[L9] Robotics (ME671): Inverse Kinematics - 2

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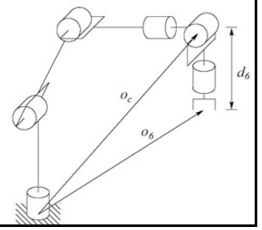
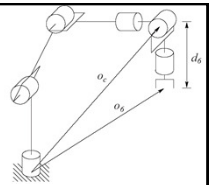
Overview: kinematic decoupling

- Systems that have an arm and a wrist
  - Such that the wrist joint axes are aligned at a point
- Inverse kinematics problem can be split into two parts:
  1. Inverse position kinematics: position of the wrist center
  2. Inverse orientation kinematics: orientation of the wrist



Kinematic Decoupling



Given the pose of tool frame ot 6 DOF robot

$$R_6^0(q_1, \dots, q_6) = R \qquad o_6^0(q_1, \dots, q_6) = o$$

1. Find q_1, q_2, q_3 such that the position of the wrist center is:

$$o_c^o = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Bigg\} \text{Inverse position kinematics}$$

2. Using q_1, q_2, q_3, determine R_3^0

3. Find: $R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$

4. Find Euler angles corresponding to the rotation matrix

$$R_6^3 = (R_3^0)^T R \Bigg\} \text{Inverse orientation kinematics}$$

Inverse position kinematics

- Inputs:

$$o = o_6^0 = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix}, R = R_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

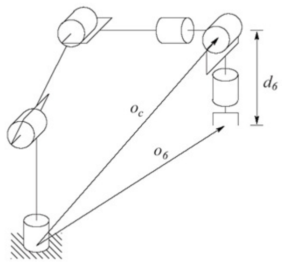
- We can write:

$$o = o_6^0 = o_c^o + R \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix} \Rightarrow o_c^o = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

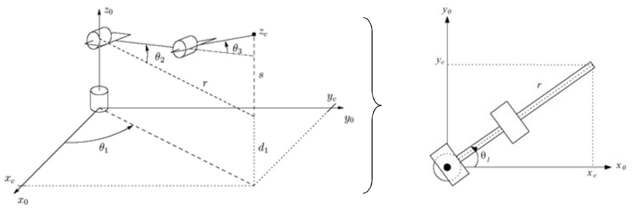
  - Substituting $o_c^0 = [x_c\ y_c\ z_c]^T$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix} - d_6 \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$



Inverse position kinematics: : RRR manipulator



- To solve for θ_1, project the arm onto the x_0, y_0 plane

$$\theta_1 = \mathrm{atan2}(x_c, y_c)$$

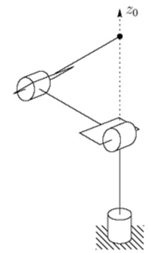
  - Can also have:

$$\theta_1 = \pi + \mathrm{atan2}(x_c, y_c)$$

This will of course change the solutions for θ_2 and θ_3

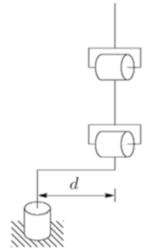
Warning: singular configurations, offsets

- If $x_c = y_c = 0$



- θ_1 is undefined
- Any value of θ_1 will work

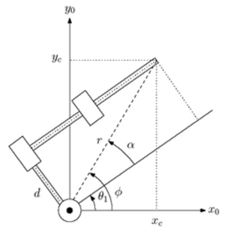
- Solution?
- If there is an offset, wrist centers cannot intersect z_0.



Left arm and right arm solutions

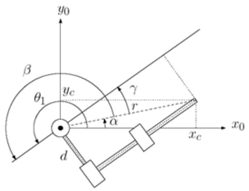
Left arm and right arm solutions

- Left arm:



$$\theta_1 = \varphi - \alpha$$
$$\varphi = a\tan 2(x_c, y_c)$$
$$\alpha = a\tan 2\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right)$$

- Right arm:

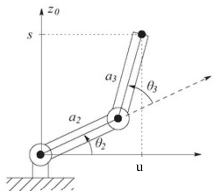


$$\theta_1 = \alpha + \beta$$
$$\alpha = \mathrm{atan2}(x_c, y_c)$$
$$\beta = \pi + \mathrm{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right)$$
$$= \mathrm{atan2}\left(-\sqrt{x_c^2 + y_c^2 - d^2}, -d\right)$$

Therefore there are in general two solutions for θ_1

Left arm and right arm solutions

- Finding θ_2 and θ_3 is identical to the planar two-link manipulator we have seen previously:

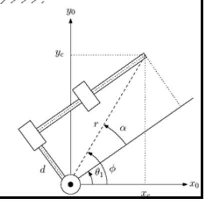


$$\cos\theta_3 = \frac{u^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$
$$u^2 = x_c^2 + y_c^2 - d^2$$
$$s = z_c - d_1$$

$$\Rightarrow \cos\theta_3 = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} \equiv D$$

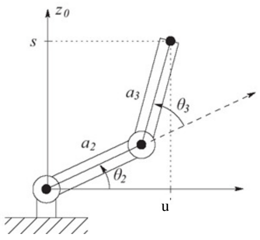
$$\sin\theta_3 = \pm\sqrt{1 - D^2}$$

- Therefore we can find two solutions for θ_3:



Left arm and right arm solutions

- The two solutions for θ_3 correspond to the elbow-down and elbow-up positions respectively



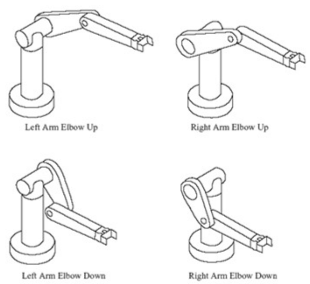
- Now solve for θ_2:

$$\theta_2 = \text{atan2}(u, s) - \text{atan2}(a_2 + a_3 c_3, a_3 s_3)$$
$$= \text{atan2}\left(\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, z_c - d_1\right) - \text{atan2}(a_2 + a_3 c_3, a_3 s_3)$$

Thus there are two solutions for the pair (θ_2, θ_3)

Elbow manipulator

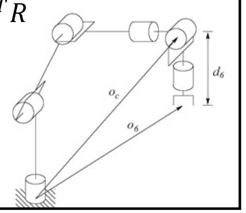
- In general, there will be a maximum of four solutions to the inverse *position* kinematics



Inverse orientation: spherical wrist

- Thus we now have $R_3^{\,0}$
- Note that: $R = R_3^0 R_6^3$
- To solve for the final three joint angles:

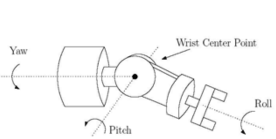
$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$$

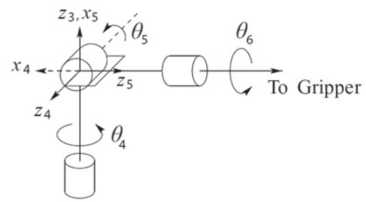


Inverse orientation: spherical wrist

- Forward kinematics of the spherical wrist is identical to a *ZYZ* Euler angle transformation:

$$T_6^3 = A_4 A_5 A_6 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 c_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Inverse orientation: spherical wrist

- The inverse orientation problem reduces to finding a set of Euler angles (θ_4, θ_5, θ_6) that satisfy:

$$R_6^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5c_6 & c_5 \end{bmatrix}$$

- To solve this, take two cases:
  1. $r_{13} = r_{23} \neq 0$, i.e., $\theta_5 \neq 0$: Nonsingular
  2. $r_{13} = r_{23} = 0$, i.e., $\theta_5 = 0$: Singular

- Nonsingular case ($\theta_5 \neq 0$, $r_{33} \neq \pm1$)

$$c_5 = r_{33},\ s_5 = \pm\sqrt{1 - r_{33}{}^2}$$
$$\theta_5 = \operatorname{atan2}\left(r_{33}, \pm\sqrt{1 - r_{33}{}^2}\right)$$

Inverse orientation: spherical wrist

- Thus there are two values for θ_5.
- Using the first ($s_5 > 0$):

$$\theta_4 = \operatorname{atan2}(r_{13}, r_{23})$$
$$\theta_6 = \operatorname{atan2}(-r_{31}, r_{32})$$

- Using the second value for θ_5 ($s_5 < 0$):

$$\theta_4 = \operatorname{atan2}(-r_{13}, -r_{23})$$
$$\theta_6 = \operatorname{atan2}(r_{31}, -r_{32})$$

Two solutions for the inverse orientation kinematics

Inverse orientation: spherical wrist

Singular case ($r_{13} = r_{23} = r_{31} = r_{32} = 0$, thus $\theta_5 = 0$, $s_5 = 0$)

- Therefore, $R_6{}^3$ has the form:

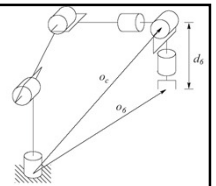
$$R_6^3 = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4c_6 - s_4s_6 & -c_4s_6 - s_4c_6 & 0 \\ s_4c_6 + c_4s_6 & -s_4s_6 + c_4c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{46} & -s_{46} & 0 \\ s_{46} & c_{46} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- We can only find the sum $\theta_4 + \theta_6$ as follows:

$$\theta_4 + \theta_6 = \operatorname{atan2}(r_{11}, r_{21}) = \operatorname{atan2}(r_{11}, -r_{12})$$

There is an infinite number of solutions

Kinematic Decoupling



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THANK YOU