Section 8 Moment Generating Function, Bi-variate Distribution

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1 Bivariate Random Variables

1. CDF: Joint Distribution Function:

$$\begin{aligned} F_{XY}(x,y) &= P(X \leq x, Y \leq y) \\ &= P(X \leq x \cap Y \leq y) \end{aligned}$$

for any pair (x,y) in the $xy-$plane.

2. pmf/pdf

DRV: Let X and Y be two drv, then their **joint probability mass function (pmf)** is defined as

$$f_{XY}(x,y) = P(X = x \ \cap \ Y = y) = P(X = x, Y = y)$$

for any point (x,y) in the xy-plane.

(i) $0 \leq f_{XY}(x,y) \leq 1$ for all (x,y).

(ii) $\sum_y \sum_x f_{XY}(x,y) = 1$.

CRV: Two rv's X and Y are said to have a continuous joint distribution if there exists a nonnegative function $f_{XY}(x,y)$ such that for any subset A on the xy-plane,

$$P[(X,Y) \in A] = \int\int_{(x,y)\in A} f_{XY}(x,y)dxdy.$$

The function $f_{XY}(x,y)$ is called the **joint probability density function (pdf).**

(i) $f_{XY}(x,y) \geq 0$;

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$.

Exampls:

1. If the joint probability distribution of X and Y is given by $f(x,y) = \frac{x+y}{42}$, for $x = 1,2,3,4$; $y = 0,1,2$

find $P(X \geq Y)$ and $P(X + Y > 4)$

2. If the joint probability distribution of X and Y is given by

$$f(x,y) = \begin{cases} 6x & 0 < x < 1,\ 0 < y < 1 - x \\ 0 & elsewhere \end{cases}$$

Find (a) $P(X + Y = 1)$;

(b) $P(X + Y < 1/2)$

3. Marginal distributions

DRV: Suppose X and Y have a joint discrete distribution with joint pmf $f_{XY}(x,y)$, then the marginal pmf of X and Y are defined as

$$f_X(x) = P(X=x) = \sum_y f_{XY}(x,y), \text{ where } -\infty < x < \infty,$$
$$f_Y(y) = P(Y=y) = \sum_x f_{XY}(x,y), \text{ where } -\infty < y < \infty.$$

(i) $0 \le f_X(x) \le 1,\ 0 \le f_Y(y) \le 1$;

(ii) $\sum_x f_X(x) = 1,\ \sum_y f_Y(y) = 1$.

CRV: Suppose X and Y have a joint continuous distribution with joint pdf $f_{XY}(x,y)$, then the marginal pdf's of X and Y are defined as

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy, \text{ where } -\infty < x < \infty,$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx, \text{ where } -\infty < y < \infty.$$

(i) $f_X(x) \ge 0,\ f_Y(y) \ge 0$;

(ii) $\int f_X(x)dx = 1,\ \int f_Y(y)dy = 1$.

Exampls:

1. Suppose that X and Y have the following joint probability distribution:

$$f(x,y) = \begin{cases} \frac{3x-y}{9} & 1 < x < 3,\ 1 < y < 2 \\ 0 & elsewhere \end{cases}$$

(a) Find the marginal distribution of X.

(b) Find the marginal distribution of Y.

2. We have a following bivariate pdf of X and Y :

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & elsewhere \end{cases}$$

(a) Find the marginal distribution of X.

(b) Find the marginal distribution of Y.