{i=1}^n$ m.e. & c.e., then

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^n P(B|A_j)P(A_j)}$$

5. Independence

$$P(A \cap B) = P(A)P(B)$$

If $P(A) > 0$, then

$$P(B) = P(B|A)$$

Remark 1 *Notation:* $P(AB) := P(A \cap B)$

2 Practice

1. Keep picking balls from a box without replacement. There are r red balls and b blue balls in the box.

(a) What is the prob that the second pick is blue?

(b) What is the prob that the third pick is blue? What is the prob that the n^{th} pick is blue? (No given information.)

2. Let $B_1, \cdots, B_k$ be mutually exclusive, and let $B = \cup_{i=1}^{k} B_i$. Suppose $P(B_i) > 0$ and $P(A|B_i) = p$ for $i = 1, \cdots, k$. Find $P(A|B)$.

3. [**Monty Hall Problem**] (Goat or Car?) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

What is the prob of car behind No.1 (or car behind No.2) given that a goat is behind No.3?

Behind door 1	Behind door 2	Behind door 3	Result if staying at door #1	Result if switching to the door offered
Goat	Goat	**Car**	Wins goat	**Wins car**
Goat	**Car**	Goat	Wins goat	**Wins car**
Car	Goat	Goat	**Wins car**	Wins goat

Figure 1: Enumeration Method.

4. [**Independence**] Suppose that the chance of you winning a weekly lottery is $\alpha = 10^{-5}$. What is the prob that you never win a lottery if you insist on buying lottery tickets for ten years (every year has 52 weeks).

When $\alpha = 10^{-5}$, $P = 0.9948$

When $\alpha = 10^{-4}$, $P = 0.9493$

5. [**Bayes' Theorem**] A publisher sends a sample statistics textbook to 80% of all statistics professors in the US. schools. 30% of the professors who received this sample textbook adopted the book, as did 10% of the professors who did not receive the sample book. What is the probability that a professor who adopts the book has received a sample book?

6. [**Bayes' Theorem**] In the final exam, there is a multiple-choice problem, i.e. picking a right answer from A, B, C, and D. Since the problem is difficult, only $\beta = 5\%$ of students can solve the it problem out. Assume that the prob of students who are able to solve the problem out picking the right answer is 0.99, and students who cannot solve it out randomly pick a answer (which means he/she picks the right answer with probability 0.25). In the students who pick the right answer, what proportion of students are not able to solve the problem ?

When $\beta = 5\%$, $P = 0.8275$

When $\beta = 90\%$, $P = 0.0273$

3 Take-Home Practice

1. Think about how to use **Bulkhead Method** (introduced in the second TA session) to figure out how many different outcomes does picking r objects out of n (unordered sampling, with replacement) have?

Answer: C_{r+n-1}^{r} or C_{r+n-1}^{n-1}

Recall: How many distinct solutions $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \ldots + x_r = n, \ x_i \geq 0, \ i = 1, \ldots, r$$

Example: Put n balls which are indistinguishable from each other into r boxes. (Some boxes might be empty.)

Answer: C_{n+r-1}^{r-1} or C_{n+r-1}^{n}

2. If you play better than your opponent, which one do you prefer, 'best of three games' or 'best of five games', (i.e. winning at least 2 games out of 3, or winning at least 3 games out of 5)?