Note
Homogeneous involutions of graded rings

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Abstract

In this note, we introduce a novel concept on graded rings, namely, homogeneous involutions.

Throughout this note, R denotes an associative ring with the center $Z(R)$, and G is a group with identity e. A ring R is G-graded if there is a family $\{R_g, g \in G\}$ of additive subgroups R_g of $(R,+)$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for every $g,h \in G$. The additive subgroup R_g called the homogeneous component of R. The set $h(R) = \bigcup_{g\in G} R_g$ is the set of homogeneous elements of R. The relation $R_g R_h \subset R_{gh}$ means that the multiplication of homogeneous elements is compatible with the group operation. A nonzero element $x \in R_g$ is said to be homogeneous of degree g, and we write $\deg(x) = g$. An element $x \in R$ has a unique decomposition $x = \sum_{g\in G} x_g$ with $x_g \in R_g$, where the sum is finite. The x_g terms called the homogeneous components of element x.

If R and T are G_1-graded and G_2-graded rings, respectively, then $R \times T$ is $G_1 \times G_2$-graded with

$$(R\times T)_{(g,h)} := \Big\{(a_g, b_h) \mid a_g \in R \text{ and } b_h \in T_h\Big\} \text{ for } (g,h) \in G_1 \times G_2.$$

Definition 0.1. *Let R be a G-graded ring. An additive mapping $*_h : R \longrightarrow R$ is called an homogeneous involution if:*

i) $(a^{*_h})^{*_h} = a$

ii) $(ab)^{*_h} = b^{*_h} a^{*_h}$

iii) $r^{*_h} \in h(R)$ *for all* $r \in h(R)$.

A G-graded ring equipped with homogeneous involution is called G-graded ring with homogeneous involution or $*_h$-G-graded ring.

Remark 0.1. *We make some remarks regarding Definition* (0.1).

i) It is clear that any homogeneous involution is involution of R.

ii) It is clear that a homogeneous involution $*_h$ is a graded anti-automorphism of order 2.

We will now consider some examples of homogeneous involutions.

Example 0.1. *1) Let's consider the ring* $R := \begin{pmatrix} k & k \\ k & k \end{pmatrix}$ *(with matrix addition and matrix multiplication), where k is a field. Then R is* $\mathbb{Z}_4$*-graded ring by:*

$$R_0 := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix},\ R_1 := \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix},\quad R_2 := 0 \ \text{ and } \ R_3 := \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}.$$

Let's consider the mapping

$$\begin{array}{cccc} *_h : & R & \longrightarrow & R \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \end{array}$$

Then $*_h$ *is a homogeneous involution of R.*

2) Let's consider the ring $R := \begin{pmatrix} k & k & k \\ k & k & k \\ k & k & k \end{pmatrix}$*. Then R is* $\mathbb{Z}_3$*-graded by:*

$$R_0 := \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix},\ R_1 := \begin{pmatrix} 0 & k & 0 \\ 0 & 0 & k \\ k & 0 & 0 \end{pmatrix} \ \text{and} \ R_2 := \begin{pmatrix} 0 & 0 & k \\ k & 0 & 0 \\ 0 & k & 0 \end{pmatrix}.$$

Define a mapping

$$\begin{array}{cccc} *_h : & R & \longrightarrow & R \\ & \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} & \longmapsto & \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \end{array}$$

Then $*_h$ *is a homogeneous involution of R.*

3) Let S be a G-graded ring with homogeneous involution $*_h$*. Then the ring* $R := A \times \mathbb{C}$ *is* $G \times \mathbb{Z}_2$*-graded ring by:*

$$R_{(e,0)} := A_e \times \mathbb{C}_0 \quad \text{and} \quad R_{(g,1)} := A_g \times \mathbb{C}_1 \quad for \ \ all \ \ g \in G \setminus \{e\}$$

where $\mathbb{C}_0 := \mathbb{R}$ *and* $\mathbb{C}_1 := i\mathbb{R}$*. Now, it is easy to see that R is a graded ring with homogeneous involution defined by:*

$$\begin{array}{cccc} \tau_h : & R & \longrightarrow & R \\ & (x, z) & \longmapsto & (x^{*_h}, \bar{z}) \end{array}$$

Here, we present an example that illustrates the existence of involutions of G-graded rings that are not homogeneous.

Example 0.2. *Let's consider* $R := \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$*, where* $\mathbb{R}$ *is the field of real numbers. Then* R *is* $\mathbb{Z}_4$*-graded by:*

$$R_0 := \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{pmatrix}, \; R_1 := \begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix}, \quad R_2 := 0 \; \textit{and} \; R_3 := \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}.$$

Let's consider the mapping defined as follows:

$$\begin{array}{cccc} *: & R & \longrightarrow & R \\ & M & \longmapsto & UM^tU^{-1} \end{array}$$

where $U := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ *(is an invertible element of* R*). Then* $*$ *is an involution of* R *but is not homogeneous involution. Indeed,* $^*(M) = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \notin \mathcal{H}(R)$*, where* $M := \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \in \mathcal{H}(R)$.