

# GENERALIZED HOMOGENEOUS DERIVATIONS OF GRADED RINGS

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## Abstract

In this paper, we introduce a novel concept in graded rings called generalized homogeneous derivations, which serve as a natural generalization of the homogeneous derivations introduced by Kanunnikov. We provide an example demonstrating that the class of homogeneous derivations is strictly contained within the class of generalized homogeneous derivations. Furthermore, we extend several significant results, originally established for prime rings, to the context of gr-prime rings.

## 1. Introduction

The concept of generalized derivations, first introduced by Brešar [4], has been widely studied since the work of Hvala [5]. Many results about derivations have been extended to the context of generalized derivations (for example, see [8], [9]). In 2018, Kanunnikov [6] introduced the concept of homogeneous derivations for graded rings. Homogeneous derivations are derivations in the classical sense that are compatible with the graded structure of the ring. The purpose of this paper is to introduce a novel concept called generalized homogeneous derivations for graded rings, which extends the notion of homogeneous derivations to the context of generalized derivations while respecting the graded structure. We establish that homogeneous derivations are a special case of generalized homogeneous derivations, thereby providing a unifying framework for the study of derivations in graded rings. As a consequence of this new concept, we also extend several important results, originally proven for prime rings, to the setting of gr-prime rings.

Throughout this paper,  $R$  denotes an associative ring with the center  $Z(R)$ , and  $G$  is a group with identity  $e$ . For  $x, y \in R$ , the symbol  $[x, y]$  (resp.  $x \circ y$ ) stands for the Lie product  $xy - yx$  (resp.  $xy + yx$  for Jordan product). A ring  $R$  is  $G$ -graded if there is a family  $\{R_g, g \in G\}$  of additive subgroups  $R_g$  of  $(R, +)$  such that  $R = \bigoplus_{g \in G} R_g$  and  $R_g R_h \subseteq R_{gh}$  for every  $g, h \in G$ . The additive subgroup  $R_g$  is called the homogeneous component of  $R$ . The set  $\mathcal{H}(R) = \bigcup_{g \in G} R_g$  is the set of homogeneous elements of  $R$ . A nonzero element  $x \in R_g$  is said to be homogeneous of degree  $g$ . An element  $x \in R$  has a unique decomposition  $x = \sum_{g \in G} x_g$  with  $x_g \in R_g$ , where the sum is finite. The  $x_g$  terms are called the homogeneous components of element  $x$ . The  $G$ -graded ring  $R$  has a trivial grading if  $R_e = R$  and  $R_g = \{0\}$  for all  $g \in G \setminus \{e\}$ . In general, there are many different  $G$ -gradings of a fixed ring  $R$ . However, as is common in the literature, we will write a  $G$ -graded ring when we consider  $R$  together with some implicit  $G$ -grading.

Let  $R$  and  $T$  be  $G_1$ -graded and  $G_2$ -graded rings, respectively, then  $R \times T$  is  $G_1 \times G_2$ -graded

by:

$$(R \times T)_{(g,h)} := \{(a_g, b_h) \mid a_g \in R \text{ and } b_h \in T_h\} \text{ for } (g,h) \in G_1 \times G_2.$$

Let  $I$  be a right (resp. left) ideal of a  $G$ -graded ring  $R$ . Then  $I$  is said to be a graded right (resp. left) ideal if  $I = \bigoplus_{g \in G} I_g$ , where  $I_g = (I \cap R_g)$  for all  $g \in G$ . That is, for  $x \in I$ ,  $x = \sum_{g \in G} x_g$ , where  $x_g \in I$  for all  $g \in G$ .

A graded ring  $R$  is said to be gr-prime if  $aRb = \{0\}$  implies that either  $a = 0$  or  $b = 0$ , for any elements  $a, b \in \mathcal{H}(R)$ . As the examples of group rings show, there exist gr-prime rings that are not prime (see [7]).

An additive mapping  $d : R \longrightarrow R$  is a derivation of a ring  $R$  if  $d$  satisfies

$$d(xy) = d(x)y + xd(y) \quad (\text{Leibniz Formula})$$

for all  $x, y \in R$ . A derivation  $d$  is called an inner derivation if there exists  $a \in R$  such that  $d(x) = [a, x]$  for all  $x \in R$ .

An additive mapping  $F : R \longrightarrow R$  is called a generalized derivation if there exists a derivation  $d$  of  $R$  such that

$$F(xy) = F(x)y + xd(y)$$

for all  $x, y \in R$ . The map  $d$  is called an associated derivation of  $F$ .

To close this introduction, we briefly outline the contents of the paper. In Section 2, we recall graded analogues of some classical results and the definition of homogeneous derivations necessary for our study. In Section 3, we introduce generalized homogeneous derivations for graded rings. We establish that this class properly extends homogeneous derivations and also forms a proper subclass of generalized derivations in the presence of non-trivial gradings. In Section 4, we extend some important results for prime rings to gr-prime rings.

## 2. Preliminaries

In this section, we present some fundamental concepts and results relating to graded rings and homogeneous derivations, which we will need for the rest of this paper. Throughout this paper,  $R$  is graded by an abelian group  $G$ .

**Lemma 2.1.** [[2], Proposition 2.1] *Let  $R$  be a gr-prime ring.*

- (1) *If  $aRb = \{0\}$ , where  $a$  or  $b \in \mathcal{H}(R)$ , then  $a = 0$  or  $b = 0$ .*
- (2) *The centralizer of any nonzero graded one-sided ideal  $I$  is equal to the center of  $R$ . In particular, if  $R$  has a nonzero central graded ideal, then it is a commutative graded ring.*

Let us recall the definition of homogeneous derivation of graded rings.

**Definition 2.1.** [6] *Let  $R$  be a  $G$ -graded ring. An additive mapping  $d : R \longrightarrow R$  is called homogeneous derivation if*

- (i)  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$
- (ii)  $d(r) \in \mathcal{H}(R)$  for all  $r \in \mathcal{H}(R)$

In the following, we present some remarks regarding Definition 2.1.

**Remark 2.1.** Let  $R$  be a  $G$ -graded ring.

- (1) The sum of two homogeneous derivations is not necessarily a homogeneous derivation of  $R$ .
- (2) For any homogeneous element  $r \in \mathcal{H}(R)$ , the commutator map  $d_r := [r, \cdot]$  is a homogeneous derivation of  $R$ , called the inner homogeneous derivation associated with  $r$ .
- (3) We denote by  $\text{Der}_G^h(R)$  the set of all homogeneous derivations of  $R$ . This set is closed under the Lie product: if  $d_1$  and  $d_2$  are homogeneous derivations of  $R$ , then their Lie bracket  $[d_1, d_2]$  is also a homogeneous derivation of  $R$ .
- (4) Let  $S$  be a  $G'$ -graded ring. Then  $R \times S$  is a  $G \times G'$ -graded ring. Moreover, if  $d_1$  and  $d_2$  are homogeneous derivations of  $R$  and  $S$  respectively, then  $(d_1, d_2)$  is a homogeneous derivation of  $R \times S$ . Conversely, any homogeneous derivation  $D$  of  $R \times S$  can be written as  $D = (d_1, d_2)$ , where  $d_1 \in \text{Der}_G^h(R)$  and  $d_2 \in \text{Der}_{G'}^h(S)$ .

**Proposition 2.1.** [[2], Lemma 2.6] Let  $R$  be a gr-prime ring and  $I$  a nonzero graded one-side ideal of  $R$ . If  $d$  is a nonzero homogeneous derivation of  $R$ , then its restriction on  $I$  is nonzero.

### 3. Generalized homogeneous derivations

We begin this section by introducing generalized homogeneous derivations of graded rings.

**Definition 3.1.** Let  $R$  be a  $G$ -graded ring. An additive mapping  $F : R \rightarrow R$  is called a generalized homogeneous derivation if there exists a homogeneous derivation  $d : R \rightarrow R$  of  $R$  such that

- (i)  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in R$
- (ii)  $F(r) \in \mathcal{H}(R)$  for all  $r \in \mathcal{H}(R)$

The map  $d$  is called an associated homogeneous derivation of  $F$ .

**Notation.** We will denote the generalized homogeneous derivation  $F$  associated with homogeneous derivation  $d$  by  $(F, d)_h$ . The set of all generalized homogeneous derivations of  $R$  will be denoted by  $\text{Der}_G^{gh}(R)$ .

In the following, we present some illustrative examples of generalized homogeneous derivations of graded rings.

**Example 3.1.** (1) Let  $R = \left\{ \begin{pmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\} \times \mathbb{C}[X]$ . Then  $R$  is  $\mathbb{Z}^2$ -graded by

$$R_{(0,0)} = \left\{ \begin{pmatrix} 0 & z_1 & 0 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix} \mid z_1, z_2 \in \mathbb{C} \right\} \times \mathbb{C}, \quad R_{(1,1)} = \left\{ \begin{pmatrix} 0 & 0 & z_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid z_3 \in \mathbb{C} \right\} \times \text{span}_{\mathbb{C}}(X),$$

$$R_{(n,n)} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \times \text{span}_{\mathbb{C}}(X^n) \text{ if } n \geq 2 \text{ and } R_{(n,n)} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \times \{0_{\mathbb{C}[X]}\} \text{ if } n < 0.$$

Let  $F$  be a mapping defined by:

$$F : \begin{matrix} R \\ \left( \begin{pmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{pmatrix}, P \right) \end{matrix} \longrightarrow \begin{matrix} R \\ \left( \begin{pmatrix} 0 & 0 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{pmatrix}, i \frac{dP}{dX} \right) \end{matrix}$$

Then  $F$  is a generalized homogeneous derivation associated with homogeneous derivation  $d$  defined by:

$$d : \begin{matrix} R \\ \left( \begin{pmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{pmatrix}, P \right) \end{matrix} \longrightarrow \begin{matrix} R \\ \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{pmatrix}, i \frac{dP}{dX} \right) \end{matrix}$$

(2) Consider the ring  $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$ . Then  $R$  is  $\mathbb{Z}_4$ -graded by

$$R_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z} \right\}, \quad R_1 = \left\{ \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}, \quad R_2 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid c \in \mathbb{Z} \right\}$$

$$\text{and } R_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid d \in \mathbb{Z} \right\}.$$

Let  $F$  be a mapping defined as follow:

$$F : \begin{matrix} R \\ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} \end{matrix} \longrightarrow \begin{matrix} R \\ \begin{pmatrix} 0 & a & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

Then  $F$  is a generalized homogeneous derivation associated with homogeneous derivation  $d$  defined by:

$$d : \begin{matrix} R \\ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} \end{matrix} \longrightarrow \begin{matrix} R \\ \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & -d \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}.$$

We now present several important remarks concerning Definition 3.1.

**Remark 3.1.** Let  $R$  be a  $G$ -graded ring,  $(F_1, d_1)_h$  and  $(F_2, d_2)_h$  be generalized homogeneous derivations of  $R$ . Then:

- (1) The sum  $F_1 + F_2$  is not necessarily a generalized homogeneous derivation of  $R$ .
- (2) For each  $r \in Z(R) \cap \mathcal{H}(R)$ ,  $rF_1$  is a generalized homogeneous derivation associated with the homogeneous derivation  $rd_1$ .

- (3) *There exists a nonzero generalized homogeneous derivation associated with the zero homogeneous derivation.*

The following example illustrates the key aspects discussed in the preceding remark and provides additional clarity.

**Example 3.2.** (1) *Let us consider the ring  $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$ . Then  $R$  is  $\mathbb{Z}_4$ -graded by*

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}, \quad R_2 = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{R} \right\} \quad \text{and} \quad R_1 = R_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

*Let  $F_1$  be a mapping given by:*

$$F_1 : \begin{matrix} R & \longrightarrow & R \\ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \end{matrix}$$

*Then  $F_1$  is a generalized homogeneous derivation associated with homogeneous derivation  $d_1$  defined by:*

$$d_1 : \begin{matrix} R & \longrightarrow & R \\ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \end{matrix}$$

*If  $r \notin \mathcal{H}(R) \cap Z(R)$ , then  $rF_1$  is not necessarily homogeneous. Indeed, for  $r = \begin{pmatrix} 2 & 9 \\ 0 & 2 \end{pmatrix}$  we have*

$$r \notin Z(R) \text{ and } rF_1 \begin{pmatrix} 2 & 0 \\ 0 & -7 \end{pmatrix} = \begin{pmatrix} 0 & -63 \\ 0 & -14 \end{pmatrix} \notin \mathcal{H}(R).$$

(2) *Let  $R$  be a  $G$ -graded ring. For each  $r_1, r_2 \in Z(R) \cap \mathcal{H}(R)$  such that  $\deg(r_1) = \deg(r_2)$ . Define  $F_{r_1, r_2} : R \longrightarrow R$  such that  $F_{r_1, r_2}(x) = r_1 x + x r_2$  for all  $x \in R$ . Then  $F_{r_1, r_2}$  is a generalized homogeneous derivation associated with homogeneous derivation  $d_{r_2} : R \longrightarrow R$  defined by  $d_{r_2}(x) = [r_2, x] = 0$ .*

**Proposition 3.1.** *Let  $R$  be a gr-prime ring and  $(F, d)_h$  a generalized homogeneous derivation of  $R$ . If  $d \neq 0$ , then  $F \neq 0$ .*

*Proof.* Assume  $F = 0$ . For any elements  $x, y \in R$ ,  $F(xy) = 0$ . Since  $F(xy) = F(x)y + xd(y)$ , we obtain  $xd(y) = 0$  for all  $x, y \in R$ . This means  $xRd(y) = \{0\}$  for all  $x, y \in R$ . In particular, for some nonzero homogeneous element  $r \in \mathcal{H}(R) \setminus \{0\}$ , we have  $rRd(y) = \{0\}$  for all  $y \in R$ . According to Lemma 2.1, we get  $d(y) = 0$  for all  $y \in R$ . Thus,  $d = 0$ .  $\square$

**Remark 3.2.** *Every homogeneous derivation is a generalized homogeneous derivation. However, the converse does not hold in general.*

The following example demonstrates that the class of generalized homogeneous derivations is a proper extension of the class of homogeneous derivations.

**Example 3.3.** Let us consider the ring  $R = \left\{ \begin{pmatrix} z_1 & z_2 \\ 0 & z_3 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$ . Then  $R$  is  $\mathbb{Z}_4$ -graded by

$$R_0 = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \mid z_1, z_2 \in \mathbb{C} \right\}, \quad R_2 = \left\{ \begin{pmatrix} 0 & z_3 \\ 0 & 0 \end{pmatrix} \mid z_3 \in \mathbb{C} \right\} \quad \text{and} \quad R_1 = R_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Let  $F$  be a mapping given by:

$$F : \begin{matrix} R & \longrightarrow & R \\ \begin{pmatrix} z_1 & z_2 \\ 0 & z_3 \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & iz_2 \\ 0 & z_3 \end{pmatrix} \end{matrix}$$

Then  $F$  is a generalized homogeneous derivation associated with homogeneous derivation  $d$  defined by:

$$d : \begin{matrix} R & \longrightarrow & R \\ \begin{pmatrix} z_1 & z_2 \\ 0 & z_3 \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & iz_2 \\ 0 & 0 \end{pmatrix} \end{matrix}$$

However,  $F$  is not homogeneous derivation of  $R$ .

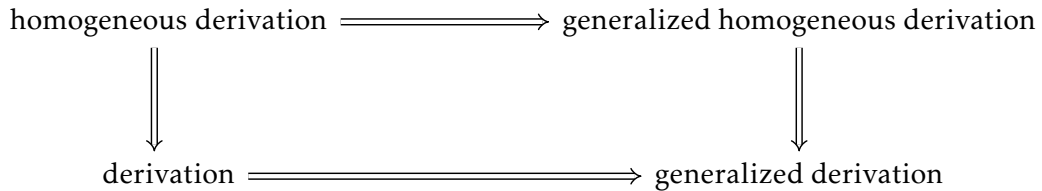
**Remark 3.3.** Every generalized homogeneous derivation is a generalized derivation. The converse, however, is false, as the following example shows.

**Example 3.4.** Let us consider  $R = M_2(\mathbb{R})$ . The ring  $R$  is  $\mathbb{Z}_4$ -graded by

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}, \quad R_3 = \left\{ \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \mid c, d \in \mathbb{R} \right\} \quad \text{and} \quad R_1 = R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Let  $x = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \in R$ . Define  $F : R \longrightarrow R$  by  $F = d_x + \psi$ , where  $d_x(y) = [x, y]$  for all  $y \in R$  and  $\psi(y) = xy$ . Then  $(F, d_x)$  is a generalized derivation of  $R$ . However,  $F$  is not a generalized homogeneous derivation. Indeed, for  $r = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{H}(R)$ , we have  $F(r) = \begin{pmatrix} 2 & 0 \\ 9 & 1 \end{pmatrix} \notin \mathcal{H}(R)$ .

We conclude this section with the following diagram.



#### 4. Some commutativity criteria involving generalized homogeneous derivations on gr-prime rings

In [3], it was shown that a prime ring  $R$  with a nonzero ideal  $I$  is commutative if it admits a generalized derivation  $F$  satisfying either

$$F(xy) \pm xy \in Z(R) \text{ or } F(x)F(y) \pm xy \in Z(R)$$

for all  $x, y \in I$ . We extend this result to gr-prime rings in the context of generalized homogeneous derivations.

**Theorem 4.1.** *Let  $R$  be a gr-prime ring and  $I$  a nonzero graded ideal of  $R$ . If  $R$  admits a generalized homogeneous derivation  $F$  associated with a nonzero homogeneous derivation  $d$  such that*

$$F(xy) \pm xy \in Z(R)$$

*for all  $x, y \in I$ , then  $R$  is a commutative graded ring.*

*Proof.* Consider the case

$$F(xy) - xy \in Z(R)$$

for all  $x, y \in I$ . The following identity is derived using the same reasoning as in the proof of [3, Theorem 2.1]:

$$[z, z_1]xyd(z) = 0$$

for all  $x, y, z, z_1 \in I$ , which obviously gives

$$[z, z_1]xRI d(z) = \{0\}$$

for all  $x, z, z_1 \in I$ . Since  $I$  is a graded ideal of  $R$ , then we have

$$[r', z_1]xRrd(r') = \{0\}$$

for all  $x, z_1 \in I$  and  $r, r' \in I \cap \mathcal{H}(R)$ . According to Lemma 2.1, we have either that

$$[z_1, r']x = 0 \text{ or } rd(r') = 0$$

for all  $x, z_1 \in I$  and  $r, r' \in I \cap \mathcal{H}(R)$ . This implies that

$$[z, z_1]I = \{0\} \text{ or } Id(z) = \{0\}$$

for all  $z, z_1 \in I$ .

Let us define  $I_1 = \{z \in I \mid [z, z_1]I = \{0\}, \text{ for all } z_1 \in I\}$  and  $I_2 = \{z \in I \mid Id(z) = \{0\}\}$ . Then  $I_1$  and  $I_2$  are additive subgroups of  $I$  with  $I = I_1 \cup I_2$ . Since a group cannot be expressed as the union of two proper subgroups, either  $I_1 = I$  or  $I_2 = I$ . We consider these cases separately.

*Case 1:* If  $[z_1, z]I = \{0\}$  for all  $z, z_1 \in I$ . Since  $I$  is an ideal, we get  $[z_1, z]RI = \{0\}$  for all  $z, z_1 \in I$ . Since  $I$  is a nonzero graded ideal,  $[z, z_1]Rr = \{0\}$  for all  $z, z_1 \in I$  and for some  $r \in I \cap \mathcal{H}(R) \setminus \{0\}$ . In light of Lemma 2.1,  $[z, z_1] = 0$  for all  $z, z_1 \in I$ . Hence  $I$  is a commutative graded ideal. Therefore,  $R$  is a commutative graded ring.

*Case 2:* If  $Id(z) = \{0\}$  for all  $z \in I$ , then  $IRd(z) = \{0\}$  for all  $z \in I$ . In particular,  $rRd(z) = \{0\}$  for

some  $r \in I \cap \mathcal{H}(R) \setminus \{0\}$ . Using Lemma 2.1, we get  $d(z) = 0$  for all  $z \in I$ . Hence  $d$  vanishes on  $I$ . In view of Proposition 2.1,  $d$  is zero on  $R$ , a contradiction. Further, in the end the second case

$$F(xy) + xy \in Z(R)$$

for all  $x, y \in I$  can be reduced to the first one considering  $-F$  instead of  $F$ .  $\square$

Next, we extend [3, Theorem 2.5] to gr-prime rings by considering a pair of generalized homogeneous derivations  $F_1$  and  $F_2$  satisfying

$$F_1(x)F_2(y) \pm xy \in Z(R)$$

for all  $x, y$  in a graded ideal  $I$  of  $R$ .

**Theorem 4.2.** *Let  $R$  be a gr-prime ring and  $I$  be a nonzero graded ideal of  $R$ . If  $R$  admits two generalized homogeneous derivations  $F_1$  and  $F_2$  associated with nonzero homogeneous derivations  $d_1$  and  $d_2$  respectively, such that*

$$F_1(x)F_2(y) \pm xy \in Z(R)$$

*for all  $x, y \in I$ , then  $R$  is a commutative graded ring.*

*Proof.* Consider the case

$$F_1(x)F_2(y) - xy \in Z(R) \tag{4.1}$$

for all  $x, y \in I$ . Writing  $yz$  instead  $y$  in (4.1), we get

$$(F_1(x)F_2(y) - xy)z + F_1(x)y d_2(z) \in Z(R) \tag{4.2}$$

for all  $x, y \in I$  and  $z \in R$ . Commuting (4.2) with  $z$ , we obtain

$$F_1(x)[y d_2(z), z] + [F_1(x), z]y d_2(z) = 0 \tag{4.3}$$

for all  $x, y \in I$  and  $z \in R$ . Putting  $y = F_1(x)y$  in (4.3), we arrive at

$$[F_1(x), z]F_1(x)y d_2(z) = 0 \tag{4.4}$$

for all  $x, y \in I$  and  $z \in R$ . This implies that

$$[F_1(x), z]F_1(x)RId_2(z) = \{0\}$$

for all  $x \in I$  and  $z \in R$ . In particular,

$$[F_1(r), r']F_1(r)RId_2(r') = \{0\}$$

for all  $r \in I \cap \mathcal{H}(R)$  and  $r' \in \mathcal{H}(R)$ . According to Lemma 2.1, we have that either

$$[F_1(r), r']F_1(r) = 0 \text{ or } Id_2(r') = \{0\}$$



for all  $r \in I \cap \mathcal{H}(R)$  and  $r' \in \mathcal{H}(R)$ . So,  $[F_1(x), z]F_1(x) = 0$  or  $Id_2(z) = \{0\}$  for all  $x \in I$  and  $z \in R$ . Set  $J_1 = \{z \in R \mid [F_1(x), z]F_1(x) = \{0\}, \text{ for all } x \in I\}$  and  $J_2 = \{z \in R \mid Id_2(z) = \{0\}\}$ . Clearly,  $J_1$  and  $J_2$  are additive subgroups of  $R$  whose union is  $R$ . Since a group cannot be a union of two of its proper subgroups, we are forced to conclude that either  $J_1 = R$  or  $J_2 = R$ . Now, if  $J_2 = R$ , then  $Id_2(z) = \{0\}$  for all  $z \in R$ . Since,  $I$  is an ideal then  $IRd_2(z) = \{0\}$  for all  $z \in R$ . In particular,  $rRd_2(z) = \{0\}$  for all  $z \in R$  and for some  $r \in I \cap \mathcal{H}(R) \setminus \{0\}$ . According to Lemma 2.1, we conclude that  $d_2(z) = 0$  for all  $z \in R$ . Hence  $d_2 = 0$  which is a contradiction. Therefore, we have  $[F_1(x), z]F_1(x) = 0$  for all  $x \in I$  and  $z \in R$ . Replacing  $z$  by  $zz'$ , we get

$$[F_1(x), z]z'F_1(x) = 0$$

for all  $x \in I$  and  $z, z' \in R$ , which implies that

$$[F_1(x), z]RF_1(x) = \{0\}$$

for all  $x \in I$  and  $z \in R$ . In particular,

$$[F_1(r), z]RF_1(r) = \{0\}$$

for all  $r \in I \cap \mathcal{H}(R)$  and  $z \in R$ . Invoking Lemma 2.1, we conclude that either

$$F_1(r) = 0 \text{ or } [F_1(r), z] = 0$$

Hence  $F_1(x) = 0$  or  $[F_1(x), z] = 0$  for all  $x \in I$  and  $z \in R$ . In both cases we have  $[F_1(x), z] = 0$  for all  $x \in I$  and  $z \in R$ . Replacing  $x$  by  $xz$ , we get

$$x[d_1(z), z] + [x, z]d_1(z) = 0 \quad (4.5)$$

for all  $x \in I$  and  $z \in R$ . Writing  $sx$  instead of  $x$  in (4.5) and using it, we arrive at

$$[s, z]xd_1(z) = 0$$

for all  $x \in I$  and  $s, z \in R$ , which implies that

$$[s, z]RI d_1(z) = \{0\}$$

for all  $s, z \in R$ . Using similar arguments as above, either  $[s, z] = 0$  or  $d_1(z) = 0$  for all  $s, z \in R$ . Since  $d_1 \neq 0$ , we have  $[s, z] = 0$  for all  $s, z \in R$ . Therefore,  $R$  is a commutative graded ring. Further, in the end the second case

$$F_1(x)F_2(y) + xy \in Z(R)$$

for all  $x, y \in I$  can be reduced to the first one considering  $-F_1$  instead of  $F$ .  $\square$

In the following example, we show that the gr-primeness hypothesis on  $R$  cannot be omitted in the above theorems.

**Example 4.1.** Let us consider the ring  $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \times \mathbb{R}[X]$ . Then  $R$  is  $\mathbb{Z}^2$ -graded by

$$R_{(0,0)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\} \times \mathbb{R}, \quad R_{(1,1)} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{R} \right\} \times \text{span}_{\mathbb{R}}(X)$$

$$R_{(n,n)} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \times \left\{ 0_{\mathbb{R}[X]} \right\} \text{ if } n < 0 \text{ and } R_{(n,n)} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \times \text{span}_{\mathbb{R}}(X^n)$$

for all  $n \geq 2$ .

$R$  is not a gr-prime ring. Let  $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\} \times \langle X^3 \rangle$ . Obviously,  $I$  is a nonzero graded ideal of  $R$ . Consider the following mappings:

$$F_1 : \begin{pmatrix} R \\ \left( \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, P \right) \end{pmatrix} \longrightarrow \begin{pmatrix} R \\ \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \frac{dP}{dX} \right) \end{pmatrix} \quad \text{and} \quad d_1 : \begin{pmatrix} R \\ \left( \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, P \right) \end{pmatrix} \longrightarrow \begin{pmatrix} R \\ \left( \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}, \frac{dP}{dX} \right) \end{pmatrix}$$

and

$$F_2 : \begin{pmatrix} R \\ \left( \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, P \right) \end{pmatrix} \longrightarrow \begin{pmatrix} R \\ \left( \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, 0_{\mathbb{R}[X]} \right) \end{pmatrix}$$

Then  $(F_1, d_1)_h$  and  $(F_2, d_2)_h$  with  $d_2 = F_2$  are generalized homogeneous derivations of  $R$ . Moreover,  $F_1(xy) \pm xy \in Z(R)$  and  $F_1(x)F_2(y) \pm xy \in Z(R)$  for all  $x, y \in I$ . However,  $R$  is a non-commutative graded ring.

At the conclusion of this paper, we present several open problems that remain unresolved, offering potential avenues for future research.

**Problem 1.** Let  $R$  be a gr-prime ring and  $I$  a nonzero graded ideal of  $R$ , let  $(F_1, d_1)_h$  and  $(F_2, d_2)_h$  be generalized homogeneous derivations of  $R$  satisfies one of the following:

- (i)  $[F_1(x), x] \in Z(R)$
- ii)  $[F_1(x), F_2(y)] \pm xy \in Z(R)$
- (iii)  $F_1(x)F_2(y) \pm [x, y] \in Z(R)$

for all  $x, y \in I$ . In light of this, what conclusions can be drawn with regard to the commutativity of  $R$ ?

**Problem 2.** Let  $R$  be a gr-prime ring of characteristic different from 2 and  $I$  a nonzero graded ideal of  $R$ . If  $R$  admits a generalized homogeneous derivations  $F_1$  and  $F_2$  associated with homogeneous derivations  $d_1$  and  $d_2$  respectively, such that

$$F_1(x)F_2(y) \pm x \circ y \in Z(R)$$

for all  $x, y \in I$ . What is the conclusion?

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