JBMO Shortlist 2023

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by Orestis_Lignos

A1 Prove that for all positive real numbers a, b, c, d,

$$\frac{2}{(a+b)(c+d)+(b+c)(a+d)} \leq \frac{1}{(a+c)(b+d)+4ac} + \frac{1}{(a+c)(b+d)+4bd}$$

and determine when equality occurs.

A2 For positive real numbers x, y, z with $xy + yz + zx = 1$, prove that

$$\frac{2}{xyz} + 9xyz \geq 7(x+y+z)$$

A3 Prove that for all non-negative real numbers x, y, z, not all equal to 0, the following inequality holds

$$\frac{2x^2-x+y+z}{x+y^2+z^2} + \frac{2y^2+x-y+z}{x^2+y+z^2} + \frac{2z^2+x+y-z}{x^2+y^2+z} \geq 3.$$

Determine all the triples (x, y, z) for which the equality holds.

Milan Mitreski, Serbia

A4 Let a, b, c, d be positive real numbers with $abcd = 1$. Prove that

$$\sqrt{\frac{a}{b+c+d^2+a^3}} + \sqrt{\frac{b}{c+d+a^2+b^3}} + \sqrt{\frac{c}{d+a+b^2+c^3}} + \sqrt{\frac{d}{a+b+c^2+d^3}} \leq 2$$

A5 Let $a \geq b \geq 1 \geq c \geq 0$ be real numbers such that $a + b + c = 3$. Show that

$$3\left(\frac{a}{b} + \frac{b}{a}\right) \geq 4c^2 + \frac{a^2}{b} + \frac{b^2}{a}$$

A6 Find the maximum constant C such that, whenever $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers satisfying $a_{n+1} - a_n = a_n(a_n+1)(a_n+2)$, we have

$$\frac{a_{2023} - a_{2020}}{a_{2022} - a_{2021}} > C.$$

A7 Let $a_1, a_2, a_3, \ldots, a_{250}$ be real numbers such that $a_1 = 2$ and

$$a_{n+1} = a_n + \frac{1}{a_n^2}$$

for every $n = 1, 2, \ldots, 249$. Let x be the greatest integer which is less than

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{250}}$$

How many digits does x have?

Proposed by Miroslav Marinov, Bulgaria

C1 Given is a square board with dimensions 2023×2023, in which each unit cell is colored blue or red. There are exactly 1012 rows in which the majority of cells are blue, and exactly 1012 columns in which the majority of cells are red.

What is the maximal possible side length of the largest monochromatic square?

C2 There are n blocks placed on the unit squares of a $n \times n$ chessboard such that there is exactly one block in each row and each column. Find the maximum value k, in terms of n, such that however the blocks are arranged, we can place k rooks on the board without any two of them threatening each other.

(Two rooks are not threatening each other if there is a block lying between them.)

C3 Alice and Bob play the following game on a 100×100 grid, taking turns, with Alice starting first. Initially the grid is empty. At their turn, they choose an integer from 1 to 100^2 that is not written yet in any of the cells and choose an empty cell, and place it in the chosen cell. When there is no empty cell left, Alice computes the sum of the numbers in each row, and her score is the maximum of these 100 numbers. Bob computes the sum of the numbers in each column, and his score is the maximum of these 100 numbers. Alice wins if her score is greater than Bob's score, Bob wins if his score is greater than Alice's score, otherwise no one wins.

Find if one of the players has a winning strategy, and if so which player has a winning strategy.

Théo Lenoir, France

C4 Anna and Bob are playing the following game: The number 2 is initially written on the blackboard. With Anna playing first, they alternately double the number currently written on the blackboard or square it.

The person who first writes on the blackboard a number greater than 2023^{10} is the winner. Determine which player has a winning strategy.

C5 Consider an increasing sequence of real numbers $a_1 < a_2 < \ldots < a_{2023}$ such that all pairwise sums of the elements in the sequence are different. For such a sequence, denote by M the number of pairs (a_i, a_j) such that $a_i < a_j$ and $a_i + a_j < a_2 + a_{2022}$. Find the minimal and the maximal possible value of M.

G1 Let ABC be a triangle with circumcentre O and circumcircle Ω. Γ is the circle passing through O, B and tangent to AB at B. Let Γ intersect Ω a second time at $P \neq B$. The circle passing through P, C and tangent to AC at C intersects with Γ at M. Prove that $|MP| = |MC|$.

G2 Let ABC be a triangle with $AB < AC$ and ω be its circumcircle. The tangent line to ω at A intersects line BC at D and let E be a point on ω such that BE is parallel to AD. DE intersects segment AB and ω at F and G, respectively. The circumcircle of BGF intersects BE at N. The line NF intersects lines AD and EA at S and T, respectively. Prove that $DGST$ is cyclic.

G3 Let A, B, C, D and E be five points lying in this order on a circle, such that $AD = BC$. The lines AD and BC meet at a point F. The circumcircles of the triangles CEF and ABF meet again at the point P.

Prove that the circumcircles of triangles BDF and BEP are tangent to each other.

G4 Let $ABCD$ be a cyclic quadrilateral, for which B and C are acute angles. M and N are the projections of the vertex B on the lines AC and AD, respectively, P and T are the projections of the vertex D on the lines AB and AC respectively, Q and S are the intersections of the pairs of lines MN and CD, and PT and BC, respectively. Prove the following statements:

a) $NS \parallel PQ \parallel AC$;
b) $NP = SQ$;
c) $NPQS$ is a rectangle if, and only if, AC is a diamteter of the circumscribed circle of quadrilateral $ABCD$.

G5 Let D, E, F be the points of tangency of the incircle of a given triangle ABC with sides BC, CA, AB, respectively. Denote by I the incenter of ABC, by M the midpoint of BC and by G the foot of the perpendicular from M to line EF. Prove that the line ID is tangent to the circumcircle of the triangle MGI.

G6 Let ABC be an acute triangle with circumcenter O. Let D be the foot of the altitude from A to BC and let M be the midpoint of OD. The points O_b and O_c are the circumcenters of triangles

AOC and AOB, respectively. If $AO = AD$, prove that points A, O_b, M and O_c are concyclic.

Marin Hristov and Bozhidar Dimitrov, Bulgaria

G7 Let D and E be arbitrary points on the sides BC and AC of triangle ABC, respectively. The circumcircle of $\triangle ADC$ meets for the second time the circumcircle of $\triangle BCE$ at point F. Line FE meets line AD at point G, while line FD meets line BE at point H. Prove that lines CF, AH and BG pass through the same point.

N1 Find all pairs (a, b) of positive integers such that $a! + b$ and $b! + a$ are both powers of 5.

Nikola Velov, North Macedonia

N2 A positive integer is called *Tiranian* if it can be written as $x^2 + 6xy + y^2$, where x and y are (not necessarily distinct) positive integers. The integer 36^{2023} is written as the sum of k Tiranian integers. What is the smallest possible value of k?

Proposed by Miroslav Marinov, Bulgaria

N3 Let A be a subset of $\{2, 3, \ldots, 28\}$ such that if $a \in A$, then the residue obtained when we divide a^2 by 29 also belongs to A.

Find the minimum possible value of $|A|$.

N4 The triangle ABC is sectioned by AD, BE and CF (where $D \in (BC), E \in (CA)$ and $F \in (AB)$) in seven disjoint polygons named *regions*. In each one of the nine vertices of these regions we write a digit, such that each nonzero digit appears exactly once. We assign to each side of a region the lowest common multiple of the digits at its ends, and to each region the greatest common divisor of the numbers assigned to its sides.

Find the largest possible value of the product of the numbers assigned to the regions.

N5 Find the largest positive integer k such that we can find a set $A \subseteq \{1, 2, \ldots, 100\}$ with k elements such that, for any $a, b \in A$, a divides b if and only if $s(a)$ divides $s(b)$, where $s(k)$ denotes the sum of the digits of k.

N6 **Version 1.** Find all primes p satisfying the following conditions:

(i) $\frac{p+1}{2}$ is a prime number.
(ii) There are at least three distinct positive integers n for which $\frac{p^2+n}{p+n^2}$ is an integer.

Version 2. Let $p \neq 5$ be a prime number such that $\frac{p+1}{2}$ is also a prime. Suppose there exist positive integers $a < b$ such that $\frac{p^2+a}{p+a^2}$ and $\frac{p^2+b}{p+b^2}$ are integers. Show that $b = (a-1)^2 + 1$.
