ECON130264.01: Section 2 Foundation of Probability Theory

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1 Basic Concepts

- Random Experiment
- Basic Outcome: the possible outcomes of a random experiment
- Sample Space: the set of all basic outcomes
- Event: a collection of basic outcomes or a subset of sample space

  Remark 1 *Relationship: Basic Outcome* $\subseteq$ *Event* $\subseteq$ *Sample Space*

1.1 Practice

A random experiment: Toss a dice until '6' appears.

1. Describe the sample space.
2. Denote the event E_n as 'The experiment stops after n tosses.' What basic outcomes does E_n contain?
3. What does the event $(\cup_{i=1}^{n} E_n)^c$ mean?

2 Set Theory

2.1 Set Operations

- Intersection: $A \cap B$
- Union: $A \cup B$

2.2 Set Relationship

- Complement: $A^c \cap A = \emptyset$, $A^c \cup A = S$
- Exclusiveness: $A \cap B = \emptyset$. In this case, A and B are called **mutually exclusive**, or **disjoint**.
- Collective Exhaustiveness: $\cup_{i=1}^{n} A_i = S$

2.3 Laws of Set Operations

- Commutativity
- Associativity
- Distributivity
- De Morgan's Laws

2.4 Practice

1. Suppose the events A and B are disjoint. Under what condition are A^c and B^c also disjoint?

   Hint: Use De Morgan's Laws.

3 Probablity Function

- Probability Space: $(S, \mathcal{B}, P)$
- Measurable Space: $(S, \mathcal{B})$
- Sample Space: S
- Borel Field: $\mathcal{B}$
- Prbobility Function: P

3.1 Countable Set

Question: Why there is always ∞ in the definition of Borel Field and Probability Function?

- If $A_1, A_2, \ldots \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i = \mathcal{B}$
- If $A_1, A_2, \ldots \in \mathcal{B}$ are disjoint, then $P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

What about the finite case?

[Definition][Countable Set] A set S is countable if it has the same cardinality (the number of elements of the set) as some subset of the set of natural numbers $N = \{0, 1, 2, 3, \ldots\}$. It simply means that every element in S has the correspondence to a different element in N.

A countable set is either a finite set or a countably infinite set. **In the case of a finite set, let infinite empty sets $\emptyset$ map to redundant elements in N.**

3.1.1 Examples

Countable sets: the set of natural numbers N, the set of rational numbers Q

Uncountable sets: the set of real numbers R, the set of complex numbers C

Remark 2 *Intuitively, the size of R is bigger than that of N, while the sizes of N and Q are the same.*

3.2 Practice

1. Show that when $P(A) = 1$, for any event B, $P(A \cap B) = P(B)$
2. Show that when $P(A) = 0$, for any event B, $P(A \cup B) = P(B)$
3. Does $P(A) = 1$ mean that A is equal to the sample space?

   Does $P(A) = 0$ mean that A is an empty set?
4. Show that

$$P\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n+1} P(\cap_{i=1}^{n} A_i)$$

   Hint: In class, we learned that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

   $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
5. Intuitively, for any sequence of events $\{A_i \in \mathcal{B}, i = 1, \ldots, n\}$,

$$P\left(\cup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

   The equation holds if and only if $\{A_i\}$'s are disjoint.

4 Combination & Permutation

The number of orderings of k items from n objects is

$$P_n^k = \frac{n!}{(n-k)!}.$$

The number of subsets (without orderings) of k items from n objects is

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

Remark 3 $C_n^k = \frac{P_n^k}{k!}$

Usually, we calculate the probability of event A by

$$P(A) = \frac{\text{\# of basic outcomes in A}}{\text{\# of basic outcomes in S}}$$

4.1 Practice

1. Suppose that a class contains 15 boys and 30 girls, and 10 students will be selected randomly to form a term. What is the probability that exactly 3 boys will be selected?
2. **[Birthday Problem]** Suppose there are k $(2 \leq k \leq 365)$ students in this class, what is the probability that at least two students have the same birthday? (That is, on the same day of

the same month, not not necessarily of the same year)?

Compute the probability when $k = 50$.

3. In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

4. How many distinct solutions $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \ldots + x_n = n, \; x_i > 0, \; i = 1, \ldots, r$$

Example: Put r balls which are indistinguishable from each other into n boxes, and there is at least one ball in each box.

5. How many distinct solutions $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \ldots + x_n = n, \; x_i \geq 0, \; i = 1, \ldots, r$$

Example: Put r balls which are indistinguishable from each other into n boxes. (Some boxes might be empty.)