INFOF422: Q1 solutions

June 1, 2023

Single positive test

By using the Bayes theorem and in particular the formula (2.6.25) with $\mathcal{E}_1 : \mathbf{x}_3 = Y$, $\mathcal{E}_2 : \mathbf{x}_4 = P$ and $\mathcal{E}_3 : \mathbf{x}_1 = Y, \mathbf{x}_2 = M$, we obtain

$$P(\underbrace{\mathbf{x}_3 = Y}_{\mathcal{E}_1} \,|\, \underbrace{\mathbf{x}_4 = P}_{\mathcal{E}_2}, \underbrace{\mathbf{x}_1 = Y, \mathbf{x}_2 = M}_{\mathcal{E}_3}) = \frac{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y, \mathbf{x}_1 = Y, \mathbf{x}_2 = M)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4 = P|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} =$$

$$\frac{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y, \mathbf{x}_1 = Y, \mathbf{x}_2 = M)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}2 = M)}{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y, \mathbf{x}_1 = Y, \mathbf{x}_2 = M)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4 = P|\mathbf{x}_3 = N, \mathbf{x}_1 = Y, \mathbf{x}_2 = M)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} =$$

$$= \frac{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4 = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}$$

where the equality $P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y, \mathbf{x}_1 = Y, \mathbf{x}_2 = M) = P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)$ is due to the conditional independence.

EXAM.2223.1s.Q1.1

$$P(\mathbf{x}_3 = Y|\mathbf{x}_4 = P, \mathbf{x}_1 = Y, \mathbf{x}_2 = M) = \frac{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4 = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} = \frac{0.9 * 0.1}{0.9 * 0.1 + 0.1 * 0.9}$$

EXAM.2223.1s.Q1.2

$$P(\mathbf{x}_3 = Y|\mathbf{x}_4 = P, \mathbf{x}_1 = Y, \mathbf{x}_2 = M) = \frac{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4 = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} = \frac{0.9 * 0.1}{0.9 * 0.1 + 0.2 * 0.9} = 0.3333$$

EXAM.2223.1s.Q1.3

$$P(\mathbf{x}_3 = Y|\mathbf{x}_4 = P, \mathbf{x}_1 = Y, \mathbf{x}_2 = M) = \frac{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4 = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4 = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} = \frac{0.9 * 0.2}{0.9 * 0.2 + 0.2 * 0.8} = 0.5294$$

Two conditionally independent positive tests

$$P(\mathbf{x}_3 = Y|\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P, \mathbf{x}_1 = Y, \mathbf{x}_2 = M) =$$

$$= \frac{P(\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y, \mathbf{x}_1 = Y, \mathbf{x}_2 = M)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y, \mathbf{x}_1 = Y, \mathbf{x}_2 = M)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = N, \mathbf{x}_1 = Y, \mathbf{x}_2 = M)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} =$$

$$= \frac{P(\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} =$$

$$\frac{P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}$$

where $P(\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y) = P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)$ because of the conditional independence of the tests.

EXAM.2223.1s.Q1.1

$$P(\mathbf{x}_3 = Y|\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P, \mathbf{x}_1 = Y, \mathbf{x}_2 = M) =$$

$$\frac{P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} =$$

$$\frac{0.9 * 0.9 * 0.1}{0.9 * 0.9 * 0.1 + 0.1 * 0.1 * 0.9} = 0.9$$

EXAM.2223.1s.Q1.2

$$P(\mathbf{x}_3 = Y|\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P, \mathbf{x}_1 = Y, \mathbf{x}_2 = M) =$$

$$\frac{P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} =$$

$$\frac{0.9 * 0.9 * 0.1}{0.9 * 0.9 * 0.1 + 0.2 * 0.2 * 0.9} = 0.69237$$

EXAM.2223.1s.Q2.3

$$P(\mathbf{x}_3 = Y|\mathbf{x}_4^{(a)} = P, \mathbf{x}_4^{(b)} = P, \mathbf{x}_1 = Y, \mathbf{x}_2 = M) =$$

$$\frac{P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)}{P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = Y)P(\mathbf{x}_3 = Y|\mathbf{x}_1 = Y, \mathbf{x}_2 = M) + P(\mathbf{x}_4^{(a)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_4^{(b)} = P|\mathbf{x}_3 = N)P(\mathbf{x}_3 = N|\mathbf{x}_1 = Y, \mathbf{x}_2 = M)} =$$

$$\frac{0.9 * 0.9 * 0.2}{0.9 * 0.9 * 0.2 + 0.2 * 0.2 * 0.8} = 0.835$$