Some notations first:

Y - outcome. D - treatment. Z - instrument. || S - missing indicator,
W - covariates for instruments. || X - covariates for NA.
W and X are non-overlapping.

# of nuisance params: r, q (q_1, q_2). $a_1, a_0, b_1, b_0, f_1, f_0, g_1, g_0$.

where. $r(W) = \mathbb{E}[Z \mid W]$.

$q_1(X) = \mathbb{E}[S \mid X]$. $\quad q_2(X, W) = \mathbb{E}[S \mid X, W]$.

$a_1(W) = \mathbb{E}[YZ \mid S=1, W]$. $\quad a_0(W) = \mathbb{E}[Y(1-Z) \mid S=1, W]$.

$b_1(X, W) = \mathbb{E}[YZ \mid S=1, X, W]$. $\quad b_0(X, W) = \mathbb{E}[Y(1-Z) \mid S=1, X, W]$.

$f_1(W) = \mathbb{E}[DZ \mid S=1, W]$. $\quad f_0(W) = \mathbb{E}[D(1-Z) \mid S=1, W]$.

$g_1(X, W) = \mathbb{E}[DZ \mid S=1, X, W]$ $\quad g_0(X, W) = \mathbb{E}[D(1-Z) \mid S=1, X, W]$.

(I think you can somehow simplify and potentially reduce # of params by 2 or smth, but I am too tired to investigate.).

Proof starts on next page.

$$\mathbb{E}[m(1,W)] = \mathbb{E}[Y \mid Z=1, W] = \mathbb{E}\left[\frac{YZ}{r(W)}\right] \quad \text{by IPW.} \quad \text{assume. } (Y,Z,W) \perp S \mid X.$$

$$= \mathbb{E}\left[\ \mathbb{E}\left[\frac{YZ}{r(W)} \,\middle|\, W, X\right]\right] \quad \text{by LIE.}$$

$$= \mathbb{E}\left[\frac{1}{r(W)} \mathbb{E}[YZ \mid W, S=1, X]\right]$$

$$= \mathbb{E}\left[\frac{1}{r(W)} \cdot \frac{YZS}{\mathbb{P}[S=1 \mid W, X]}\right]$$

$$= \mathbb{E}\left[\frac{YZS}{r(W)\, q(W,X)}\right] \rightarrow \text{interchangeable. } \text{"}q(X).$$

$$\Rightarrow g^{1,1} = \frac{YZS}{r(W)\, q(X,W)}.$$

let $A = (Y, Z, S, W, X)$.

$$g^{1,1}_{2,1}(A; r, q) = \frac{YZS}{q(X,W)} \cdot \left(-\frac{1}{[r(W)]^2}\right).$$

$$\mathbb{E}[g^{1,1}_{2,1}(A, \eta_{1,1})] = \mathbb{E}\left[-\frac{YZS}{[r(W)]^2 q(X,W)} \,\middle|\, W\right]$$

$$= -\frac{1}{[r(W)]^2} \mathbb{E}\left[\frac{YZS}{q(X,W)} \,\middle|\, W\right]$$

$$= -\frac{1}{[r(W)]^2} \mathbb{E}\left[\ \mathbb{E}\left[\frac{YZS}{q(X,W)} \,\middle|\, X, W\right] \,\middle|\, W\right]$$

$$= -\frac{1}{[r(W)]^2} \mathbb{E}\left[\mathbb{E}[YZ \mid S=1, X, W] \mid W\right]$$

$$= -\frac{1}{[r(W)]^2}\,\mathbb{E}[YZ \mid S=1, W]. \quad =: \frac{a(W)}{[r(W)]^2}.$$

$$g^{1,1}_{2,2}(A; r, q) = \frac{YZS}{r(W)} - \frac{1}{[q(X,W)]^2}.$$

$$\mathbb{E}[g^{1,1}_{2,2}(A; \eta_{1,1})] = \mathbb{E}\left[-\frac{YZS}{r(W)\,[q(X,W)]^2} \;\middle|\; X, W \right]$$

$$= -\frac{1}{r(W)\,[q(X,W)]^2}\,\mathbb{E}[YZS \mid X, W]$$

$$= -\frac{\mathbb{E}[YZ \mid S=1, X, W]}{r(W)\,[q(X,W)]} =: \frac{-\,b(X,W)}{r(W)\,q(X,W)} = -\frac{b_1(X,W)}{r(W)\cdot q(X)}.$$

This allows to define. in the actual code, use as much params as possible?

$$g^{1,1,0}(A; r, q) = \frac{YZS}{r(W)\,\underset{q(X).}{q(X,W)}} - [Z - r(W)]\cdot\frac{a(W)}{[r(W)]^2}$$

$$- [S - \underset{q(X).}{q(X,W)}]\cdot\frac{b_1(X,W)}{r(W)\cdot \underset{q(X).}{q(X,W)}}$$

Need to verify N.O. Not hard to see that $\mathbb{E}[g^{1,1,0}(A; r, q)] = m_1$.

concentrate on the orthogonal part.

4 nuisance params: r, q, a, b.

since we're using r.

$$h_r(\alpha) = \mathbb{E}[g^{1.1.0}(A;\; r + \alpha(\tilde{r} - r), q, a, b)].$$

$$= \mathbb{E}\left[\frac{YZS}{[r+\alpha(\hat{r}-r)]\cdot q} - \frac{Z\cdot a}{[r+\alpha(\hat{r}-r)]^2} + \frac{a}{[r+\alpha(\hat{r}-r)]} - \frac{(S-q)\cdot b}{[r+\alpha(\hat{r}-r)]\, q} \right]$$

$$h_r'(\alpha)\Big|_{\alpha=0} = \mathbb{E}\left[-\frac{YZS(\hat{r}-r)}{q\cdot r^2} + \frac{2Z\cdot a(\hat{r}-r)}{r^3} - \frac{a(\hat{r}-r)}{r^2} + \frac{(S-q)\cdot b(\hat{r}-r)}{q\cdot r^2} \right]$$

$$-YZS[Z|W] + Z\cdot[YZ|W]\cdot[S|W,X] - [YZ|W]\cdot[Z|W]\cdot[S|W,X]$$ $\overset{\shortparallel}{0}$ by LIE.

$$= 0!$$

$$h_q(\alpha) = \mathbb{E}[g^{1,1,0}(A; r, q+\alpha(\hat{q}-q), a, b)].$$

$$= \mathbb{E}\left[\frac{YZS}{r[q+\alpha(\hat{q}-q)]} - \cdots - \frac{S\cdot b}{r[q+\alpha(\hat{q}-q)]} + \cdots \right].$$

$$h_q'(\alpha)\Big|_{\alpha=0} = \mathbb{E}\left[-\frac{YZS(\hat{q}-q)}{r\, q^2} + \frac{S\cdot b(\hat{q}-q)}{r\cdot q^2} \right] = 0 \text{ by LIE.}$$

$$h_a(\alpha) = \mathbb{E}[g^{1,1,0}(A; r, q, a+\alpha(\hat{a}-a), b)].$$

$$= \mathbb{E}\left[-\frac{(Z-r)\cdot[a+\alpha(\hat{a}-a)]}{r} \right] = 0 \Rightarrow h_a'(\alpha)\Big|_{\alpha=0} = 0.$$

$$h_b(\alpha) = \mathbb{E}[g^{1,1,0}(A; r, q, a, b+\alpha(\hat{b}-b))].$$

$$= \mathbb{E}\left[-(S-q)\cdot\frac{[b+\alpha(\hat{b}-b)]}{r\cdot q} \right] = 0. \Rightarrow h_b'(\alpha)\Big|_{\alpha=0} = 0.$$

Similarly

$$\mathbb{E}[m(0,W)] = \mathbb{E}\left[\frac{Y(1-Z)}{1-r(W)}\right] \text{ by IPW.}$$

$$= \mathbb{E}\left[\mathbb{E}\left[\frac{Y(1-Z)}{1-r(W)}\middle| W, X\right]\right]. \quad \text{by LIE.}$$

$$= \mathbb{E}\left[\frac{1}{1-r(W)}\mathbb{E}[Y(1-Z)|W, S=1, X]\right]$$

$$= \mathbb{E}\left[\frac{1}{1-r(W)} \cdot \frac{Y(1-Z)S}{\mathbb{P}[S=1|W,X]}\right].$$

$$= \mathbb{E}\left[\frac{Y(1-Z)S}{r(W)\, q(W,X)}\right]. \quad q(X).$$

$$\Rightarrow g^{1,0} = \frac{Y(1-Z)S.}{[1-r(W)]q(X,W).} = \frac{Y(1-Z)\cdot S}{[1-r(W)]q(X).}$$

let $A = (Y, Z, S, W, X)$.

$$g^{1,0}_{2,1}(A; r, q) = \frac{Y(1-Z)S}{q(X,W)} - \frac{1}{[1-r(W)]^2}(-1) = \frac{Y(1-Z)S}{[1-r(W)]^2 q(X,W).}$$

$$\mathbb{E}[g^{1,0}_{2,1}(A, \eta_{1,1})] = \mathbb{E}\left[\frac{Y(1-Z)S}{[1-r(W)]^2 q(X,W).}\middle| W\right]$$

$$= \frac{1}{[1-r(W)]^2}\mathbb{E}\left[\frac{Y(1-Z)S}{q(X,W)}\middle| W\right]$$

$$= \frac{1}{[1-r(W)]^2}\mathbb{E}\left[\frac{1}{q(X,W)}\mathbb{E}[Y(1-Z)S|X,W] \middle| W\right]$$

$$= \frac{1}{[1-r(W)]^2}\mathbb{E}\left[\mathbb{E}[Y(1-Z)|S=1, X, W]|W\right]$$

$$= \frac{1}{[1-r(W)]^2}\mathbb{E}[Y(1-Z)|S=1, W]. =: \frac{a_0(W).}{[1-r(W)]^2.}$$

$$g^{1,0}_{2,2}(A; r, q) = \frac{Y(1-Z)S}{[1-r(W)]} \cdot \frac{1}{q(X,W)^2}.$$

$$\mathbb{E}[g^{1,0}_{2,2}(A; r, q)] = \mathbb{E}\left[- \frac{Y(1-Z)\cdot S}{[1-r(W)]\,[q(X,W)]^2} \,\middle|\, X, W \right].$$

$$= - \frac{1}{[1-r(W)]\,q(X,W)} \mathbb{E}[Y(1-Z) \mid S=1, X, W].$$

$$=: \frac{-b_0(X,W)}{[1-r(W)]\cdot q(X,W)}.$$

probb define
$q_1 = q(X,W)$. $q_2 = q(X)$
in code.

This allows us to define.

$$g^{1,0,0}(A; r, q) = \frac{Y(1-Z)S.}{[1-r(W)]q(X,W)} - [Z - r(W)]\cdot \frac{a_0(W)}{[1-r(W)]^2}$$
$$- [S - q(X,W)]\cdot \frac{b_0(X,W)}{[1-r(W)]q(X,W)}.$$

check N.O.

$$h_r(\alpha) = \mathbb{E}\left[\frac{Y(1-Z)S}{[\overset{1-}{r}+\alpha(\hat{r}-r)]q} - \frac{a_0[Z - r - \alpha(\hat{r}-r)]}{[1-r-\alpha(\hat{r}-r)]^2} - (S-q)\frac{b}{q}\,\frac{1}{[r+\alpha(\hat{r}-r)]}. \right]$$

$$h'_r(\alpha)\Big|_{\alpha=0.} = \mathbb{E}\left[- \frac{Y(1-Z)S}{(1-r)^2\cdot q}^{(\hat{r}-r)} - a_0\left[\frac{(1-r)^{\cancel{2}}\cdot(-1) - (Z-r)\cdot 2\,\cancel{(1-r)}}{(1-r)^{\cancel{4}3}}\right] + (S-q)\frac{b}{q}\,\frac{1}{(1-r)^2} \right].$$

$=0$ by all kinds of LIE.

By similar thoughts. need $(Y, D, Z, W) \perp S \mid X$.
in similar veins. yields. $\underbrace{Y, D, Z, W, S, X}_{A.}$

$$g^{2,1,0}(A; r, q) = \frac{DZS}{r(W)\cdot q(X,W)} - [Z - r(W)]\cdot \frac{\overset{\sim f_1(W)}{\mathbb{E}[DZ \mid S=1, W]}}{[r(W)]^2} - [S - q(X,W)]\cdot \frac{\overset{\sim g_1(W)}{\mathbb{E}[DZ \mid S=1, X, W]}}{r(W)\, q(X,W)}.$$

$$g^{2,0,0}(A; r, q) = \frac{D(1-Z)S}{[1-r(W)]\, q(X,W)} - [Z - r(W)]\cdot \frac{\overset{\sim f_0(W)}{\mathbb{E}[D(1-Z) \mid S=1, W]}}{[1-r(W)]^2}$$
$$- [S - q(X,W)]\cdot \frac{\overset{\sim g_0(W)}{\mathbb{E}[D(1-Z) \mid S=1, X, W]}}{[1-r(W)]\, q(X,W)}$$

# of nuisance params: $r, q\ (q_1, q_2), a_1, a_0, b_1, b_0, f_1, f_0, g_1, g_0$.

where. $r(W) = \mathbb{E}[Z \mid W]$.

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$b_1(X,W) = \mathbb{E}[YZ \mid S=1, X, W]$. $\qquad b_0(X,W) = \mathbb{E}[Y(1-Z) \mid S=1, X, W]$.

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$g_1(X,W) = \mathbb{E}[DZ \mid S=1, X, W]$ $\qquad g_0(X,W) = \mathbb{E}[D(1-Z) \mid S=1, X, W]$.

Also not hard to see that the score function.

$g^0 = g^{1,1,0} - g^{1,0,0} - (g^{2,1,0} - g^{2,0,0})\, \tau_{LATE}$.

is N.O. wrt. τ_{LATE}. ∎