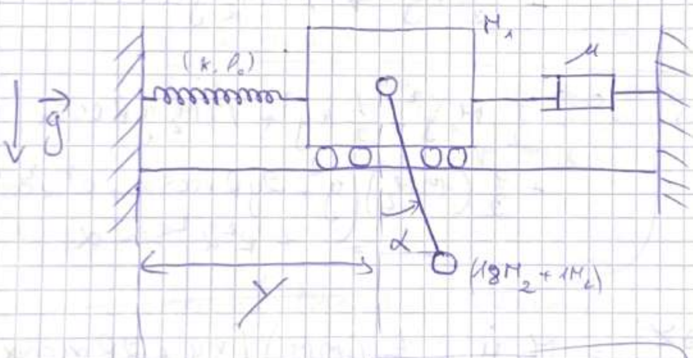
python/
Matlab/
Simulink;

Numerical Methods

I-) Runge-Kutta's Methods.

- Cart-spring-pendulum-Damper system;



- EQts of Mvt ;

$$\boxed{\mathcal{L} = T - V}$$

- $T_\Sigma = T = T_p + T_c$
- $T_c = \frac{1}{2} M_1 \dot{y}^2$; $T_p = \frac{1}{2}(19M_2)\Big(\overbrace{(\dot{y} + R\dot{\alpha}\cos\alpha)^2}^{\text{Horizontal}} + \underbrace{(R\dot{\alpha}\sin\alpha)^2}_{\text{vertical}}\Big)$
- $V_\Sigma = V = V_p + V_R$

$$= -19M_2 g R \cos\alpha + \frac{1}{2} K y^2$$

$\Rightarrow D = \frac{1}{2}\mu\dot{y}^2$; $\mathcal{L} = \frac{1}{2}M_1\dot{y}^2 + \frac{1}{2}(19M_2)((\dot{y}$

$$\Rightarrow \left\| \begin{array}{l} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0 \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = 0 \end{array} \right.$$

$\Leftrightarrow$ ~~$\frac{d}{dt}($~~ (We write the Lagrangian $\downarrow$)

$$\boxed{\mathcal{L} = T - V = \frac{1}{2}M_1\dot{y}^2 + \frac{1}{2}(19M_2)\left((\dot{y} + R\dot{\alpha}\cos\alpha)^2 + (R\dot{\alpha}\sin\alpha)^2\right) - \frac{1}{2}ky^2 + 19M_2 gR\cos\alpha}$$

$$\Leftrightarrow \mathcal{L} = \frac{1}{2}M_1\dot{y}^2 - \frac{1}{2}ky^2 + 19M_2 gR\cos\alpha + \frac{1}{2}(19M_2)\left(\dot{y}^2 + 2\dot{y}R\dot{\alpha}\cos\alpha + R^2\dot{\alpha}^2\cos^2\alpha + R^2\dot{\alpha}^2\sin^2\alpha\right)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{y}} = M_1\dot{y} + \frac{1}{2}(19M_2)\left(2\dot{y} + 2R\dot{\alpha}\cos\alpha\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = M_1\ddot{y} + \frac{1}{2}(19M_2)\left(2\ddot{y} + 2R(\ddot{\alpha}\cos\alpha - \dot{\alpha}^2\sin\alpha)\right)$$

$$\Rightarrow -\frac{\partial \mathcal{L}}{\partial y} = -(-ky) = ky$$

$$\Leftrightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial D}{\partial \dot{y}}$$

$\Leftrightarrow \quad M_1\ddot{y} + 19M_2\left(\ddot{y} + R\left(\ddot{\alpha}\cos\alpha - \dot{\alpha}^2\sin\alpha\right)\right) + ky = -\mu\dot{y}$

$$\boxed{(M_1 + 19M_2)\ddot{y} + 19M_2R\ddot{\alpha}\cos\alpha - 19M_2R\dot{\alpha}^2\sin\alpha + ky + \mu\dot{y} = 0}$$

$$\Rightarrow \left(\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\right) - \frac{\partial\mathcal{L}}{\partial\alpha} = -\frac{\partial D}{\partial\dot{\alpha}}\right) \quad (\text{DiHo})$$

$\Rightarrow$ ~~At~~ the end we find:

$$\begin{cases} R\ddot{\alpha} + \ddot{y}\cos\alpha = -g\sin\alpha \\ (M_1 + 19M_2)\ddot{y} + M_2R\ddot{\alpha}\cos\alpha - M_2R\dot{\alpha}^2\sin\alpha + ky + \mu\dot{y} = 0 \end{cases}$$

Q1/ Order of the Diff system?

$$\Rightarrow \left\| \begin{array}{l} \ddot{\alpha} \rightarrow (2) \\ \ddot{y} \rightarrow (2) \end{array} \right\} \oplus \rightarrow \boxed{4}$$

$\Rightarrow$ 4^{th} order Diff system

Q2) Forme résolue ? (Resolved form)

$$\Rightarrow \begin{cases} \ddot{\alpha} = -\dfrac{(g\sin\alpha + \ddot{y}\cos\alpha)}{R} \\ \ddot{y} = \left(-\mu\dot{y} - Ky + M_2 R\dot{\alpha}^2\sin\alpha - M_2 R\ddot{\alpha}\cos\alpha\right) \Big/ M_1 + M_2 \end{cases}$$

($2^{nd} - 1 = 1^{st}$)

Q3) Always ~~turn~~ back into a 1^{st} order ODE : (Cauchy's Problem) :

$$\boxed{\begin{aligned} & Z' = f(Z, t) \\ & Z(t_0) = Z_0 \end{aligned}}$$

State variables :

$$Z_1 = y \qquad Z_2 = \alpha \qquad Z_3 = \dot{y} \qquad Z_4 = \dot{\alpha}$$

(same order !)

Q4)

$$\begin{pmatrix} y \\ \dot{y} \\ \alpha \\ \dot{\alpha} \end{pmatrix} \begin{matrix} z[0] \\ z[1] \\ z[2] \\ z[3] \end{matrix}$$

$\Rightarrow$ Numerical Resolution :

① — Code the z' function

(VS)

$$\left(\begin{cases} z' = f(t,z) \\ z(t_0) = z_0 \end{cases}\right)$$

— Decouple the system

a :

$$\boxed{\ddot{\alpha} = \frac{\left(-g\sin\alpha - \ddot{y}\cos\alpha\right)}{R}}$$

do

$$\Longrightarrow \quad \ddot{y} = \frac{-\mu\dot{y} - Ky + M_2 R\dot{\alpha}^2\sin\alpha - M_2 R\cos\alpha \times \left(\frac{-g\sin\alpha - \ddot{y}\cos\alpha}{R}\right)}{M_1 + 19 M_2}$$

~~What you should use!~~

$\ddot{\alpha}$

$$\ddot{y} - \frac{M_2 \cancel{R}\cos\alpha\left(g\sin\alpha + \ddot{y}\cos\alpha\right)}{\cancel{R}\left(M_1 + 19M_2\right)} = \frac{M_2 R\dot{\alpha}^2\sin\alpha - \mu\dot{y} - Ky}{M_1 + 19M_2}$$

$$\begin{cases} (M_1 + 19M_2)\ddot{y} - M_2\cos\alpha\left(g\sin\alpha + \ddot{y}\cos\alpha\right) = M_2 R\dot{\alpha}^2\sin\alpha - \mu\dot{y} - Ky \\ \left(M_1 + 19M_2 - M_2\cos^2\alpha\right)\ddot{y} - M_2 g\cos\alpha\sin\alpha = \end{cases}$$

$$\boxed{\ddot{y}\left((M_1 + 19M_2) - M_2\cos^2\alpha\right) = M_2 g\cos\alpha\sin\alpha + M_2 R\dot{\alpha}^2\sin\alpha - \mu\dot{y} - Ky}$$

- then we calculate the Derivatives

(2) We return np.array([derivatives])

Now initial condition

At $t = 0s$:

$y = 1m$

$\alpha = 0°$

$\dot{y} = 0 m/s$

$\dot{\alpha} = 0 rad/s$

$y = 1m$

$\dot{y} = 0 m/s$

$\alpha = 0°$

$\dot{\alpha} = 0 rad/s$

$\boxed{\text{We implement these as a vector (array)}}$

$\Rightarrow$ (i) 6) RK2 (Heun's method)

RK4:

$$\boxed{Z_{k+1} = Z_k + b_1 k_1 + b_2 k_2 + \ldots + b_m k_m}$$

- $k_1 = f(t_k, Z_k)$
- $K_2 = f\left(t_k + \frac{h}{2}, Z_k + \frac{h}{2} K_1\right)$
- $K_3 = f\left(t_k + \frac{h}{2}, Z_k + \frac{h}{2} K_2\right)$
- $K_4 = f\left(t_k + h, Z_k + h K_3\right)$

$$\Rightarrow Z_{k+1} = Z_k + \frac{h}{6}\left(K_1 + 2 K_2 + 2 K_3 + K_4\right)$$

$$Z_{k+1} = Z_k + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Approximating: $Z(t+h) \sim Z(t)$ with 4 intermediate slopes pondered

RK2:

$$\boxed{Z_{k+1} = Z_k + \frac{h}{2}\left(K_1 + K_2\right)}$$

$K_1 = f(t_k, Z_k)$

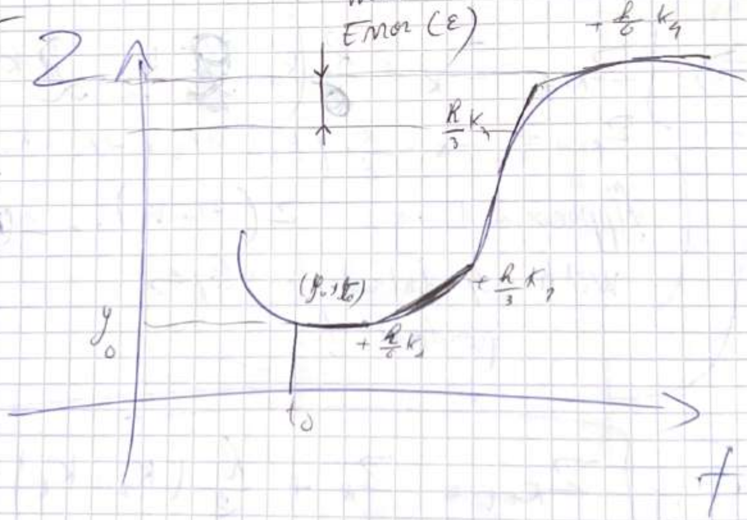
$K_2 = f(t_k + h, Z_k + h K_1)$

It comes from the scheme (implicit) of **Crank - Nicholson**

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)}$$

$$y_{approx} = y_n + h\, f(t_{n+1}$$

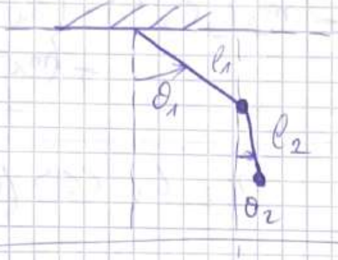
RK4



II) Double pendulum :

(chaotic system)

$E_{II} = 0\,J$



(unpredictable (Highly sensitive to I.C.))

$$\Rightarrow \boxed{\begin{aligned}\mathcal{L} = &\frac{1}{2}(m_1+m_2)\, l_1^2\, \dot\theta_1^2 + \frac{1}{2} m_2\, l_2^2\, \dot\theta_2^2 \\ &+ m_2\, l_1 l_2\, \dot\theta_1 \dot\theta_2 \cos(\theta_2-\theta_1) + l_1 g (m_1+m_2) \times \cos\theta_1 \\ &+ m_2\, l_2\, g \cos\theta_2\end{aligned}}$$

$$V = (m_2+m_1) g l_1 \cos(\theta_1+\theta_2) - m_2 g l_2 \cos\theta_2$$

Eqt of mvt : (θ_1, θ_2)

$$(m_1+m_2) l_1 \ddot\theta_1 + m_2 L_2 \ddot\theta_2 \cos(\theta_1-\theta_2) + m_2 L_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g \sin\theta_1 = 0$$

$$m_2 L_2 \ddot\theta_2 + m_2 L_1 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_1 L_1 \dot\theta_1^2 \sin(\theta_1-\theta_2) + m_2 g \sin\theta_2 = 0$$

Q1) ORDER? $\Rightarrow$ 4th order Diff syste

Q2) Resolved for $\Rightarrow$

$$\Rightarrow \left\| \begin{array}{l} m_2 L_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) = -m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) \\ \qquad - (m_1+m_2) g \sin\theta_1 \\ \qquad - (m_1+m_2) L_1 \ddot{\theta}_1 \\ \Rightarrow \ddot{\theta}_2 = \dfrac{-m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) - (m_1+m_2) g \sin\theta_1 - (m_1+m_2) L_1 \ddot{\theta}_1}{m_2 L_2 \cos(\theta_1-\theta_2)} \end{array} \right.$$

$$\Rightarrow m_2 L_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) = m_1 L_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) - m_2 g \sin\theta_2 - m_2 L_2 \ddot{\theta}_2$$

$$\Rightarrow \ddot{\theta}_1 = \frac{m_1 L_1 \sin(\theta_1-\theta_2) \dot{\theta}_1^2 - m_2 g \sin\theta_2 - m_2 L_2 \ddot{\theta}_2}{m_2 L_1 \cos(\theta_1-\theta_2)}$$

Q3) Canonical form (1st order ODE)

$$\left\| \begin{array}{l} z' = f(t, z) \\ z(t_0) = z(0) \end{array} \right.$$

En Python

$$\theta_1 \to Z[0]$$
$$\theta_2 \to Z[1]$$
$$\dot{\theta}_1 \to Z[2]$$
$$\dot{\theta}_2 \to Z[3]$$

$$\Rightarrow \left\| \begin{array}{l} z_1 = \theta_1 \\ z_2 = \theta_2 \\ z_3 = \dot{\theta}_1 \\ z_4 = \dot{\theta}_2 \end{array} \right.$$

CIs. $\left\| \begin{matrix} m_1 \\ m_2 \\ L_1 \\ L_2 \\ g \end{matrix} \right.$; $\wedge\ t = 0$: $\left\| \begin{matrix} \theta_1 = \frac{\pi}{2} \\ \theta_2 = 0 \end{matrix} \right.$; $\dot{\theta}_1 = \dot{\theta}_2 = 0$ rad/s

Approche diff.

Instead of :

$\Longrightarrow$ $\left\| \begin{matrix} f(t, Z) = \cancel{Z'} \\ \frac{dZ}{dt} = Z' = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ f_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \\ f_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \end{pmatrix} \end{matrix} \right.$

III) Forced + Damped pendulum.

(Système Non-linéaire chaos)

$\vec{g}$

(possible Transition to chaos $F > 1{,}5$)

Eq. of Mvt

$$\boxed{\ddot{\theta} + 2\alpha\dot{\theta} + \omega_0^2 \sin\theta = F\cos(\omega t)}$$

$$\alpha = 0{,}1\ s^{-1}$$
$$\omega_0 = \sqrt{\frac{g}{L}}\ ; \quad \Gamma = 1{,}5\ rad/s^2$$
$$g = 9{,}81 \qquad \omega = 2\ rad/s$$

Q1) Ordre $\Rightarrow$ 2nd ordre système :

Q2) $$\boxed{\ddot{\theta} = -2\alpha\dot{\theta} - \omega_0^2 \sin\theta + \Gamma\cos(\omega t)}$$

(Resolved for)

Q3) $$\begin{cases} z' = f(t,z) \\ z(t_0) = z_0 \end{cases} \quad \text{(Pb de Cauchy)}$$

SV $\Rightarrow$ $$\Rightarrow \begin{cases} \omega = \dot{\theta} \\ \dot{\omega} = -2\alpha\omega - \omega_0^2 \sin\theta + \Gamma\cos(\omega t) \end{cases}$$

CI :

$$\boxed{\begin{array}{l} \theta(0) = 0{,}1\ rad/s \\ \dot{\omega}(0) = 0\ rad/s \end{array}}$$

Rq : chaos $\Rightarrow$ $$\begin{cases} \Gamma > 1{,}5\ ; \quad \omega = \frac{2}{3} \\ \alpha = 0{,}5\ ; \end{cases}$$

IV) Satellites (Orbital Movement)

EQ :
$$\begin{cases} \ddot{x} = \dfrac{-GM_x}{(x^2+y^2)^{3/2}} \\ \ddot{y} = \dfrac{-GM_y}{(x^2+y^2)^{3/2}} \end{cases}$$

CIs :

$x(0) = R = 6{,}771 \times 10^6\ m$ (Alt : 400 km)

$y(0) = 0$

$V_x(0) = 0$

$V_y(0) = \sqrt{\frac{GM}{R}} \simeq 7662\ m/s$

Q1) Order ? $\Rightarrow$ 4^{th} order

Q2) Resolved for ? $\Rightarrow$ / Already

Q3) Pb de Cauchy :
$$\begin{cases} Z' = f(t, Z) \\ Z(t_0) = Z_0 \end{cases}$$

$\left(\frac{dz}{dt} = \begin{pmatrix} \dot{o}_1 \\ \dot{o}_2 \\ f(o_1, o_2, o_3, o_4) \\ f(\text{[illegible]} \end{pmatrix} \right)$

$Z_1 = x$

$Z_2 = y$

$Z_3 = \dot{x}$

$Z_4 = \dot{y}$

python

x	z[0]
$\dot{x}$	z[1]
y	z[2]
$\dot{y}$	z[3]

* Adams – Bashforth's Methods :

* Principle

AB2 :

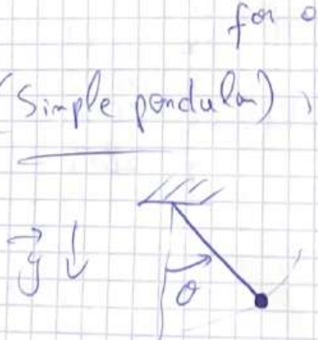
$$\boxed{Y_{n+1} = Y_n + \frac{h}{2}\left(3f(t_n, y_n) - f(t_{n-1}, y_{n-1})\right)}$$

AB4 :

$$\boxed{y_{n+1} = y_n + \frac{h}{24}\left(55f(t_n, y_n) - 59f(t_{n-1}, y_{n-1}) + 37f(t_{n-2}, y_{n-2}) - 9f(t_{n-3}, y_{n-3})\right)}$$

→ Pros : less pricey than RK for ODEs

Example (Simple pendulum) :



: $\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$

$\Rightarrow$ Cauchy's Pb :

$$\begin{cases} z' = f(t, z) \\ z(t_0) = z_0 \end{cases}$$

(1^{st} order ODE)

$\Rightarrow$ ~~Let~~ Let be :

$$\omega = \dot{\theta}$$

$$\dot{\omega} = -\frac{g}{\ell} \sin(\theta)$$

$$SV \Rightarrow \begin{pmatrix} z_1 = \theta \\ z_2 = \dot{\theta} \end{pmatrix}$$

S - A + F = 2

T - A + S = 1 - χ^1