CSCI 335 Theory of Computing

Lecture 16 – Decidability II: Countable and Uncountable Sets

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Learning Outcomes

Learning Outcomes

By the end of this section you will be able to:

- Establish equivalence between infinite sets using bijective maps
- Identify countable and uncountable sets
- Get acquainted with the diagonalization method

Outline

- Correspondence or bijective maps
- Countable and uncountable sets
- The integer numbers are countable
- The rational numbers are countable
- The real numbers are uncountable

Correspondence (bijective) map

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DEFINITION 4.12

Assume that we have sets A and B and a function f from A to B. Say that f is ***one-to-one*** if it never maps two different elements to the same place—that is, if $f(a) \neq f(b)$ whenever $a \neq b$. Say that f is ***onto*** if it hits every element of B—that is, if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$. Say that A and B are the ***same size*** if there is a one-to-one, onto function $f: A \longrightarrow B$. A function that is both one-to-one and onto is called a ***correspondence***. In a correspondence, every element of A maps to a unique element of B and each element of B has a unique element of A mapping to it. A correspondence is simply a way of pairing the elements of A with the elements of B.

Countable and uncountable sets

The size of infinity

How to compare the relative sizes of infinite sets?

Cantor (~1890s) had the following idea.

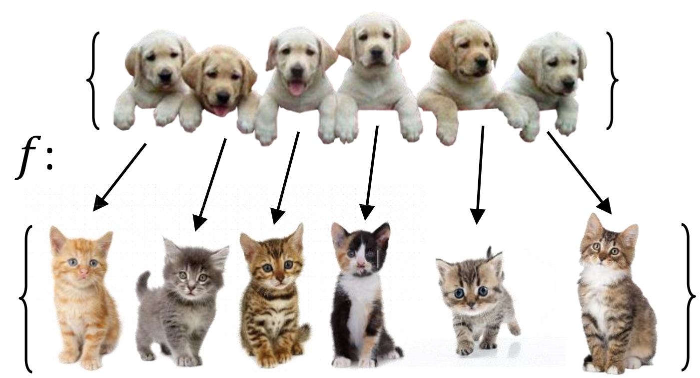
Defn: Say that set A and B <u>have the same size</u> if there is
a $\underbrace{\text{one-to-one}}_{\substack{x \neq y \to \\ f(x) \neq f(y) \\ \text{“injective”}}}$ and $\underbrace{\text{onto}}_{\substack{\text{Range } (f) = B \\ \text{“surjective”}}}$ function $f : A \to B$

We call such an f a <u>1-1 correspondence</u>

Informally, two sets have the same size if we can pair up their members.

This definition works for finite sets.

Apply it to infinite sets too.



The integer numbers are countable

The integer numbers are countable

Let $\mathbb{N} = \{1,2,3, \dots\}$ and let $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Show $\mathbb{N}$ and $\mathbb{Z}$ have the same size.

Proof: A correspondence can be established between $\mathbb{N}$ and $\mathbb{Z}$ as follows.

n	$f(n)$
1	0
2	-1
3	1
4	-2
5	2
6	-3
7	3
$\vdots$	$\vdots$

The rational numbers are countable

The positive rational numbers $\mathbb{Q}^+$ are countable

Let $\mathbb{Q}^+ = \{ {}^{m}/_{n} \mid m, n \in \mathbb{N}\}$

Show $\mathbb{N}$ and $\mathbb{Q}^+$ have the same size.

Proof: A correspondence can be established between $\mathbb{N}$ and $\mathbb{Q}^+$as follows.

$\mathbb{Q}^+$	1	2	3	4	…
1	1/1	1/2	1/3	1/4	
2	2/1	2/2	2/3	2/4	…
3	3/1	3/2	3/3	3/4	
4	4/1	4/2	4/3	4/4	
⋮		⋮			

	n	$f(n)$	
$\mathbb{N}$	1	1/1	$\mathbb{Q}^+$
	2	2/1	
	3	1/2	
	4	3/1	
	5	3/2	
	6	2/3	
	7	1/3	
	⋮	⋮	

The real numbers are uncountable

Uncountability of real numbers $\mathbb{R}$: Proof by diagonalization

Let $\mathbb{R}$ = all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f : \mathbb{N} \rightarrow \mathbb{R}$

n	$f(n)$
1	2.718281828…
2	3.141592653…
3	0.000000000…
4	1.414213562…
5	0.142857242…
6	0.207879576…
7	1.234567890…
⋮	⋮

Diagonalization

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0.\underset{\neq 7}{8}\,\underset{\neq 4}{5}\,\underset{\neq 0}{1}\,\underset{\neq 2}{6}\,\underset{\neq 5}{1}\,\underset{\neq 9}{8}\,\underset{\neq 8}{2}\ldots$$

differs from the n^{th} number in the n^{th} digit so cannot be the n^{th} number for any n.

Hence x is not paired with any n. It is missing from the list.

Therefore f is not a 1-1 correspondence.

Summary

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- Sizes of two infinite sets are considered to be the same if they have a 1-1 correspondence (or bijective map)
- The integer numbers are countable
- The rational numbers are countable
- The real numbers are uncountable

Readings

- Undecidability: Section 4.2 (Sipser 2013)