Section 12 Practice

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1 Practice

[3σ Principle] Suppose $X \sim N(\mu, \sigma^2)$, calculate
a. $P(|X-\mu| < \sigma)$
b. $P(|X-\mu| < 2\sigma)$
c. $P(|X-\mu| < 3\sigma)$

5.40. [#4.50, p.201] Suppose (X, Y) has a bivariate normal PDF

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}.$$

Show that $\mathrm{corr}(X,Y) = \rho$, and $\mathrm{corr}(X^2, Y^2) = \rho^2$. (Hint: Conditional expectation will simplify calculations.)

5.56. [#4.58, page 202] For any two random variables X and Y with finite variances, prove that
(1) $\mathrm{cov}(X,Y) = \mathrm{cov}(X, E(Y|X))$.
(2) X and $Y - E(Y|X)$ are uncorrelated.
(3) $\mathrm{var}[Y - E(Y|X)] = E[\mathrm{var}(Y|X)]$.

5.50. Let X and Y be two random variables and $0 < \sigma_X^2 < \infty$. Show that if $E(Y|X) = \alpha_o + \alpha_1 X$, then $\alpha_1 = \mathrm{cov}(X,Y)/\sigma_X^2$.

5.51. Suppose $E(Y|X)$ is a linear function of X, i.e., $E(Y|X) = a + bX$ for some constants a, b. Find the expressions of a and b in terms of $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$ and $\text{cov}(X, Y)$. Is it true that when $E(Y|X)$ is linear in X, then $E(Y|X)$ does not depend on X if and only if $\text{cov}(X, Y) = 0$? Give your reasoning.

1.1 Take-Home Practice

Suppose (X, Y) has a joint pdf

$$f(x, y) = \frac{1}{\Gamma(k_1)\Gamma(k_2)} x^{k_1-1} (y-x)^{k_2-1} e^{-y}$$

where $k_1, k_2 > 0$, $0 < x \leq y < \infty$. Find the marginal distributions of X and Y.

5.48. Suppose $Y = \alpha_0 + \alpha_1 X + \varepsilon\sqrt{\beta_0 + \beta_1 X^2}$, where ε and X are mutually independent random variables, with $E(\varepsilon) = 0$ and $\text{var}(\varepsilon) = 1$. Find (1) $E(Y|X)$; (2) $\text{var}(Y|X)$.

[3σ]

We know $P(X<x) = \Phi(x)$. if $X \sim N(0,1)$.

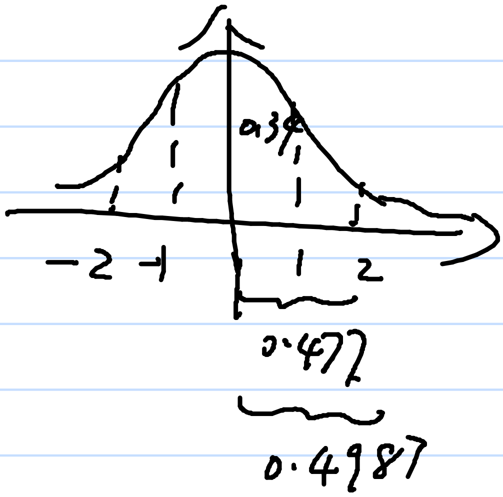
If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$.

a. $P(|X-\mu| < \sigma)$

$= P\left(\frac{|X-\mu|}{\sigma} < 1\right)$

$= P(-1 < Z < 1)$ where $Z \sim N(0,1)$.

$= 0.68$



b. $P(|X-\mu| < 2\sigma)$

$= P(-2<Z<2) = 0.95$

c. $P(|X-\mu| < 3\sigma)$

$= P(-3 < Z < 3)$

$= 0.995$

99.5% of data will be in.

$[\mu - 3\sigma, \mu + 3\sigma]$

5.40

$$\operatorname{cor}(X,Y) = \rho$$

$$\operatorname{corr}(X^2,Y^2) = \frac{\operatorname{Cov}(X^2,Y^2)}{\operatorname{Var}(X^2)\operatorname{Var}(Y^2)}$$

$$= \frac{E(X^2Y^2) - EX^2EY^2}{\sqrt{(EX^4-(EX^2)^2)(EY^4-(EY^2)^2)}}$$

$$= \frac{(2\rho^2+1) - 1\times 1}{\sqrt{(3-1)(3-1)}} = \rho^2$$

Prove. $E(X^2Y^2) = 2\rho^2+1$

$$EX^2Y^2 = E(X^2E(Y^2|X))$$

$$= E\{X^2[\operatorname{Var}(Y|X) + (EY|X)^2]\}$$

$$= E\{X^2((1-\rho^2) + (\rho X)^2)\}$$

$$= E\{X^2(1-\rho^2) + \rho^2X^4\}$$

$$= (1-\rho^2) + 3\rho^2$$

$$= 2\rho^2+1$$

5.56

(1) Right $= Cov(X, E(Y|X))$

$= E(X E(Y|X)) - EX\,EY.$

$= E(E(XY|X)) - EX\,EY$

$= EXY - EX\,EY$

(2) $Cov(X, Y - E(Y|X))$

$= E\{X(Y - E(Y|X))\} - EX(\cancel{EY - EY})$

$= EXY - E\{X E(Y|X)\}$

$= EXY - E\{E(XY|X)\}$

$= EXY - EXY$

$= 0$

(3) $Var[Y - E(Y|X)]$

$= \underline{Var Y} + Var[E(Y|X)] - 2cov[Y, E(Y|X)]$

$= E[Var(Y|X)] + 2Var[E(Y|X)] - 2cov[Y, E(Y|X)]$

Prove $Var(E(Y|X)) = Cov(Y, E(Y|X))$

$cov(Y, E(Y|X)) = E[Y E(Y|X)] - EY\,E[E(Y|X)]$

$= E[Y E(Y|X)] - (EY)^2$

$Var[E(Y|X)] = E[(E(Y|X))^2] - \{E(E(Y|X))\}^2$

$= E[(E(Y|X))^2] - (EY)^2$

Prove $E[Y E(Y|X)] = E[(E(Y|X))^2]$

$E[Y E(Y|X)] = E\{ E[Y E(Y|X) | X] \}$

$= E\{ E(Y|X) E(Y|X) \}.$

Intuition?

5.50

$$E(Y|X) = \alpha_0 + \alpha_1 X \longrightarrow 2 \text{ unknown parameters } \alpha_1, \alpha_0.$$

$$Var[E(Y|X)] = \alpha_1^2 \, Var X$$

$$\alpha_1^2 = \frac{Var[E(Y|X)]}{Var X}$$

#5.56(3)

$$= \frac{cov(Y, E(Y|X))}{Var X}$$

$$= \frac{cov(Y, \alpha_0 + \alpha_1 X)}{Var X}$$

$$= \frac{\alpha_1 \, cov(X, Y)}{Var X}$$

$$\Rightarrow \alpha_1 = \frac{cov(X, Y)}{Var X}$$

5.51

$$E(Y|X) = a + bX$$

① $$b = \frac{cov(X,Y)}{\sigma_X^2}$$ From #5.50

② Since. $\mu_Y = a + b\mu_X$

$$a = \mu_Y - b\mu_X$$

$$= \mu_Y - \frac{cov(X,Y)}{\sigma_X^2}\mu_X$$

when $E(Y|X) = a + bX$,

$$b = 0 \iff cov(X,Y) = 0.$$

Yes!