ECON130264.01: Section 4 Random Variable

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1 Independence

1. Independence

$$P(A \cap B) = P(A)P(B)$$

If $P(A) > 0$, then

$$P(B) = P(B|A)$$

2. Jointly Independent for 3 events

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \\ P(A \cap B \cap C) &= P(A)P(B)P(C) \end{aligned}$$

1.1 Example

Let $\Omega = \{1, 2, 3, 4\}$, and $P(i) = \frac{1}{4},\ i = 1, 2, 3, 4$. Let $A = \{1, 2\}, B = \{1, 3\}, C = \{2, 3\}$.

A, B, C are pairwise independent but not jointly independent.

2 Random Variable

2.1 Discrete Random Variable

[**Prob (Mass) Function**] of a discrete R.V. X is defined as

$$f_X(x) = P(X = x), \ \forall x \in \mathbb{R}$$

Properties of pmf:

(i) $0 \leq P(X = x) \leq 1$ for all $x \in \mathbb{R}$.

(ii) $\sum_x P(X = x) = 1$.

Relationship between pmf and cdf:

pmf→cdf: $F_X(x) = P(X \leq x) = \sum_{y \leq x} P(X = y)$;

cdf→pmf: $P(X = x_i) = \begin{cases} F(x_i) & i = 1 \\ F(x_i) - F(x_{i-1}) & i > 1. \end{cases}$

Remark 1 *pmf and cdf are in one-to-one correspondence.*

2.1.1 Practice

1. Suppose random variable X has a cdf

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.4 & -2 \leq x < 0 \\ 0.5 & 0 \leq x < 1 \\ 0.8 & 1 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

Find

(a) the probability (mass) function $f(x) = P(X = x)$

(b) $P(-1 < X < 4)$

2.2 Continuous Random Variable

[**Prob Density Function**] Suppose the (cumulative) distribution function $F_X(x)$ of a continuous R.V. X is absolutely continuous, then there exists a prob density function $f_X(x)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du, \ \forall x \in (-\infty, \infty).$$

Remark 2 *Normally, for a C.R.V. X, $P(X = x) = 0$ for any x.*

Properties of pdf:

(i) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$;

(ii) $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

Remark 3 *Can $f_X(x) > 1$? Yes.*

Relationship between pdf and cdf:

pdf→cdf: $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)du$;

cdf→pdf: $f_X(x) = \frac{dF_X(x)}{dx}$.

Remark 4 *pdf and cdf are in one-to-one correspondence.*

2.2.1 Practice

1. For each of the following, determine the value of c that makes $f_X(x)$ a pdf.

(a) $f_X(x) = c(x+1)^2$, $-1 < x < 1$

(b) $f_X(x) = ce^{-2x}$, $0 < x < \infty$

Remark 5 *$f_X(x) = \lambda e^{-\lambda x}, 0 < x < \infty$ is called exponential distribution, which is a common distribution.*

(c) $f_X(x) = c\cos x, 0 < x < \frac{\pi}{2}$

Remark 6 *When the type of $f_X(x)$ (like exponential, triangular, or composite) is given, then $f_X(x)$ can be determined (because we only need to calculate the constant c).*

2.
Find $f_Y(y)$, the pdf of Y, where

$$F_Y(y) = P(Y \leq y) = \begin{cases} 1 - \frac{1}{y^2}, & 1 \leq y < \infty \\ 0 & otherwise. \end{cases}$$

3. Suppose a C.R.V. X has a pdf $f_X(x) = \begin{cases} |x| & for \ -1 < x < 1 \\ 0 & otherwise \end{cases}$,

Find $P(X < \frac{1}{2})$.