6

Inverse Kinematics

6.1 Decoupling Technique

Determination of joint variables in terms of the end-effector position and orientation is called *inverse kinematics.* Mathematically, inverse kinematics is searching for the elements of vector $\mathbf{q}$ when a transformation is given as a function of the joint variables $q_1, q_2, q_3, \cdots$.

$$ {}^0T_n = {}^0T_1(q_1)\,{}^1T_2(q_2)\,{}^2T_3(q_3)\,{}^3T_4(q_4) \cdots {}^{n-1}T_n(q_n) \tag{6.1} $$

Computer controlled robots are usually actuated in the joint variable space, however objects to be manipulated are usually expressed in the global coordinate frame. Therefore, carrying kinematic information, back and forth, between joint space and Cartesian space, is a need in robot applications. To control the configuration of the end-effector to reach an object, the inverse kinematics solution must be solved. Hence, we need to know what the required values of joint variables are, to reach a desired point in a desired orientation.

The result of forward kinematics of a 6 DOF robot is a 4×4 transformation matrix

$$ \begin{aligned} {}^0T_6 &= {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6 \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{6.2} $$

where 12 elements are trigonometric functions of six unknown joint variables. However, since the upper left 3×3 submatrix of (6.2) is a rotation matrix, only three elements of them are independent. This is because of the orthogonality condition (2.129). Hence, only six equations out of the 12 equations of (6.2) are independent.

Trigonometric functions inherently provide multiple solutions. Therefore, multiple configurations of the robot are expected when the six equations are solved for the unknown joint variables.

It is possible to decouple the inverse kinematics problem into two subproblems, known as *inverse position* and *inverse orientation* kinematics. The practical consequence of such a decoupling is the allowance to break the problem into two independent problems, each with only three unknown parameters. Following the decoupling principle, the overall transformation matrix of a robot can be decomposed to a translation and a rotation

$$\begin{aligned} {}^{0}T_{6} &= {}^{0}D_{6}\,{}^{0}R_{6} \\ &= \begin{bmatrix} \mathbf{I} & {}^{0}\mathbf{d}_{6} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{0}R_{6} & \mathbf{0} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} {}^{0}R_{6} & {}^{0}\mathbf{d}_{6} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{6.3}$$

The translation matrix ${}^{0}D_{6}$ can be solved for wrist position variables, and the rotation matrix ${}^{0}R_{6}$ can be solved for wrist orientation variables.

Proof. Most robots have a wrist made of three revolute joints with intersecting and orthogonal axes at the wrist point. Taking advantage of having a spherical wrist, we can decouple the kinematics of the wrist and manipulator by decomposing the overall forward kinematics transformation matrix ${}^{0}T_{6}$ into the wrist orientation and wrist position

$$\begin{aligned} {}^{0}T_{6} &= {}^{0}T_{3}\,{}^{3}T_{6} \\ &= \begin{bmatrix} {}^{0}R_{3} & {}^{0}\mathbf{d}_{6} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{3}R_{6} & \mathbf{0} \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{6.4}$$

where the wrist orientation matrix is

$${}^{3}R_{6} = {}^{0}R_{3}^{T}\,{}^{0}R_{6} = {}^{0}R_{3}^{T} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{6.5}$$

and the wrist position vector is

$${}^{0}\mathbf{d}_{6} = \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix}. \tag{6.6}$$

The wrist position vector includes the manipulator joint variables only. Hence, to solve the inverse kinematics of such a robot, we must solve ${}^{0}T_{3}$ for position of the wrist point, and then solve ${}^{3}T_{6}$ for orientation of the wrist.

The components of the wrist position vector ${}^{0}\mathbf{d}_{6} = \mathbf{d}_{w}$ provides three equations for the three unknown manipulator joint variables. Solving $\mathbf{d}_{w}$, for manipulator joint variables, leads to calculating ${}^{3}R_{6}$ from (6.5). Then, the wrist orientation matrix ${}^{3}R_{6}$ can be solved for wrist joint variables.

In case we include the tool coordinate frame in forward kinematics, the decomposition must be done according to the following equation to exclude the effect of tool distance d_6 from the robot's kinematics.

$$\begin{aligned} {}^0T_7 &= {}^0T_3\,{}^3T_7 \\ &= {}^0T_3\,{}^3T_6\,{}^6T_7 \\ &= \begin{bmatrix} {}^0R_3 & \mathbf{d}_w \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^3R_6 & \mathbf{0} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} & 0 \\ \mathbf{I} & 0 \\ & d_6 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{6.7}$$

In this case, inverse kinematics starts from determination of ${}^0T_6$, which can be found by

$$\begin{aligned} {}^0T_6 &= {}^0T_7\,{}^6T_7^{-1} \qquad (6.8) \\ &= {}^0T_7 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= {}^0T_7 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

■

Example 163 *Inverse kinematics of an articulated robot.*

The forward kinematics of the articulated robot, illustrated in Figure 6.1, was found in Example 151, where the overall transformation matrix of the end-effector was found, based on the wrist and arm transformation matrices.

$$\begin{aligned} {}^0T_7 &= T_{arm}T_{wrist} \\ &= {}^0T_3\,{}^3T_7 \end{aligned} \tag{6.9}$$

The wrist transformation matrix T_{wrist} is described in (5.73) and the manipulator transformation matrix, T_{arm} is found in (5.80). However, according to a new setup coordinate frame, as shown in Figure 6.1, we have a 6R robot with a six links configuration

1	$R\vdash R(90)$
2	$R\|R(0)$
3	$R\vdash R(90)$
4	$R\vdash R(-90)$
5	$R\vdash R(90)$
6	$R\|R(0)$

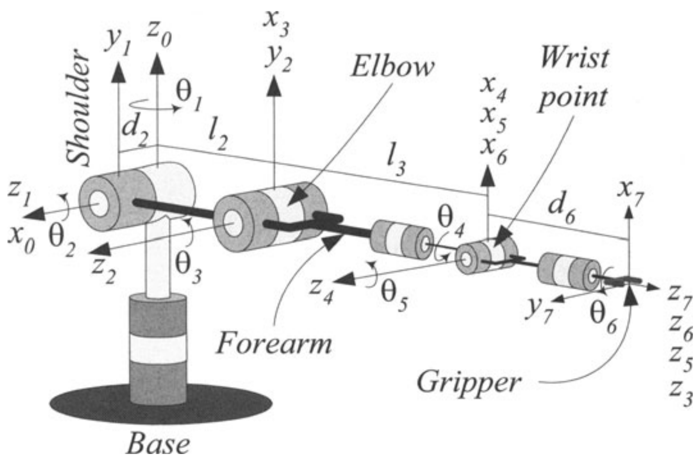

FIGURE 6.1. A 6 DOF articulated manipulator.

and a displacement T_{Z,d_6}. *Therefore, the individual links' transformation matrices are*

$$^{0}T_{1}=\begin{bmatrix}\cos\theta_1 & 0 & \sin\theta_1 & 0\\ \sin\theta_1 & 0 & -\cos\theta_1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{6.10}$$

$$^{1}T_{2}=\begin{bmatrix}\cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2\\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2\\ 0 & 0 & 1 & d_2\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{6.11}$$

$$^{2}T_{3}=\begin{bmatrix}\cos\theta_3 & 0 & \sin\theta_3 & 0\\ \sin\theta_3 & 0 & -\cos\theta_3 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{6.12}$$

$$^{3}T_{4}=\begin{bmatrix}\cos\theta_4 & 0 & -\sin\theta_4 & 0\\ \sin\theta_4 & 0 & \cos\theta_4 & 0\\ 0 & -1 & 0 & l_3\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{6.13}$$

$$^{4}T_{5}=\begin{bmatrix}\cos\theta_5 & 0 & \sin\theta_5 & 0\\ \sin\theta_5 & 0 & -\cos\theta_5 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{6.14}$$

$$ {}^5T_6 = \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.15} $$

$$ {}^6T_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.16} $$

and the tool transformation matrix in the base coordinate frame is

$$ \begin{aligned} {}^0T_7 &= {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6\,{}^6T_7 \qquad (6.17) \\ &= {}^0T_3\,{}^3T_6\,{}^6T_7 \\ &= \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} $$

where

$$ {}^0T_3 = \begin{bmatrix} c\theta_1 c(\theta_2+\theta_3) & s\theta_1 & c\theta_1 s(\theta_2+\theta_3) & l_2 c\theta_1 c\theta_2 + d_2 s\theta_1 \\ s\theta_1 c(\theta_2+\theta_3) & -c\theta_1 & s\theta_1 s(\theta_2+\theta_3) & l_2 c\theta_2 s\theta_1 - d_2 c\theta_1 \\ s(\theta_2+\theta_3) & 0 & -c(\theta_2+\theta_3) & l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.18} $$

$$ {}^3T_6 = \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_6 s\theta_4 - c\theta_4 c\theta_5 s\theta_6 & c\theta_4 s\theta_5 & 0 \\ c\theta_5 c\theta_6 s\theta_4 + c\theta_4 s\theta_6 & c\theta_4 cc\theta_6 - c\theta_5 s\theta_4 s\theta_6 & s\theta_4 s\theta_5 & 0 \\ -c\theta_6 s\theta_5 & s\theta_5 s\theta_6 & c\theta_5 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.19} $$

and

$$ \begin{aligned} t_{11} &= c\theta_1\left(c\left(\theta_2+\theta_3\right)\left(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6\right) - c\theta_6 s\theta_5 s\left(\theta_2+\theta_3\right)\right) \\ &\quad + s\theta_1\left(c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4\right) \qquad (6.20) \\ t_{21} &= s\theta_1\left(c\left(\theta_2+\theta_3\right)\left(-s\theta_4 s\theta_6 + c\theta_4 c\theta_5 c\theta_6\right) - c\theta_6 s\theta_5 s\left(\theta_2+\theta_3\right)\right) \\ &\quad - c\theta_1\left(c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4\right) \qquad (6.21) \\ t_{31} &= s\left(\theta_2+\theta_3\right)\left(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6\right) + c\theta_6 s\theta_5 c\left(\theta_2+\theta_3\right) \qquad (6.22) \end{aligned} $$

$$ \begin{aligned} t_{12} &= c\theta_1\left(s\theta_5 s\theta_6 s\left(\theta_2+\theta_3\right) - c\left(\theta_2+\theta_3\right)\left(c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6\right)\right) \\ &\quad + s\theta_1\left(c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6\right) \qquad (6.23) \\ t_{22} &= s\theta_1\left(s\theta_5 s\theta_6 s\left(\theta_2+\theta_3\right) - c\left(\theta_2+\theta_3\right)\left(c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6\right)\right) \\ &\quad + c\theta_1\left(-c\theta_4 c\theta_6 + c\theta_5 s\theta_4 s\theta_6\right) \qquad (6.24) \\ t_{32} &= -s\theta_5 s\theta_6 c\left(\theta_2+\theta_3\right) - s\left(\theta_2+\theta_3\right)\left(c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6\right) \qquad (6.25) \end{aligned} $$

$$\begin{aligned}
t_{13} &= s\theta_1 s\theta_4 s\theta_5 + c\theta_1 \left(c\theta_5 s\left(\theta_2+\theta_3\right) + c\theta_4 s\theta_5 c\left(\theta_2+\theta_3\right)\right) && (6.26)\\
t_{23} &= -c\theta_1 s\theta_4 s\theta_5 + s\theta_1 \left(c\theta_5 s\left(\theta_2+\theta_3\right) + c\theta_4 s\theta_5 c\left(\theta_2+\theta_3\right)\right) && (6.27)\\
t_{33} &= c\theta_4 s\theta_5 s\left(\theta_2+\theta_3\right) - c\theta_5 c\left(\theta_2+\theta_3\right) && (6.28)
\end{aligned}$$

$$\begin{aligned}
t_{14} &= d_6\left(s\theta_1 s\theta_4 s\theta_5 + c\theta_1\left(c\theta_4 s\theta_5 c\left(\theta_2+\theta_3\right) + c\theta_5 s\left(\theta_2+\theta_3\right)\right)\right)\\
&\quad + l_3 c\theta_1 s\left(\theta_2+\theta_3\right) + d_2 s\theta_1 + l_2 c\theta_1 c\theta_2 && (6.29)\\
t_{24} &= d_6\left(-c\theta_1 s\theta_4 s\theta_5 + s\theta_1\left(c\theta_4 s\theta_5 c\left(\theta_2+\theta_3\right) + c\theta_5 s\left(\theta_2+\theta_3\right)\right)\right)\\
&\quad + s\theta_1 s\left(\theta_2+\theta_3\right) l_3 - d_2 c\theta_1 + l_2 c\theta_2 s\theta_1 && (6.30)\\
t_{34} &= d_6\left(c\theta_4 s\theta_5 s\left(\theta_2+\theta_3\right) - c\theta_5 c\left(\theta_2+\theta_3\right)\right)\\
&\quad + l_2 s\theta_2 + l_3 c\left(\theta_2+\theta_3\right). && (6.31)
\end{aligned}$$

Solution of the inverse kinematics problem starts with the wrist position vector $\mathbf{d}$, *which is* $\begin{bmatrix} t_{14} & t_{24} & t_{34} \end{bmatrix}^T$ *of* ${}^0T_7$ *for* $d_6 = 0$

$$\mathbf{d} = \begin{bmatrix} c\theta_1\left(l_3 s\left(\theta_2+\theta_3\right) + l_2 c\theta_2\right) + d_2 s\theta_1 \\ s\theta_1\left(l_3 s\left(\theta_2+\theta_3\right) + l_2 c\theta_2\right) - d_2 c\theta_1 \\ l_3 c\left(\theta_2+\theta_3\right) + l_2 s\theta_2 \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}. \tag{6.32}$$

Theoretically, we must be able to solve Equation (6.32) for the three joint variables θ_1, θ_2, *and* θ_3. *It can be seen that*

$$d_x \sin\varphi_1 - d_y \cos\theta_1 = d_2 \tag{6.33}$$

which provides

$$\theta_1 = 2\,\text{atan2}(d_x \pm \sqrt{d_x^2 + d_y^2 - d_2^2}, d_2 - d_y). \tag{6.34}$$

Equation (6.34) has two solutions for $d_x^2 + d_y^2 > d_2^2$, *one solution for* $d_x^2 + d_y^2 = d_2^2$, *and no real solution for* $d_x^2 + d_y^2 < d_2^2$.
Combining the first two elements of $\mathbf{d}$ *gives*

$$l_3 \sin\left(\theta_2+\theta_3\right) = \pm\sqrt{d_x^2 + d_y^2 - d_2^2} - l_2\cos\theta_2 \tag{6.35}$$

then, the third element of $\mathbf{d}$ *may be utilized to find*

$$l_3^2 = \left(\pm\sqrt{d_x^2 + d_y^2 - d_2^2} - l_2\cos\theta_2\right)^2 + \left(d_z - l_2\sin\theta_2\right)^2 \tag{6.36}$$

which can be rearranged to the following form

$$a\cos\theta_2 + b\sin\theta_2 = c \tag{6.37}$$

$$\begin{aligned}
a &= 2l_2\sqrt{d_x^2 + d_y^2 - d_2^2} && (6.38)\\
b &= 2l_2 d_z && (6.39)\\
c &= d_x^2 + d_y^2 + d_z^2 - d_2^2 + l_2^2 - l_3^2. && (6.40)
\end{aligned}$$

with two solutions

$$\begin{aligned}
\theta_2 &= \operatorname{atan2}(\frac{c}{r}, \pm\sqrt{1-\frac{c^2}{r^2}}) - \operatorname{atan2}(a,b) && (6.41)\\
\theta_2 &= \operatorname{atan2}(\frac{c}{r}, \pm\sqrt{r^2-c^2}) - \operatorname{atan2}(a,b) && (6.42)\\
r^2 &= a^2+b^2. && (6.43)
\end{aligned}$$

Summing the squares of the elements of $\mathbf{d}$ *gives*

$$d_x^2 + d_y^2 + d_z^2 = d_2^2 + l_2^2 + l_3^2 + 2l_2l_3\sin(2\theta_2+\theta_3) \tag{6.44}$$

that provides

$$\theta_3 = \arcsin\left(\frac{d_x^2+d_y^2+d_z^2-d_2^2-l_2^2-l_3^2}{2l_2l_3}\right) - 2\theta_2. \tag{6.45}$$

Having θ_1, θ_2, and θ_3 means we can find the wrist point in space. However, because the joint variables in ${}^0T_3$ and in ${}^3T_6$ are independent, we should find the orientation of the end-effector by solving ${}^3T_6$ or ${}^3R_6$ for θ_4, θ_5, and θ_6.

$$\begin{aligned}
{}^3R_6 &= \begin{bmatrix} c\theta_4c\theta_5c\theta_6 - s\theta_4s\theta_6 & -c\theta_6s\theta_4 - c\theta_4c\theta_5s\theta_6 & c\theta_4s\theta_5 \\ c\theta_5c\theta_6s\theta_4 + c\theta_4s\theta_6 & c\theta_4cc\theta_6 - c\theta_5s\theta_4s\theta_6 & s\theta_4s\theta_5 \\ -c\theta_6s\theta_5 & s\theta_5s\theta_6 & c\theta_5 \end{bmatrix}\\
&= \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} && (6.46)
\end{aligned}$$

The angles θ_4, θ_5, and θ_6 can be found by examining elements of ${}^3R_6$

$$\theta_4 = \operatorname{atan2}(s_{23}, s_{13}) \tag{6.47}$$

$$\theta_5 = \operatorname{atan2}\left(\sqrt{s_{13}^2+s_{23}^2}, s_{33}\right) \tag{6.48}$$

$$\theta_6 = \operatorname{atan2}(s_{32}, -s_{31}). \tag{6.49}$$

Example 164 *Solution of equation $a\cos\theta + b\sin\theta = c$.*

The first type of trigonometric equation

$$a\cos\theta + b\sin\theta = c \tag{6.50}$$

can be solved by introducing two new variables r and ϕ such that

$$\begin{aligned}
a &= r\sin\phi && (6.51)\\
b &= r\cos\phi && (6.52)
\end{aligned}$$

and

$$r = \sqrt{a^2 + b^2} \tag{6.53}$$
$$\phi = \operatorname{atan2}(a, b). \tag{6.54}$$

Substituting the new variables show that

$$\sin(\phi + \theta) = \frac{c}{r} \tag{6.55}$$
$$\cos(\phi + \theta) = \pm\sqrt{1 - \frac{c^2}{r^2}}. \tag{6.56}$$

Hence, the solutions of the problem are

$$\theta = \operatorname{atan2}(\frac{c}{r}, \pm\sqrt{1 - \frac{c^2}{r^2}}) - \operatorname{atan2}(a, b) \tag{6.57}$$

and

$$\theta = \operatorname{atan2}(\frac{c}{r}, \pm\sqrt{r^2 - c^2}) - \operatorname{atan2}(a, b). \tag{6.58}$$

Therefore, the equation $a\cos\theta + b\sin\theta = c$ *has two solutions if* $r^2 = a^2 + b^2 > c^2$, *one solution if* $r^2 = c^2$, *and no solution if* $r^2 < c^2$.

Example 165 *Meaning of the function* $\tan_2^{-1}\frac{y}{x} = \operatorname{atan2}(y, x)$.

In robotic calculation, specially in solving inverse kinematics problems, we need to find an angle based on the sin *and* cos *functions of the angle. However,* $\tan^{-1}$ *cannot show the effect of the individual sign for the numerator and denominator. It always represents an angle in the first or fourth quadrant. To overcome this problem and determine the joint angles in the correct quadrant, the* atan2 *function is introduced.*

$$\begin{aligned} \operatorname{atan2}(y, x) &= \tan^{-1}\frac{y}{x} && \text{if} \quad y > 0 \\ &= \tan^{-1}\frac{y}{x} + \pi \operatorname{sign} y && \text{if} \quad y < 0 \\ &= \frac{\pi}{2}\operatorname{sign} x && \text{if} \quad y = 0 \end{aligned} \tag{6.59}$$

In this text, whether it has been mentioned or not, wherever $\tan^{-1}\frac{y}{x}$ *is used, it must be calculated based on* $\operatorname{atan2}(y, x)$.

6.2 Inverse Transformation Technique

Assume we have the transformation matrix ${}^0T_6$ indicating the global position and the orientation of the end-effector of a 6 DOF robot in the base frame B_0. Furthermore, assume the geometry and individual transformation matrices ${}^0T_1(q_1)$, ${}^1T_2(q_2)$, ${}^2T_3(q_3)$, ${}^3T_4(q_4)$, ${}^4T_5(q_5)$, and ${}^5T_6(q_6)$ are given as functions of joint variables.

According to forward kinematics,

$$
\begin{aligned}
{}^0T_6 &= {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6 \qquad (6.60)\\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

We can solve the inverse kinematics problem by solving the following equations for the unknown joint variables:

$$
\begin{aligned}
{}^1T_6 &= {}^0T_1^{-1}\,{}^0T_6 && (6.61)\\
{}^2T_6 &= {}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 && (6.62)\\
{}^3T_6 &= {}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 && (6.63)\\
{}^4T_6 &= {}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 && (6.64)\\
{}^5T_6 &= {}^4T_5^{-1}\,{}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 && (6.65)\\
\mathbf{I} &= {}^5T_6^{-1}\,{}^4T_5^{-1}\,{}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 && (6.66)
\end{aligned}
$$

Proof. We multiply both sides of the transformation matrix ${}^0T_6$ by ${}^0T_1^{-1}$ to obtain

$$
\begin{aligned}
{}^0T_1^{-1}\,{}^0T_6 &= {}^0T_1^{-1}\left({}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6\right)\\
&= {}^1T_6. \qquad (6.67)
\end{aligned}
$$

Note that ${}^0T_1^{-1}$ is the mathematical inverse of the 4×4 matrix ${}^0T_1$, and not an inverse transformation. So, ${}^0T_1^{-1}$ must be calculated by a mathematical matrix inversion.

The left-hand side of the Equation (6.67) is a function of q_1. However, the elements of the matrix ${}^1T_6$ on the right-hand side are either zero, constant, or functions of q_2, q_3, q_4, q_5, and q_6. The zero or constant elements of the right-hand side provides the required algebraic equation to be solved for q_1.

Then, we multiply both sides of (6.67) by ${}^1T_2^{-1}$ to obtain

$$
\begin{aligned}
{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 &= {}^1T_2^{-1}\,{}^0T_1^{-1}\left({}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6\right)\\
&= {}^2T_6. \qquad (6.68)
\end{aligned}
$$

The left-hand side of the this equation is a function of q_2, while the elements of the matrix ${}^2T_6$, on the right hand side, are either zero, constant, or functions of q_3, q_4, q_5, and q_6. Equating the associated element, with constant or zero elements on the right-hand side, provides the required algebraic equation to be solved for q_2.

Following this procedure, we can find the joint variables q_3, q_4, q_5, and q_6 by using the following equalities respectively.

$$
\begin{aligned}
&{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6\\
&= {}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\left({}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6\right) \qquad (6.69)\\
&= {}^3T_6.
\end{aligned}
$$

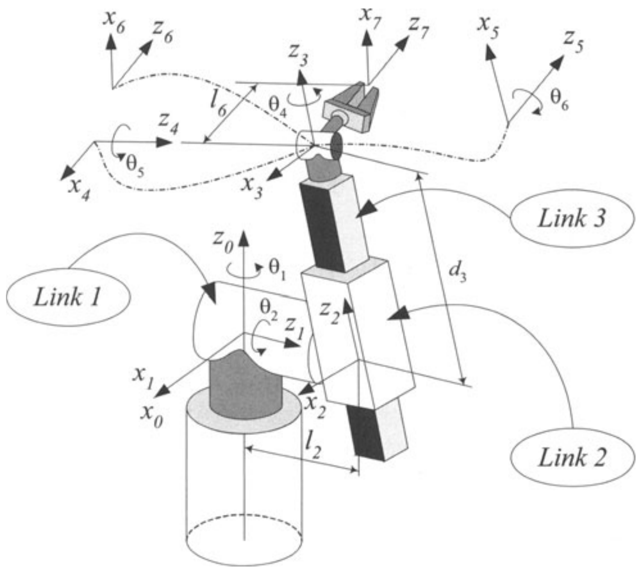

FIGURE 6.2. A spherical robot, made of a spherical manipulator attached to a spherical wrist.

$$\begin{aligned}
&{}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 \\
&= {}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\left({}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6\right) \\
&= {}^4T_6.
\end{aligned} \tag{6.70}$$

$$\begin{aligned}
&{}^4T_5^{-1}\,{}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 \\
&= {}^4T_5^{-1}\,{}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\left({}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6\right) \\
&= {}^5T_6.
\end{aligned} \tag{6.71}$$

$$\begin{aligned}
&{}^5T_6^{-1}\,{}^4T_5^{-1}\,{}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 \\
&= {}^5T_6^{-1}\,{}^4T_5^{-1}\,{}^3T_4^{-1}\,{}^2T_3^{-1}\,{}^1T_2^{-1}\,{}^0T_1^{-1}\left({}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6\right) \\
&= \mathbf{I}.
\end{aligned} \tag{6.72}$$

The *inverse transformation technique* may sometimes be called *Pieper technique.* ■

Example 166 *Inverse kinematics for a spherical robot.*

Transformation matrices of the spherical robot shown in Figure 6.2 are

$$ {}^0T_1 = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^1T_2 = \begin{bmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$
{}^{2}T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}T_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}T_5 = \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{5}T_6 = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.73}
$$

Therefore, the position and orientation of the end-effector for a set of joint variables, which solves the forward kinematics problem, can be found by matrix multiplication

$$
\begin{aligned}
{}^{0}T_6 &= {}^{0}T_1\,{}^{1}T_2\,{}^{2}T_3\,{}^{3}T_4\,{}^{4}T_5\,{}^{5}T_6 \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{6.74}
$$

where the elements of ${}^{0}T_6$ are the same as the elements of the matrix in Equation (5.101).

Multiplying both sides of the (6.74) by ${}^{0}T_1^{-1}$ provides

$$
\begin{aligned}
{}^{0}T_1^{-1}\,{}^{0}T_6 &= \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{6.75}
$$

where

$$
\begin{aligned}
f_{1i} &= r_{1i}\cos\theta_1 + r_{2i}\sin\theta_1 && (6.76) \\
f_{2i} &= -r_{3i} && (6.77) \\
f_{3i} &= r_{2i}\cos\theta_1 - r_{1i}\sin\theta_1 && (6.78) \\
i &= 1, 2, 3, 4.
\end{aligned}
$$

Based on the given transformation matrices, we find that

$$
\begin{aligned}
{}^{1}T_6 &= {}^{1}T_2\,{}^{2}T_3\,{}^{3}T_4\,{}^{4}T_5\,{}^{5}T_6 \\
&= \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{6.79}
$$

$$f_{11} = -c\theta_2 s\theta_4 s\theta_6 + c\theta_6 \left(-s\theta_2 s\theta_5 + c\theta_2 c\theta_4 c\theta_5\right) \tag{6.80}$$
$$f_{21} = -s\theta_2 s\theta_4 s\theta_6 + c\theta_6 \left(c\theta_2 s\theta_5 + c\theta_4 c\theta_5 s\theta_2\right) \tag{6.81}$$
$$f_{31} = c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4 \tag{6.82}$$

$$f_{12} = -c\theta_2 c\theta_6 s\theta_4 - s\theta_6 \left(-s\theta_2 s\theta_5 + c\theta_2 c\theta_4 c\theta_5\right) \tag{6.83}$$
$$f_{22} = -c\theta_6 s\theta_2 s\theta_4 - s\theta_6 \left(c\theta_2 s\theta_5 + c\theta_4 c\theta_5 s\theta_2\right) \tag{6.84}$$
$$f_{32} = c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6 \tag{6.85}$$

$$f_{13} = c\theta_5 s\theta_2 + c\theta_2 c\theta_4 s\theta_5 \tag{6.86}$$
$$f_{23} = -c\theta_2 c\theta_5 + c\theta_4 s\theta_2 s\theta_5 \tag{6.87}$$
$$f_{33} = s\theta_4 s\theta_5 \tag{6.88}$$

$$f_{14} = d_3 s\theta_2 \tag{6.89}$$
$$f_{24} = -d_3 c\theta_2 \tag{6.90}$$
$$f_{34} = l_2. \tag{6.91}$$

The only constant element of the matrix (6.79) is $f_{34} = l_2$*, therefore,*

$$r_{24} \cos\theta_1 - r_{14} \sin\theta_1 = l_2. \tag{6.92}$$

This kind of trigonometric equation frequently appears in robotic inverse kinematics, which has a systematic method of solution. We assume

$$r_{14} = r\cos\phi \tag{6.93}$$
$$r_{24} = r\sin\phi \tag{6.94}$$

where

$$r = \sqrt{r_{14}^2 + r_{24}^2} \tag{6.95}$$
$$\phi = \tan^{-1}\frac{r_{24}}{r_{14}} \tag{6.96}$$

and therefore, Equation (6.92) becomes

$$\frac{l_2}{r} = \sin\phi\cos\theta_1 - \cos\phi\sin\theta_1 \tag{6.97}$$
$$= \sin(\phi - \theta_1) \tag{6.98}$$

showing that

$$\pm\sqrt{1 - (l_2/r)^2} = \cos(\phi - \theta_1). \tag{6.99}$$

Hence, the solution of Equation (6.92) for θ_1 *is*

$$\theta_1 = \tan^{-1}\frac{r_{24}}{r_{14}} - \tan^{-1}\frac{l_2}{\pm\sqrt{r^2 - l_2^2}}. \tag{6.100}$$

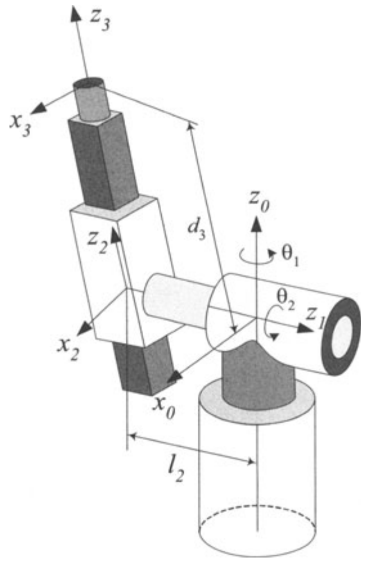

FIGURE 6.3. Left shoulder configuration of a spherical robot.

The $(-)$ *sign corresponds to a left shoulder configuration of the robot, as shown in Figure 6.3, and the* $(+)$ *sign corresponds to the right shoulder configuration.*

The elements f_{14} *and* f_{24} *of matrix (6.79) are functions of* θ_2 *and* θ_2 *only.*

$$\begin{aligned} f_{14} &= d_3 \sin\theta_2 = r_{14}\cos\theta_1 + r_{24}\sin\theta_1 && (6.101) \\ f_{24} &= -d_3\cos\theta_2 = -r_{34} && (6.102) \end{aligned}$$

Hence, it is possible to use them and find θ_2

$$\theta_2 = \tan^{-1}\frac{r_{14}\cos\theta_1 + r_{24}\sin\theta_1}{r_{34}} \tag{6.103}$$

where θ_1 *must be substituted from (6.100).*

In the next step, we find the third joint variable d_3 *from*

$${}^1T_2^{-1}\,{}^0T_1^{-1}\,{}^0T_6 = {}^2T_6 \tag{6.104}$$

where

$${}^1T_2^{-1} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & -l_2 \\ \sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.105}$$

and

$$
{}^{2}T_{6} = \begin{bmatrix} -s\theta_4 s\theta_6 + c\theta_4 c\theta_5 c\theta_6 & -c\theta_6 s\theta_4 - c\theta_4 c\theta_5 s\theta_6 & c\theta_4 s\theta_5 & 0 \\ c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4 & c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6 & s\theta_4 s\theta_5 & 0 \\ -c\theta_6 s\theta_5 & s\theta_5 s\theta_6 & c\theta_5 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.106}
$$

Employing the elements of the matrices on both sides of Equation (6.104) shows that the element $(3,4)$ *can be utilized to find* d_3*.*

$$
d_3 = r_{34}\cos\theta_2 + r_{14}\cos\theta_1\sin\theta_2 + r_{24}\sin\theta_1\sin\theta_2 \tag{6.107}
$$

Since there is no other element in Equation (6.104) to be a function of another single variable, we move to the next step and evaluate θ_4 *from*

$$
{}^{3}T_{4}^{-1}\,{}^{2}T_{3}^{-1}\,{}^{1}T_{2}^{-1}\,{}^{0}T_{1}^{-1}\,{}^{0}T_{6} = {}^{4}T_{6} \tag{6.108}
$$

because ${}^{2}T_{3}^{-1}\,{}^{1}T_{2}^{-1}\,{}^{0}T_{1}^{-1}\,{}^{0}T_{6} = {}^{3}T_{6}$ *provides no new equation. Evaluating* ${}^{4}T_{6}$

$$
{}^{4}T_{6} = \begin{bmatrix} \cos\theta_5\cos\theta_6 & -\cos\theta_5\sin\theta_6 & \sin\theta_5 & 0 \\ \cos\theta_6\sin\theta_5 & -\sin\theta_5\sin\theta_6 & -\cos\theta_5 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.109}
$$

and the left-hand side of (6.108) utilizing

$$
{}^{2}T_{3}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.110}
$$

and

$$
{}^{3}T_{4}^{-1} = \begin{bmatrix} \cos\theta_4 & \sin\theta_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\sin\theta_4 & \cos\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.111}
$$

shows that

$$
{}^{3}T_{4}^{-1}\,{}^{2}T_{3}^{-1}\,{}^{1}T_{2}^{-1}\,{}^{0}T_{1}^{-1}\,{}^{0}T_{6} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.112}
$$

where

$$\begin{aligned}
g_{1i} &= -r_{3i}c\theta_4 s\theta_2 + r_{2i}\left(c\theta_1 s\theta_4 + c\theta_2 c\theta_4 s\theta_1\right) \\
&\quad + r_{1i}\left(-s\theta_1 s\theta_4 + c\theta_1 c\theta_2 c\theta_4\right) && (6.113) \\
g_{2i} &= d_3\delta_{4i} - r_{31}c\theta_2 - r_{11}c\theta_1 s\theta_2 - r_{21}s\theta_1 s\theta_2 && (6.114) \\
g_{3i} &= r_{31}s\theta_2 s\theta_4 + r_{21}\left(c\theta_1 c\theta_4 - c\theta_2 s\theta_1 s\theta_4\right) \\
&\quad + r_{11}\left(-c\theta_4 s\theta_1 - c\theta_1 c\theta_2 s\theta_4\right) && (6.115) \\
i &= 1, 2, 3, 4.
\end{aligned}$$

The symbol δ_{4i} *indicates the Kronecker delta and is equal to*

$$\begin{aligned}
\delta_{4i} &= 1 \quad \text{if} \quad i = 4 \\
&= 0 \quad \text{if} \quad i \neq 4.
\end{aligned} \tag{6.116}$$

Therefore, we can find θ_4 *by equating the element* $(3,3)$, θ_5 *by equating the elements* $(1,3)$ *or* $(2,3)$, *and* θ_6 *by equating the elements* $(3,1)$ *or* $(3,2)$. *Starting from element* $(3,3)$

$$r_{13}\left(-c\theta_4 s\theta_1 - c\theta_1 c\theta_2 s\theta_4\right) + r_{23}\left(c\theta_1 c\theta_4 - c\theta_2 s\theta_1 s\theta_4\right) + r_{33}s\theta_2 s\theta_4 = 0 \tag{6.117}$$

we find θ_4

$$\theta_4 = \tan^{-1}\frac{-r_{13}s\theta_1 + r_{23}c\theta_1}{c\theta_2\left(r_{13}c\theta_1 + r_{23}s\theta_1\right) - r_{33}s\theta_2} \tag{6.118}$$

which, based on the second value of θ_1, *can also be equal to*

$$\theta_4 = \frac{\pi}{2} + \tan^{-1}\frac{-r_{13}s\theta_1 + r_{23}c\theta_1}{c\theta_2\left(r_{13}c\theta_1 + r_{23}s\theta_1\right) - r_{33}s\theta_2}. \tag{6.119}$$

Now we use elements $(1,3)$ *and* $(2,3)$,

$$\begin{aligned}
\sin\theta_5 &= r_{23}\left(\cos\theta_1 \sin\theta_4 + \cos\theta_2 \cos\theta_4 \sin\theta_1\right) - r_{33}\cos\theta_4 \sin\theta_2 \\
&\quad + r_{13}\left(\cos\theta_1 \cos\theta_2 \cos\theta_4 - \sin\theta_1 \sin\theta_4\right) && (6.120) \\
-\cos\theta_5 &= -r_{33}\cos\theta_2 - r_{13}\cos\theta_1 \sin\theta_2 - r_{23}\sin\theta_1 \sin\theta_2 && (6.121)
\end{aligned}$$

to find θ_5

$$\theta_5 = \tan^{-1}\frac{\sin\theta_5}{\cos\theta_5}. \tag{6.122}$$

Finally, θ_6 *can be found from the elements* $(3,1)$ *and* $(3,2)$

$$\begin{aligned}
\sin\theta_6 &= r_{31}\sin\theta_2 \sin\theta_4 + r_{21}\left(\cos\theta_1 \cos\theta_4 - \cos\theta_2 \sin\theta_1 \sin\theta_4\right) \\
&\quad + r_{11}\left(-\cos\theta_4 \sin\theta_1 - \cos\theta_1 \cos\theta_2 \sin\theta_4\right) && (6.123)
\end{aligned}$$

$$\begin{aligned}
\cos\theta_6 &= r_{32}\sin\theta_2 \sin\theta_4 + r_{22}\left(\cos\theta_1 \cos\theta_4 - \cos\theta_2 \sin\theta_1 \sin\theta_4\right) \\
&\quad + r_{12}\left(-\cos\theta_4 \sin\theta_1 - \cos\theta_1 \cos\theta_2 \sin\theta_4\right) && (6.124)
\end{aligned}$$

$$\theta_6 = \tan^{-1}\frac{\sin\theta_6}{\cos\theta_6}. \tag{6.125}$$

Example 167 *Inverse of Euler angles transformation matrix.*

The global rotation matrix based on Euler angles has been found in Equation (2.64).

$$\begin{aligned}
{}^{G}R_{B} &= \left[A_{z,\psi}\, A_{x,\theta}\, A_{z,\varphi}\right]^{T} \\
&= R_{Z,\varphi}\, R_{X,\theta}\, R_{Z,\psi} \\
&= \begin{bmatrix} c\varphi c\psi - c\theta s\varphi s\psi & -c\varphi s\psi - c\theta c\psi s\varphi & s\theta s\varphi \\ c\psi s\varphi + c\theta c\varphi s\psi & -s\varphi s\psi + c\theta c\varphi c\psi & -c\varphi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
\end{aligned} \tag{6.126}$$

Premultiplying ${}^{G}R_{B}$ *by* $R_{Z,\varphi}^{-1}$, *gives*

$$\begin{aligned}
&\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{G}R_{B} \\
&= \begin{bmatrix} r_{11}c\varphi + r_{21}s\varphi & r_{12}c\varphi + r_{22}s\varphi & r_{13}c\varphi + r_{23}s\varphi \\ r_{21}c\varphi - r_{11}s\varphi & r_{22}c\varphi - r_{12}s\varphi & r_{23}c\varphi - r_{13}s\varphi \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \cos\theta\sin\psi & \cos\theta\cos\psi & -\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}.
\end{aligned} \tag{6.127}$$

Equating the elements $(1,3)$ *of both sides*

$$r_{13}\cos\varphi + r_{23}\sin\varphi = 0 \tag{6.128}$$

gives

$$\varphi = \operatorname{atan2}\left(r_{13}, -r_{23}\right). \tag{6.129}$$

Having φ *helps us to find* ψ *by using elements* $(1,1)$ *and* $(1,2)$

$$\cos\psi = r_{11}\cos\varphi + r_{21}\sin\varphi \tag{6.130}$$

$$-\sin\psi = r_{12}\cos\varphi + r_{22}\sin\varphi \tag{6.131}$$

therefore,

$$\psi = \operatorname{atan2}\frac{-r_{12}\cos\varphi - r_{22}\sin\varphi}{r_{11}\cos\varphi + r_{21}\sin\varphi}. \tag{6.132}$$

In the next step, we may postmultiply ${}^{G}R_{B}$ *by* $R_{Z,\psi}^{-1}$*, to provide*

$$\begin{aligned}
&{}^{G}R_{B}\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} r_{11}c\psi - r_{12}s\psi & r_{12}c\psi + r_{11}s\psi & r_{13} \\ r_{21}c\psi - r_{22}s\psi & r_{22}c\psi + r_{21}s\psi & r_{23} \\ r_{31}c\psi - r_{32}s\psi & r_{32}c\psi + r_{31}s\psi & r_{33} \end{bmatrix} \\
&= \begin{bmatrix} \cos\varphi & -\cos\theta\sin\varphi & \sin\theta\sin\varphi \\ \sin\varphi & \cos\theta\cos\varphi & -\cos\varphi\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}. \qquad (6.133)
\end{aligned}$$

The elements $(1,1)$ *on both sides make an equation to find* ψ*.*

$$r_{31}\cos\psi - r_{31}\sin\psi = 0 \qquad (6.134)$$

Therefore, it is possible to find ψ *from the following equation:*

$$\psi = \operatorname{atan2}(r_{31}, r_{31}). \qquad (6.135)$$

Finally, θ *can be found using elements* $(3,2)$ *and* $(3,3)$

$$\begin{aligned}
r_{32}c\psi + r_{31}s\psi &= \sin\theta \qquad (6.136) \\
r_{33} &= \cos\theta \qquad (6.137)
\end{aligned}$$

which give

$$\theta = \operatorname{atan2}\frac{r_{32}\cos\psi + r_{31}\sin\psi}{r_{33}}. \qquad (6.138)$$

Example 168 *Inverse kinematics for a 2R planar manipulator.*

Figure 5.9 illustrates a 2R planar manipulator with two $R \| R$ *links according to the coordinate frames setup shown in the figure.*

The forward kinematics of the manipulator was found to be

$$\begin{aligned}
{}^{0}T_{2} &= {}^{0}T_{1}\,{}^{1}T_{2} \qquad (6.139) \\
&= \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & l_1c\theta_1 + l_2c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & l_1s\theta_1 + l_2s(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

The inverse kinematics of planar robots are generally easier to find analytically. In this case, we can see that the position of the tip point of the manipulator is at

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \end{bmatrix} \qquad (6.140)$$

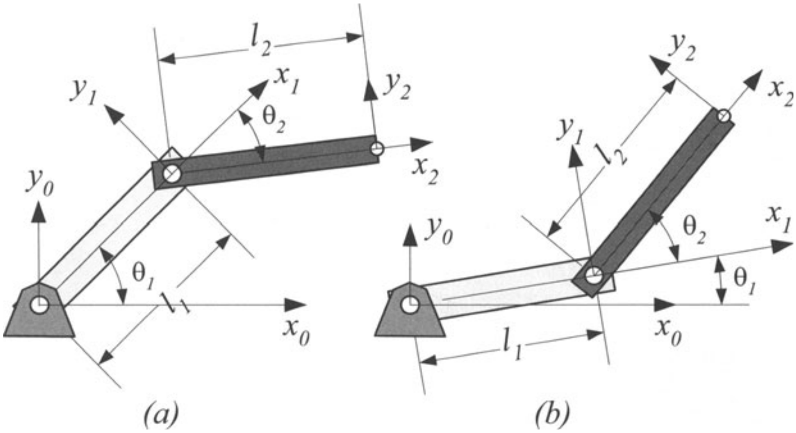

FIGURE 6.4. Illustration of a 2R planar manipulator in two possible configurations: (a) elbow up and (b) elbow down.

therefore

$$X^2 + Y^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos\theta_2 \tag{6.141}$$

and

$$\cos\theta_2 = \frac{X^2 + Y^2 - l_1^2 - l_2^2}{-2l_1 l_2}. \tag{6.142}$$

However, we should avoid using arcsin *and* arccos *because of the inaccuracy. So, we employ the half angle formula*

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} \tag{6.143}$$

to find θ_2 using an atan2 *function*

$$\theta_2 = \pm 2\,\text{atan2}\sqrt{\frac{(l_1+l_2)^2 - (X+Y)^2}{(X^2+Y^2) - (l_1^2+l_2^2)}}. \tag{6.144}$$

The $\pm$ is because of the square root, which generates two solutions. These two solutions are called elbow up and elbow down, as shown in Figure 6.4.

The first joint variable θ_1 can be found from

$$\theta_1 = \text{atan2}\frac{X(l_1 + l_2\cos\theta_2) + Yl_2\sin\theta_2}{Y(l_1 + l_2\cos\theta_2) - Xl_2\sin\theta_2}. \tag{6.145}$$

The two different solutions for θ_1 correspond to the elbow up and elbow down configurations.

Example 169 ★ *Inverse kinematics and nonstandard DH frames.*

Consider a 3 DOF planar manipulator shown in Figure 5.4. The nonstandard DH transformation matrices of the manipulator are

$$ {}^{0}T_{1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.146} $$

$$ {}^{1}T_{2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.147} $$

$$ {}^{2}T_{3} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & l_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.148} $$

$$ {}^{3}T_{4} = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.149} $$

Solution of the inverse kinematics problem is a mathematical problem and none of the standard or nonstandard DH methods for defining link frames provide any simplicity.

To calculate the inverse kinematics, we start with calculating the forward kinematics transformation matrix ${}^{0}T_{4}$

$$ \begin{aligned} {}^{0}T_{4} &= {}^{0}T_{1}\,{}^{1}T_{2}\,{}^{2}T_{3}\,{}^{3}T_{4} \qquad (6.150) \\ &= \begin{bmatrix} \cos\theta_{123} & -\sin\theta_{123} & 0 & l_1\cos\theta_1 + l_2\cos\theta_{12} + l_3\cos\theta_{123} \\ \sin\theta_{123} & \cos\theta_{123} & 0 & l_1\sin\theta_1 + l_2\sin\theta_{12} + l_3\sin\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} $$

where we used the following short notation to simplify the equation.

$$ \theta_{ijk} = \theta_i + \theta_j + \theta_k \tag{6.151} $$

Examining the matrix ${}^{0}T_{4}$ *indicates that*

$$ \theta_{123} = \operatorname{atan2}(r_{21}, r_{11}). \tag{6.152} $$

The next equation

$$
\begin{aligned}
{}^{0}T_4\,{}^{3}T_4^{-1} &= {}^{0}T_1\,{}^{1}T_2\,{}^{2}T_3 \qquad (6.153)\\
\begin{bmatrix} r_{11} & r_{12} & 0 & r_{14}-l_3r_{11} \\ r_{21} & r_{22} & 0 & r_{24}-l_3r_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & l_1c\theta_1+l_2c\theta_{12} \\ s\theta_{123} & c\theta_{123} & 0 & l_1s\theta_1+l_2s\theta_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

shows that

$$
\begin{aligned}
\theta_2 &= \arccos\frac{f_1^2+f_2^2-l_1^2-l_2^2}{2l_1l_2} \qquad (6.154)\\
\theta_1 &= \operatorname{atan2}\left(f_2f_3-f_1f_4\,,\,f_1f_3+f_2f_4\right)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1 &= r_{14}-l_3r_{11}\\
&= c\theta_1\left(l_2c\theta_2+l_1\right)-s\theta_1\left(l_2s\theta_2\right) \qquad (6.155)\\
&= c\theta_1f_3-s\theta_1f_4
\end{aligned}
$$

$$
\begin{aligned}
f_2 &= r_{24}-l_3r_{21}\\
&= s\theta_1\left(l_2c\theta_2+l_1\right)+c\theta_1\left(l_2s\theta_2\right) \qquad (6.156)\\
&= s\theta_1f_3+c\theta_1f_4.
\end{aligned}
$$

Finally, the angle θ_3 is

$$
\theta_3=\theta_{123}-\theta_1-\theta_2. \qquad (6.157)
$$

6.3 Iterative Technique

The inverse kinematics problem can be interpreted as searching for the solution q_k of a set of nonlinear algebraic equations

$$
\begin{aligned}
{}^{0}T_n &= \mathbf{T}(\mathbf{q}) \qquad (6.158)\\
&= {}^{0}T_1(q_1)\,{}^{1}T_2(q_2)\,{}^{2}T_3(q_3)\,{}^{3}T_4(q_4)\cdots{}^{n-1}T_n(q_n)\\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

or

$$
r_{ij}=r_{ij}(q_k) \qquad k=1,2,\cdots n. \qquad (6.159)
$$

where n is the number of DOF. However, maximum $m=6$ out of 12 equations of (6.158) are independent and can be utilized to solve for joint variables q_k. The functions $\mathbf{T}(\mathbf{q})$ are transcendental, which are given explicitly based on forward kinematic analysis.

Numerous methods are available to find the zeros of Equation (6.158). However, the methods are, in general, *iterative*. The most common method is known as the *Newton-Raphson method.*

In the *iterative technique*, to solve the kinematic equations

$$\mathbf{T}(\mathbf{q}) = 0 \tag{6.160}$$

for variables $\mathbf{q}$, we start with an initial guess

$$\mathbf{q}^\star = \mathbf{q} + \delta\mathbf{q} \tag{6.161}$$

for the joint variables. Using the forward kinematics, we can determine the configuration of the end-effector frame for the guessed joint variables.

$$\mathbf{T}^\star = \mathbf{T}(\mathbf{q}^\star) \tag{6.162}$$

The difference between the configuration calculated with the forward kinematics and the desired configuration represents an *error*, called *residue*, which must be minimized.

$$\delta\mathbf{T} = \mathbf{T} - \mathbf{T}^\star \tag{6.163}$$

A first order Taylor expansion of the set of equations is

$$\begin{aligned} \mathbf{T} &= \mathbf{T}(\mathbf{q}^\star + \delta\mathbf{q}) && (6.164)\\ &= \mathbf{T}(\mathbf{q}^\star) + \frac{\partial \mathbf{T}}{\partial \mathbf{q}}\delta\mathbf{q} + O(\delta\mathbf{q}^2). && (6.165) \end{aligned}$$

Assuming $\delta\mathbf{q} << \mathbf{I}$ allows us to work with a set of linear equations

$$\delta\mathbf{T} = \mathbf{J}\,\delta\mathbf{q} \tag{6.166}$$

where $\mathbf{J}$ is the Jacobian matrix of the set of equations

$$\mathbf{J}(\mathbf{q}) = \left[\frac{\partial T_i}{\partial q_j}\right] \tag{6.167}$$

that implies

$$\delta\mathbf{q} = \mathbf{J}^{-1}\,\delta\mathbf{T}. \tag{6.168}$$

Therefore, the unknown variables $\mathbf{q}$ are

$$\mathbf{q} = \mathbf{q}^\star + \mathbf{J}^{-1}\,\delta\mathbf{T}. \tag{6.169}$$

We may use the values obtained by (6.169) as a new approximation to repeat the calculations and find newer values. Repeating the methods can be summarized in the following iterative equation to converge to the exact value of the variables.

$$\mathbf{q}^{(i+1)} = \mathbf{q}^{(i)} + \mathbf{J}^{-1}(\mathbf{q}^{(i)})\ \delta\mathbf{T}(\mathbf{q}^{(i)}) \tag{6.170}$$

This iteration technique can be set in an algorithm for easier numerical calculations.

Algorithm 6.1. Inverse kinematics iteration technique.

1. *Set the initial counter* $i = 0$.
2. *Find or guess an initial estimate* $\mathbf{q}^{(0)}$.
3. *Calculate the residue* $\delta\mathbf{T}(\mathbf{q}^{(i)}) = \mathbf{J}(\mathbf{q}^{(i)})\ \delta\mathbf{q}^{(i)}$.

   If every element of $\mathbf{T}(\mathbf{q}^{(i)})$ *or its norm* $\left\|\mathbf{T}(\mathbf{q}^{(i)})\right\|$ *is less than a tolerance,* $\left\|\mathbf{T}(\mathbf{q}^{(i)})\right\| < \epsilon$ *then terminate the iteration. The* $\mathbf{q}^{(i)}$ *is the desired solution.*
4. *Calculate* $\mathbf{q}^{(i+1)} = \mathbf{q}^{(i)} + \mathbf{J}^{-1}(\mathbf{q}^{(i)})\ \delta\mathbf{T}(\mathbf{q}^{(i)})$.
5. *Set* $i = i + 1$ *and return to step 3.*

The tolerance ϵ can equivalently be set up on variables

$$\mathbf{q}^{(i+1)} - \mathbf{q}^{(i)} < \epsilon \tag{6.171}$$

or on Jacobian

$$\mathbf{J} - \mathbf{I} < \epsilon. \tag{6.172}$$

Example 170 *Inverse kinematics for a 2R planar manipulator.*

In Example 168 we have seen that the tip point of a 2R planar manipulator can be described by

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} l_1 c\theta_1 + l_2 c\,(\theta_1 + \theta_2) \\ l_1 s\theta_1 + l_2 s\,(\theta_1 + \theta_2) \end{bmatrix}. \tag{6.173}$$

To solve the inverse kinematics of the manipulator and find the joint coordinates for a known position of the tip point, we define

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \tag{6.174}$$

$$\mathbf{T} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{6.175}$$

therefore, the Jacobian of the equations is

$$\begin{aligned}
\mathbf{J}(\mathbf{q}) &= \left[\frac{\partial T_i}{\partial q_j}\right] = \begin{bmatrix} \frac{\partial X}{\partial \theta_1} & \frac{\partial X}{\partial \theta_2} \\ \frac{\partial Y}{\partial \theta_1} & \frac{\partial Y}{\partial \theta_2} \end{bmatrix} \\
&= \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}. \qquad (6.176)
\end{aligned}$$

The inverse of the Jacobian is

$$\mathbf{J}^{-1} = \frac{-1}{l_1 l_2 s\theta_2} \begin{bmatrix} -l_2 c(\theta_1+\theta_2) & -l_2 s(\theta_1+\theta_2) \\ l_1 c\theta_1 + l_2 c(\theta_1+\theta_2) & l_1 s\theta_1 + l_2 s(\theta_1+\theta_2) \end{bmatrix} \qquad (6.177)$$

and therefore, the iterative formula (6.170) is set up as

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(i+1)} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(i)} + \mathbf{J}^{-1}\left(\begin{bmatrix} X \\ Y \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix}^{(i)}\right). \qquad (6.178)$$

Let's assume

$$l_1 = l_2 = 1$$

$$\mathbf{T} = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and start from a guess value

$$\mathbf{q}^{(0)} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(0)} = \begin{bmatrix} \pi/3 \\ -\pi/3 \end{bmatrix}$$

for which

$$\begin{aligned}
\delta\mathbf{T} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \cos\pi/3 + \cos(\pi/3 + -\pi/3) \\ \sin\pi/3 + \sin(\pi/3 + -\pi/3) \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2}\sqrt{3} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3} + 1 \end{bmatrix}.
\end{aligned}$$

The Jacobian and its inverse for these values are

$$\mathbf{J} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$$

$$\mathbf{J}^{-1} = \begin{bmatrix} -\frac{2}{3}\sqrt{3} & 0 \\ \sqrt{3} & 1 \end{bmatrix}$$

and therefore,

$$\begin{aligned}
\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(1)} &= \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(0)} + \mathbf{J}^{-1}\,\delta\mathbf{T} \\
&= \begin{bmatrix} \pi/3 \\ -\pi/3 \end{bmatrix} + \begin{bmatrix} -\frac{2}{3}\sqrt{3} & 0 \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3}+1 \end{bmatrix} \\
&= \begin{bmatrix} 1.624\,5 \\ -1.779\,2 \end{bmatrix}.
\end{aligned}$$

Based on the iterative technique, we can find the following values and find the solution in a few iterations.

Iteration 1.

$$\mathbf{J} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$$

$$\delta\mathbf{T} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3}+1 \end{bmatrix}$$

$$\mathbf{q}^{(1)} = \begin{bmatrix} 1.624\,5 \\ -1.779\,2 \end{bmatrix}$$

Iteration 2.

$$\mathbf{J} = \begin{bmatrix} -0.844 & 0.154 \\ 0.934 & 0.988 \end{bmatrix}$$

$$\delta\mathbf{T} = \begin{bmatrix} 6.516 \times 10^{-2} \\ 0.155\,53 \end{bmatrix}$$

$$\mathbf{q}^{(2)} = \begin{bmatrix} 1.583 \\ -1.582 \end{bmatrix}$$

Iteration 3.

$$\mathbf{J} = \begin{bmatrix} -1.00 & -.433 \times 10^{-3} \\ .988 & .999 \end{bmatrix}$$

$$\delta\mathbf{T} = \begin{bmatrix} .119 \times 10^{-1} \\ -.362 \times 10^{-3} \end{bmatrix}$$

$$\mathbf{q}^{(3)} = \begin{bmatrix} 1.570795886 \\ -1.570867014 \end{bmatrix}$$

Iteration 4.

$$\mathbf{J} = \begin{bmatrix} -1.000 & 0.0 \\ 0.998\,50 & 1.0 \end{bmatrix}$$

$$\delta\mathbf{T} = \begin{bmatrix} -.438 \times 10^{-6} \\ .711 \times 10^{-4} \end{bmatrix}$$

$$\mathbf{q}^{(4)} = \begin{bmatrix} 1.570796329 \\ -1.570796329 \end{bmatrix}$$

The result of the fourth iteration $\mathbf{q}^{(4)}$ *is close enough to the exact value* $\mathbf{q} = \begin{bmatrix} \pi/2 & -\pi/2 \end{bmatrix}^T$.

6.4 ★ Comparison of the Inverse Kinematics Techniques

6.4.1 ★ Existence and Uniqueness of Solution

It is clear that when the desired tool frame position ${}^0\mathbf{d}_7$ is outside the working space of the robot, there can not be any real solution for the joint variables of the robot. In this condition, the overall resultant of the terms under square root signs would be negative. Furthermore, even when the tool frame position ${}^0\mathbf{d}_7$ is within the working space, there may be some tool orientations ${}^0\mathbf{R}_7$ that are not achievable without breaking joint constraints and violating one or more joint variable limits. Therefore, existing solutions for inverse kinematics problem generally depends on the geometric configuration of the robot.

The normal case is when the number of joints is six. Then, provided that no DOF is redundant and the configuration assigned to the end-effectors of the robot lies within the workspace, the inverse kinematics solution exists in finite numbers. The different solutions correspond to possible configurations to reach the same end-effector configuration.

Generally speaking, when the solution of the inverse kinematics of a robot exists, they are not unique. Multiple solutions appear because a robot can reach to a point within the working space in different configurations. Every set of solutions is associated to a particular configuration. The elbow-up and elbow-down configuration of the 2R manipulator in Example 168 is a simple example.

The multiplicity of the solution depends on the number of joints of the manipulator and their type. The fact that a manipulator has multiple solutions may cause problems since the system has to be able to select one of them. The criteria on which to base a decision may vary, but a very reasonable choice consists of choosing the closest solution to the current configuration.

When the number of joints is less than six, no solution exists unless freedom is reduced in the same time in the task space, for example, by constraining the tool orientation to certain directions.

When the number of joints exceeds six, the structure becomes redundant and an infinite number of solutions exists to reach the same end-effector configuration within the robot workspace. Redundancy of the robot architecture is an interesting feature for systems installed in a highly constrained environment. From the kinematic point of view, the difficulty lies in formulating the environment constraints in mathematical form, to ensure the uniqueness of the solution to the inverse kinematic problem.

6.4.2 ★ Inverse Kinematics Techniques

The inverse kinematics problem of robots can be solved by several methods, such as *decoupling*, *inverse transformation*, *iterative*, *screw algebra*, *dual matrices*, *dual quaternions*, and *geometric techniques*. The decoupling and inverse transform technique using 4×4 homogeneous transformation matrices suffers from the fact that the solution does not clearly indicate how to select the appropriate solution from multiple possible solutions for a particular configuration. Thus, these techniques rely on the skills and intuition of the engineer. The iterative solution method often requires a vast amount of computation and moreover, it does not guarantee convergence to the correct solution. It is especially weak when the robot is close to the singular and degenerate configurations. The iterative solution method also lacks a method for selecting the appropriate solution from multiple possible solutions.

Although the set of nonlinear trigonometric equations is typically not possible to be solved analytically, there are some robot structures that are *solvable* analytically. The sufficient condition of solvability is when the 6 DOF robot has three consecutive revolute joints with axes intersecting in one point. The other property of inverse kinematics is ambiguity of a solution in singular points. However, when closed-form solutions to the arm equation can be found, they are seldom unique.

Example 171 ★ *Iteration technique and n-m relationship.*

1- Iteration method when $n = m$.

When the number of joint variables n is equal to the number of independent equations generated in forward kinematics m, then provided that the Jacobian matrix remains non singular, the linearized equation

$$\delta \mathbf{T} = \mathbf{J}\, \delta \mathbf{q} \tag{6.179}$$

has a unique set of solutions and therefore, the Newton-Raphson technique may be utilized to solve the inverse kinematics problem.

The cost of the procedure depends on the number of iterations to be performed, which depends upon different parameters such as the distance between the estimated and effective solutions, and the condition number of the Jacobian matrix at the solution. Since the solution to the inverse kinematics problem is not unique, it may generate different configurations according to the choice of the estimated solution. No convergence may be observed if the initial estimate of the solution falls outside the convergence domain of the algorithm.

2- Iteration method when $n > m$.

When the number of joint variables n is more than the number of independent equations m, then the problem is an overdetermined case for which no solution exists in general because the number of joints is not enough to generate an arbitrary configuration for the end-effector. A solution can be generated, which minimizes the position error.

3- Iteration method when $n < m$.

When the number of joint variables n is less than the number of independent equations m, then the problem is a redundant case for which an infinite number of solutions are generally available.

6.5 ★ Singular Configuration

Generally speaking, for any robot, redundant or not, it is possible to discover some configurations, called *singular configurations*, in which the number of DOF of the end-effector is inferior to the dimension in which it generally operates. Singular configurations happen when:

1. Two axes of prismatic joints become parallel
2. Two axes of revolute joints become identical.

At singular positions, the end-effector loses one or more degrees of freedom, since the kinematic equations become linearly dependent or certain solutions become undefined. Singular positions must be avoided as the velocities required to move the end-effector become theoretically infinite.

The singular configurations can be determined from the Jacobian matrix. The Jacobian matrix $\mathbf{J}$ relates the infinitesimal displacements of the end-effector

$$\delta \mathbf{X} = [\delta X_1, \cdots \delta X_m] \tag{6.180}$$

to the infinitesimal joint variables

$$\delta \mathbf{q} = [\delta q_1, \cdots \delta q_n] \tag{6.181}$$

and has thus dimension $m \times n$, where n is the number of joints, and m is the number of end-effector DOF.

When n is larger than m and $\mathbf{J}$ has full rank, then there are $m - n$ redundancies in the system to which $m - n$ arbitrary variables correspond.

The Jacobian matrix $\mathbf{J}$ also determines the relationship between end-effector velocities $\dot{\mathbf{X}}$ and joint velocities $\dot{\mathbf{q}}$

$$\dot{\mathbf{X}} = \mathbf{J}\,\dot{\mathbf{q}}. \tag{6.182}$$

This equation can be interpreted as a linear mapping from an m-dimensional vector space $\mathbf{X}$ to an n-dimensional vector space $\mathbf{q}$. The subspace $\mathbb{R}(\mathbf{J})$ is the *range space* of the linear mapping, and represents all the possible end-effector velocities that can be generated by the n joints in the current configuration. $\mathbf{J}$ has full row-rank, which means that the system does not present any singularity in that configuration, then the range space $\mathbb{R}(\mathbf{J})$ covers the entire vector space $\mathbf{X}$. Otherwise, there exists at least one direction in which the end-effector cannot be moved.

The null space $\mathbb{N}(\mathbf{J})$ represents the solutions of $\mathbf{J}\,\dot{\mathbf{q}} = 0$. Therefore, any vector $\dot{\mathbf{q}} \in \mathbb{N}(\mathbf{J})$ does not generate any motion for the end-effector.

If the manipulator has full rank, the dimension of the null space is then equal to the number $m - n$ of redundant DOF. When $\mathbf{J}$ is degenerate, the dimension of $\mathbb{R}(\mathbf{J})$ decreases and the dimension of the null space increases by the same amount. Therefore,

$$\dim \mathbb{R}(\mathbf{J}) + \dim \mathbb{N}(\mathbf{J}) = n. \tag{6.183}$$

Configurations in which the Jacobian no longer has full rank, corresponds to singularities of the robot, which are generally of two types:

1. *Workspace boundary singularities* are those occurring when the manipulator is fully stretched out or folded back on itself. In this case, the end effector is near or at the workspace boundary.

2. *Workspace interior singularities* are those occurring away from the boundary. In this case, generally two or more axes line up.

Mathematically, singularity configurations can be found by calculating the conditions that make

$$|\mathbf{J}| = 0 \tag{6.184}$$

or

$$\left|\mathbf{J}\mathbf{J}^T\right| = 0. \tag{6.185}$$

Identification and avoidance of singularity configurations are very important in robotics. Some of the main reasons are:

1. Certain directions of motion may be unattainable.

2. Some of the joint velocities are infinite.

3. Some of the joint torques are infinite.

4. There will not exist a unique solution to the inverse kinematics problem.

Detecting the singular configurations using the Jacobian determinant may be a tedious task for complex robots. However, for robots having a spherical wrist, it is possible to split the singularity detection problem into two separate problems:

1. Arm singularities resulting from the motion of the manipulator arms.

2. Wrist singularities resulting from the motion of the wrist.

6.6 Summary

Inverse kinematics refers to determining the joint variables of a robot for a given position and orientation of the end-effector frame. The forward kinematics of a 6 DOF robot generates a 4×4 transformation matrix

$$\begin{aligned} {}^0T_6 &= {}^0T_1\, {}^1T_2\, {}^2T_3\, {}^3T_4\, {}^4T_5\, {}^5T_6 \\ &= \begin{bmatrix} {}^0R_6 & {}^0\mathbf{d}_6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{6.186}$$

where only six elements out of the 12 elements of ${}^0T_6$ are independent. Therefore, the inverse kinematics reduces to finding the six independent elements for a given ${}^0T_6$ matrix.

Decoupling, inverse transformation, and iterative techniques are three applied methods for solving the inverse kinematics problem. In decoupling technique, the inverse kinematics of a robot with a spherical wrist can be decoupled into two subproblems: inverse position and inverse orientation kinematics. Practically, the tools transformation matrix ${}^0T_7$ is decomposed into three submatrices ${}^0T_3$, ${}^3T_6$, and ${}^6T_7$.

$${}^0T_6 = {}^0T_3\, {}^3T_6\, {}^6T_7 \tag{6.187}$$

The matrix ${}^0T_3$ positions the wrist point and depends on the three manipulator joints' variables. The matrix ${}^3T_6$ is the wrist transformation matrix and the ${}^6T_7$ is the tools transformation matrix.

In inverse transformation technique, we extract equations with only one unknown from the following matrix equations, step by step.

$${}^1T_6 = {}^0T_1^{-1}\, {}^0T_6 \tag{6.188}$$

$${}^2T_6 = {}^1T_2^{-1}\, {}^0T_1^{-1}\, {}^0T_6 \tag{6.189}$$

$${}^3T_6 = {}^2T_3^{-1}\, {}^1T_2^{-1}\, {}^0T_1^{-1}\, {}^0T_6 \tag{6.190}$$

$${}^4T_6 = {}^3T_4^{-1}\, {}^2T_3^{-1}\, {}^1T_2^{-1}\, {}^0T_1^{-1}\, {}^0T_6 \tag{6.191}$$

$${}^5T_6 = {}^4T_5^{-1}\, {}^3T_4^{-1}\, {}^2T_3^{-1}\, {}^1T_2^{-1}\, {}^0T_1^{-1}\, {}^0T_6 \tag{6.192}$$

$$\mathbf{I} = {}^5T_6^{-1}\, {}^4T_5^{-1}\, {}^3T_4^{-1}\, {}^2T_3^{-1}\, {}^1T_2^{-1}\, {}^0T_1^{-1}\, {}^0T_6 \tag{6.193}$$

The iterative technique is a numerical method seeking to find the joint variable vector $\mathbf{q}$ for a set of equations $\mathbf{T}(\mathbf{q}) = 0$.

Exercises

1. Notation and symbols.

   Describe the meaning of

   a- $\operatorname{atan2}(a, b)$ b- ${}^{0}T_{n}$ c- $\mathbf{T}(\mathbf{q})$ d- w e- $\mathbf{q}$ f- $\mathbf{J}$.

2. 3R planar manipulator inverse kinematics.

   Figure 5.21 illustrates an R‖R‖R planar manipulator. The forward kinematics of the manipulator generates the following matrices. Solve the inverse kinematics and find θ_1, θ_2, θ_3.

$$
{}^{2}T_{3} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & l_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & l_3\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}T_{2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}T_{1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3. Spherical wrist inverse kinematics.

   Figure 5.22 illustrates a schematic of a spherical wrist with following transformation matrices. Assume that the frame B_3 is the base frame. Solve the inverse kinematics and find θ_4, θ_5, θ_6.

$$
{}^{3}T_{4} = \begin{bmatrix} \cos\theta_4 & 0 & -\sin\theta_4 & 0 \\ \sin\theta_4 & 0 & \cos\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}T_{5} = \begin{bmatrix} \cos\theta_5 & 0 & \sin\theta_5 & 0 \\ \sin\theta_5 & 0 & -\cos\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{5}T_{6} = \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

4. SCARA robot inverse kinematics.

   Consider the R‖R‖R‖P robot shown in Figure 5.27 with the following forward kinematics solution. Solve the inverse kinematics and find θ_1, θ_2, θ_3, d.

$$^0T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^3T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} ^0T_4 &= {^0T_1}\,{^1T_2}\,{^2T_3}\,{^3T_4} \\ &= \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & l_1c\theta_1 + l_2c\theta_{12} \\ s\theta_{123} & c\theta_{123} & 0 & l_1s\theta_1 + l_2s\theta_{12} \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \theta_{123} &= \theta_1 + \theta_2 + \theta_3 \\ \theta_{12} &= \theta_1 + \theta_2 \end{aligned}$$

5. R⊢R‖R articulated arm inverse kinematics.

   Figure 5.25 illustrates 3 DOF R⊢R‖R manipulator. Use the following transformation matrices and solve the inverse kinematics for θ_1, θ_2, θ_3.

$$^0T_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

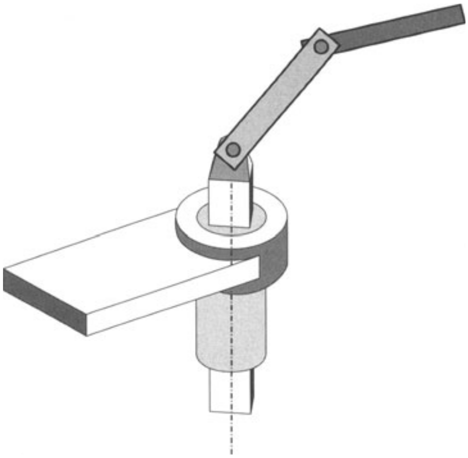

FIGURE 6.5. A PRRR manipulator.

$$
{}^{2}T_{3} = \begin{bmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0 \\ \sin\theta_3 & 0 & -\cos\theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

6. Kinematics of a PRRR manipulator.

   A PRRR manipulator is shown in Figure 6.5. Set up the links's coordinate frame according to standard DH rules. Determine the class of each link. Find the links' transformation matrices. Calculate the forward kinematics of the manipulator. Solve the inverse kinematics problem for the manipulator.

7. ★ Space station remote manipulator system inverse kinematics.

   Shuttle remote manipulator system (SRMS) is shown in Figure 5.28 schematically. The forward kinematics of the robot provides the following transformation matrices. Solve the inverse kinematics for the SRMS.

$$
{}^{0}T_{1} = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}T_{2} = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^2T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & a_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & a_3\sin\theta_3 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3T_4 = \begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & a_4\cos\theta_4 \\ \sin\theta_4 & \cos\theta_4 & 0 & a_4\sin\theta_4 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^4T_5 = \begin{bmatrix} \cos\theta_5 & 0 & \sin\theta_5 & 0 \\ \sin\theta_5 & 0 & -\cos\theta_5 & 0 \\ 0 & 1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^5T_6 = \begin{bmatrix} \cos\theta_6 & 0 & -\sin\theta_6 & 0 \\ \sin\theta_6 & 0 & \cos\theta_6 & 0 \\ 0 & -1 & 0 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^6T_7 = \begin{bmatrix} \cos\theta_7 & -\sin\theta_7 & 0 & 0 \\ \sin\theta_7 & \cos\theta_7 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$