Some examples of homogeneous derivations

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A ring R is G-graded if there is a family $\{R_g, g \in G\}$ of additive subgroups R_g of $(R,+)$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for every $g,h \in G$. The additive subgroup R_g called the homogeneous component of R. The set $\mathcal{H}(R) = \bigcup_{g\in G} R_g$ is the set of homogeneous elements of R. Throughout this note, R is assumed to be graded by an abelian group G.

Definition 1. *Let R be a G-graded ring. An additive mapping $d : R \longrightarrow R$ is called homogeneous derivation if*

(i) $d(xy) = d(x)y + xd(y)$ *for all* $x,y \in R$.

(ii) $d(r) \in \mathcal{H}(R)$ *for all* $r \in \mathcal{H}(R)$.

Examples 1. *1) Let k be a field and $k[X]$ be a polynomial ring. Then $k[X]$ is $\mathbb{Z}$-graded by:*

$$K[X]_n := span_k\{X^n\} \;\; if \;\; n \geq 0 \;\; and \;\; k[X]_n := \{0_{k[X]}\} \;\; if \;\; n < 0.$$

Set $D := \frac{d}{dX}$ the usual derivation, then D is a homogeneous derivation.

2) Let $R = \begin{pmatrix} k & k \\ k & k \end{pmatrix}$ (with matrix addition and matrix multiplication), where k is a field. Then R is $\mathbb{Z}_4$-graded ring by:

$$R_0 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \;\; R_1 = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}, \quad R_2 = 0 \;\; and \;\; R_3 = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}.$$

Consider the mapping

$$\begin{array}{cccc} d: & R & \longrightarrow & R \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \end{array}$$

Then d is a homogeneous derivation of R.

3) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\}$. Then R is $\mathbb{Z}_2$-graded by

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, a,b \in \mathbb{R} \right\} \;\; and \;\; R_1 = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \,\middle|\, c \in \mathbb{R} \right\}.$$

Consider the following mappings:

$$\begin{array}{cccc} d_1: & R & \longrightarrow & R \\ & \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \end{array} \quad and \quad \begin{array}{cccc} d_2: & R & \longrightarrow & R \\ & \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & a+b-c \\ 0 & 0 \end{pmatrix} \end{array}.$$

Then d_1 and d_2 are homogeneous derivations of R.

4) *Consider* $R=\mathbb{C}[X]\times M_2(\mathbb{C})$. *R is a* $\mathbb{Z}$*-graded by*

$$R_0=\mathbb{C}\times\left\{\begin{pmatrix} z & 0\\ 0 & z'\end{pmatrix}\mid z,z'\in\mathbb{C}\right\},\quad R_1=span_{\mathbb{C}}(X)\times\left\{\begin{pmatrix} 0 & z\\ z' & 0\end{pmatrix}\mid z,z'\in\mathbb{C}\right\},$$

$R_n=\left\{\left(0,\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\right)\right\}$ *if* $n<0$ *and* $R_n=span_{\mathbb{C}}(X^n)\times\left\{\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\right\}$ *if* $n\geq 2$. *Define a map* $d:R\longrightarrow R$ *by:*

$$d\left(P,\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=\left(D(P),\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\right)\ \text{for all } P\in\mathbb{C}[X] \text{ and } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{C}),$$

where $D:=\frac{d}{dX}$ *the usual derivation of* $\mathbb{C}[X]$. *Then d is a homogeneous derivation of R.*

5) *Let* $R=\left\{\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\middle| a,b\in\mathbb{R}\right\}\times\mathbb{R}[X]$. *Then R is* $\mathbb{Z}$*-graded by*

$$R_0=\left\{\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}\middle| a\in\mathbb{R}\right\}\times\mathbb{R},\quad R_1=\left\{\begin{pmatrix} 0 & b\\ 0 & 0\end{pmatrix}\middle| b\in\mathbb{R}\right\}\times span_{\mathbb{R}}(X)$$

$R_n=\left\{\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\right\}\times span_{\mathbb{R}}(X^n)$ *if* $n\geq 2$ *and* $R_n=\left\{\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\right\}\times\left\{0_{\mathbb{R}[X]}\right\}$ *if* $n<0$.
Consider the following mappings:

$$\begin{array}{cccc} d_1: & R & \longrightarrow & R\\ & \left(\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix},P\right) & \longmapsto & \left(\begin{pmatrix} 0 & 5b\\ 0 & 0\end{pmatrix},\frac{dP}{dX}\right)\end{array}$$

and

$$\begin{array}{cccc} d_2: & R & \longrightarrow & R\\ & \left(\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix},P\right) & \longmapsto & \left(\begin{pmatrix} 0 & a+2b\\ 0 & 0\end{pmatrix},0_{\mathbb{R}[X]}\right)\end{array}$$

Then d_1 *and* d_2 *are homogeneous derivations of R.*

6) *Let* $R=M_2(\mathbb{C})\times M_2(\mathbb{C})$, *where* $\mathbb{C}$ *is the field of complex numbers. Then R is* $\mathbb{Z}_4$*-graded by*

$$R_0=\left\{\left(\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix},\begin{pmatrix} c & 0\\ 0 & d\end{pmatrix}\right)\mid a,b,c,d\in\mathbb{C}\right\},R_1=\left\{\left(\begin{pmatrix} 0 & 0\\ e & 0\end{pmatrix},\begin{pmatrix} 0 & 0\\ f & 0\end{pmatrix}\right)\mid e,f\in\mathbb{C}\right\},$$

$$R_3=\left\{\left(\begin{pmatrix} 0 & g\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & h\\ 0 & 0\end{pmatrix}\right)\mid g,h\in\mathbb{C}\right\}\ \text{and}\ R_2=\left\{\left(\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\right)\right\}.$$

Consider the following mappings:

$$\begin{array}{cccc} \delta_1: & R & \longrightarrow & R\\ & (x,y) & \longmapsto & (d_1(x),0)\end{array}\ \text{and}\ \begin{array}{cccc} \delta_2: & R & \longrightarrow & R\\ & (x,y) & \longmapsto & (0,d_2(y))\end{array}$$

where $d_1\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 0 & b\\ -c & 0\end{pmatrix}$ *and* $d_2\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 0 & -ib\\ ic & 0\end{pmatrix}$
for all $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{C})$.

7) *Let R be a G-graded ring. For each For each $r \in \mathcal{H}(R)$, the map $d_r := [r,.]$ is a homogeneous derivation of R.*

8) *If d_1 and d_2 are two homogeneous derivations of a G-graded ring R, then $[d_1, d_2]$ is also a homogeneous derivation of R.*

9) *Let R and S be graded rings, where R is G_1-graded and S is G_2-graded. Then the direct product $R \times S$ admits a natural $G_1 \times G_2$-grading. Moreover, if d_1 and d_2 are homogeneous derivations on R and S respectively, then the mapping (d_1, d_2) defined of $R \times S$ is also a homogeneous derivation. Conversely, every homogeneous derivation of $R \times S$ arises in this manner.*