CSCI 335 Theory of Computing

Lecture 18 – Decidability IV: The reducibility method

DATE : December 19 2022

Learning Outcomes

Learning Outcomes

By the end of this section you will be able to:

- Use the undecidability of the acceptance problem of TMs (A_{TM}) to show that some other languages are undecidable
- Understand the reducibility method for proving undecidability

Outline

- Undecidability of the TM halting problem ($HALT_{\text{TM}}$)
- The concept of reducibility
- Undecidability of the TM emptiness problem (E_{TM})
- Undecidability of the TM equivalence problem (EQ_{TM})

Undecidability of the TM halting problem ($HALT_{\mathrm{TM}}$)

Undecidability of the TM halting problem ($HALT_{TM}$)

If we know that some problem (say A_{TM}) is undecidable, we can use that to show other problems are undecidable.

Defn: $HALT_{\mathrm{TM}} = \{\langle M, w\rangle |\ M$ halts on input $w\}$

Recall Theorem: $HALT_{\mathrm{TM}}$ is undecidable

Proof by contradiction, showing that A_{TM} is reducible to $HALT_{\mathrm{TM}}$:

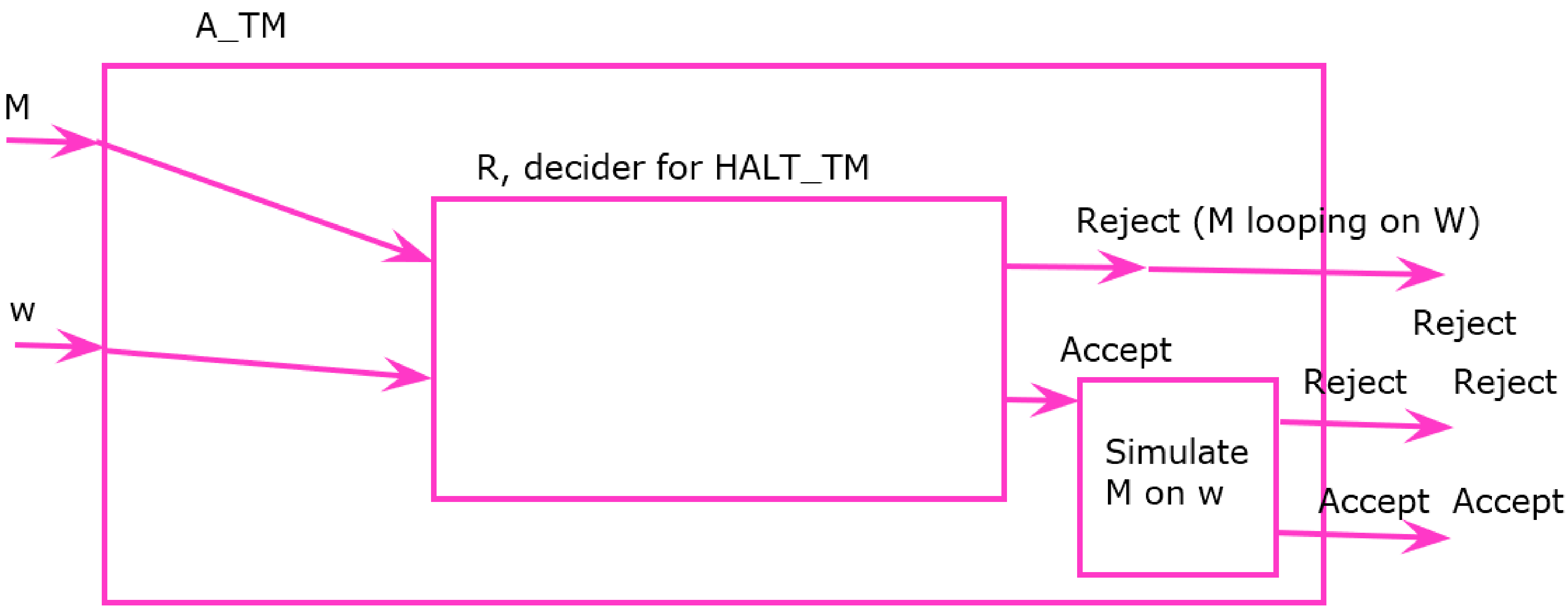
Assume that $HALT_{\mathrm{TM}}$ is decidable and show that A_{TM} is decidable (false!).
Let TM R decide $HALT_{\mathrm{TM}}$.
Construct TM S deciding A_{TM}.

$S =$ "On input $\langle M, w\rangle$

1. Use R to test if M on w halts. If not, *reject*.
2. Simulate M on w until it halts (as guaranteed by R).
3. If M has accepted then *accept*.
   If M has rejected then *reject*.

TM S decides A_{TM}, a contradiction. Therefore $HALT_{\mathrm{TM}}$ is undecidable.

Undecidability of the TM halting problem ($HALT_{TM}$)



The concept of reducibility

The concept of reducibility

If we have two languages (or problems) A and B, then
A is reducible to B means that we can use B to solve A.

Example 1: Measuring the area of a rectangle is reducible to measuring the lengths of its sides.

Example 2: We showed that A_{NFA} is reducible to A_{DFA} .

Example 3: Finding your way around a new city is reducible to the problem of obtaining a map of the city

If A is reducible to B then solving B gives a solution to A.

- then B is easy $\rightarrow$ A is easy.
- then A is hard $\rightarrow$ B is hard.

this is the form we will use

The concept of reducibility

Check-in 9.1

Is Biology reducible to Physics?

(a) Yes, all aspects of the physical world may be explained in terms of Physics, at least in principle.

(b) No, some things in the world, maybe life, the brain, or consciousness,
are beyond the realm pf Physics.

(c) I’m on the fence on this question!

The concept of reducibility

To prove B is undecidable:

- Show undecidable A is reducible to B. (often A is A_{TM})
- Template: Assume TM R decides B.
  Construct TM S deciding A. Contradiction.

Why do we use the term "reduce"?

When we reduce A to B, we show how to solve A by using B and conclude that A is no harder than B.

Possibility 1: We bring A's difficulty down to B's difficulty.

Possibility 2: We bring B's difficulty up to A's difficulty.

Undecidability of the TM emptiness problem (E_{TM})

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Let $E_{\text{TM}} = \{\langle M\rangle |\ M$ is a TM and $L(M) = \emptyset\}$

Theorem: E_{TM} is undecidable

Proof by contradiction. Show that A_{TM} is reducible to E_{TM}.

Assume that E_{TM} is decidable and show that A_{TM} is decidable (false!).

Let TM R decide E_{TM}.

Construct TM S deciding A_{TM}.

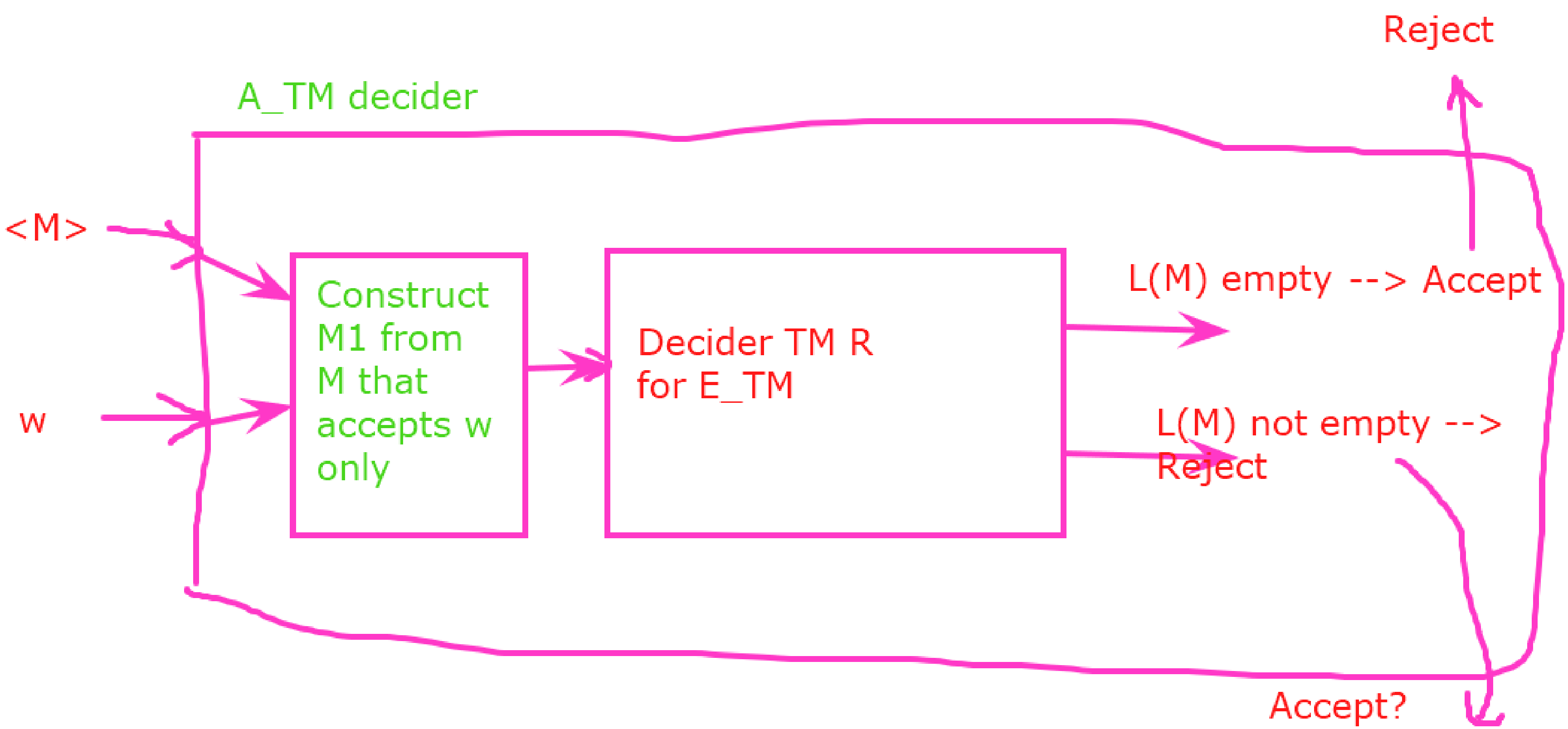
$S =$ "On input $\langle M, w\rangle$

1. Transform M to new TM $M_w =$ "On input x
   1. If $x \neq w$, *reject*.
   2. else run M on w
   3. *Accept* if M accepts."
2. Use R to test whether $L(M_w) = \emptyset$
3. If YES [so M rejects w] then *reject*.
   If NO [so M accepts w] then *accept*.

M_w works like M except that it always rejects strings x where $x \neq w$.

So $L(M_w) = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ rejects } w \end{cases}$

Undecidability of the TM emptiness problem (E_{TM})



Undecidability of the TM equivalence problem (EQ_{TM})

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EQ_{TM} is undecidable.

PROOF IDEA Show that if EQ_{TM} were decidable, E_{TM} also would be decidable by giving a reduction from E_{TM} to EQ_{TM}. The idea is simple. E_{TM} is the problem of determining whether the language of a TM is empty. EQ_{TM} is the problem of determining whether the languages of two TMs are the same. If one of these languages happens to be $\emptyset$, we end up with the problem of determining whether the language of the other machine is empty—that is, the E_{TM} problem. So in a sense, the E_{TM} problem is a special case of the EQ_{TM} problem wherein one of the machines is fixed to recognize the empty language. This idea makes giving the reduction easy.

Undecidability of the TM equivalence problem (EQ_{TM})

PROOF We let TM R decide EQ_{TM} and construct TM S to decide E_{TM} as follows.

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. If R accepts, *accept*; if R rejects, *reject*."

If R decides EQ_{TM}, S decides E_{TM}. But E_{TM} is undecidable by Theorem 5.2, so EQ_{TM} also must be undecidable.

Undecidability of the TM equivalence problem (EQ_{TM})

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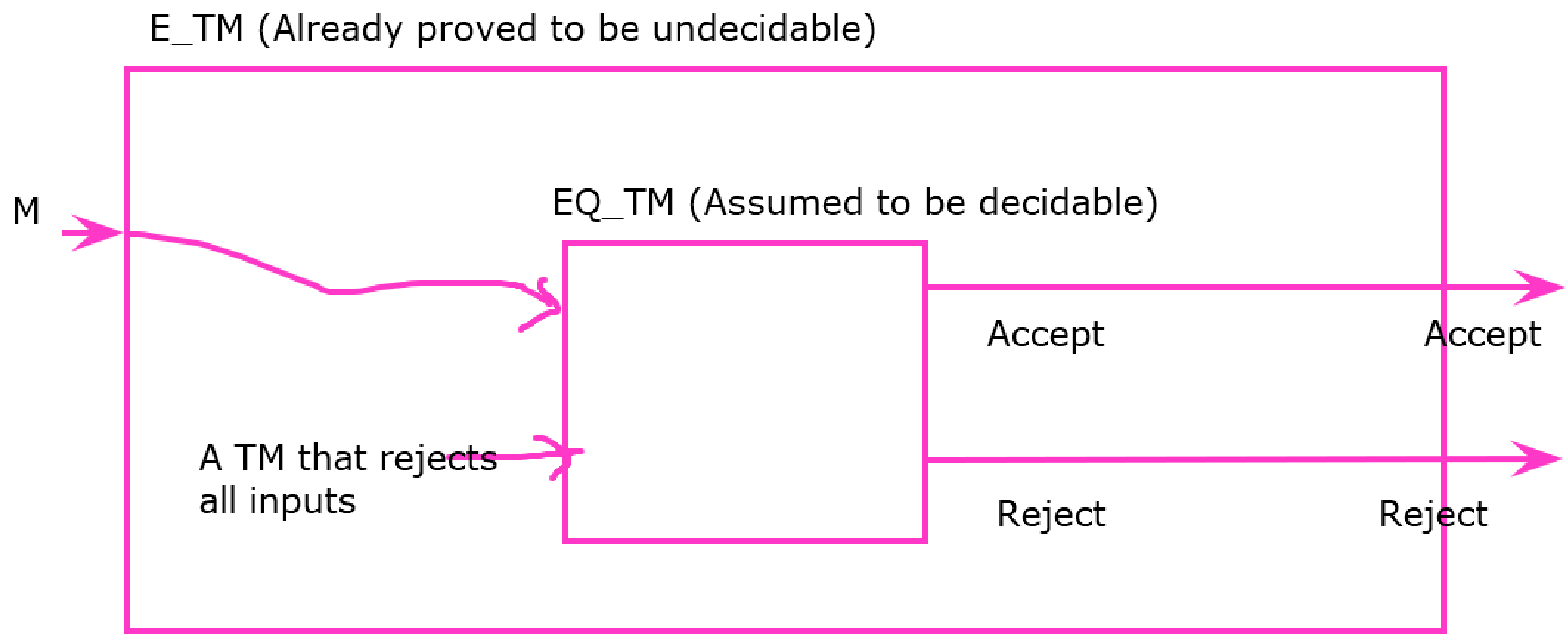

Summary

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- The TM halting problem ($HALT_{\mathrm{TM}}$) is undecidable
- The concept of reducibility
- The TM emptiness problem (E_{TM}) is undecidable
- The TM equivalence problem (EQ_{TM}) is undecidable

Readings

- Reducibility: Section 5.1 (Sipser 2013)
- What is the halting problem? https://www.codingninjas.com/codestudio/library/halting-problem-in-the-theory-of-computation
- Halting problem https://en.wikipedia.org/wiki/Halting_problem