Compilation of Combinatorics Problems

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June 7, 2022

Contents

§1 PRMO/IOQM Problems

Problem (PRMO 2019 P15)

In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected.

Problem (PRMO 2021 Part-B P2)

Find all natural numbers n for which there exists a permutation σ of $1, 2, \cdots, 1$ such that

$$\sum_{i=0}^{n} \sigma(i)(-2)^{i-1} = 0$$

Note: A permutation of $1, 2, , n$ is a bijective function from $1, 2, \cdots, n$ to itself.

§2 RMO Questions

Problem (RMO 2015)

Suppose 28 objects are placed around a circle at equal distances. In how many ways I can choose 3 objects from among them so that no 2 of the 3 chosen are adjacent nor diametrically opposite?

§3 AMC Problems

Problem (AMC 10A, 2020)

A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N?

Problem (AMC 10A, 2020)

A positive integer divisor of 12! is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as

$$\frac{m}{n}$$

where m and n are relatively prime positive integers. What is m+n?

Problem (AMC 10A, 2019)
For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

Problem (AMC 10A, 2019)
A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

Problem (AMC 10A, 2019)
The numbers 1,2,...,9 are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

Problem (AMC 10A, 2018)
How many ways can a student schedule 3 mathematics courses – algebra, geometry, and number theory – in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

Problem
AMC 10B, 2002] Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

Problem (AMC 12B, 2002)
How many different integers can be expressed as the sum of three distinct members of the set 1,4,7,10,13,16,19?

Problem (AMC 8, 2019)
Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

Problem (AMC 10B, 2020)

How many distinguishable arrangements are there of 1 brown tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable.)

Problem (AMC 10A, 2006)

How many four-digit positive integers have at least one digit that is a 2 or a 3?

Problem (AMC 10B, 2017)

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

Problem (AMC 10A, 2004)

Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

Problem (AMC 12A, 2010)

A 16-step path is to go from (-4, -4) to (4,4) with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square

$$-2 \leq x \leq 2, -2 \leq y \leq 2$$

at each step?

Problem (AMC 10A, 2016)

For some particular value of n, when

$$(a+b+c+d+1)^n$$

is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c, and d, each to some positive power. What is n?

Problem (AMC 10A, 2006)
Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

Problem (AMC 10/12 Mock, 2013)
A regular 2013-gon is placed in the plane in general position. A family of lines is drawn on this plane such that no two lines are parallel, and for every vertex A of the 2013-gon, there exists a line that passes through A. What is the smallest number of possible lines in the family?

§4 AIME Problems

Problem (AIME, 2009)

A game show offers a contestant three prizes A B and C each of which is worth a whole number of dollars from 1to9999 inclusive. The contestant wins the prizes by correctly guessing the price of each prize in the order A B and C. As a hint the digits of three prizes are given. On a particular day the digits given were 1,1,1,1,3,3,3. Find the total number of possible guesses for all three prizes consistent with the hint.

1. 110
2. 420
3. 430
4. 111

Problem (AIME I, 2015)

In a drawer Sandy has 5 pairs of socks, each pair a different color. on monday sandy selects two individual socks at random from the 10 socks in the drawer. On tuesday Sandy selects 2 of the remaining 8 socks at random and on wednesday two of the remaining 6 socks at random. The probability that wednesday is the first day Sandy selects matching socks is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Problem (AIME, 2018)

Find the number of functions f from $0, 1, 2, 3, 4, 5, 6$ to the integers such that $f(0) = 0$, $f(6) = 12$, and

$$|xy| \leq |f(x)f(y)| \leq 3|xy|$$

$\forall x, y \in 0, 1, 2, 3, 4, 5, 6$.

Problem (AIME, 2006)

Let $(a_1, a_2, a_3, \cdots, a_{12})$ be a permutation of $(1, 2, 3, \cdots, 12)$ for which $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and $a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}$.

An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.

§5 National Olympiads

Problem (Australian Math Olympiad P4, 2022)

Let S be the set of points (i,j) in the plane with

$$i, j \in \{1, 2, \ldots, 2022\}$$

and i>j. We would like to colour each point in S either red or blue, so that whenever points (i,j) and (j,k) have the same colour, the point (i,k) also has that colour.How many such colorings of S are there.

Problem (Australian Math Olympiad P6, 2022)

In the country of Biplania there are two airline companies. Redwing and Blueways. There are 20 cities in Biplania, including, Abadu and Ethora. Between any pair of cities, exactly one of the two airlines has direct flights and that airline flies in both directions between the two cities. Arthur only flies on Redwing. He can travel from Abadu to Ethora in exactly four flights but not fewer. Martha only flies on Blueways.Show that Martha can travel between any two cities in Biplania in at most two flights.

Problem (China TST, P6, 2020)

Let m be a positive integer, and $A_1, A_2, \ldots, A_m$ (not necessarily different) be m subsets of a finite set A. It is known that for any nonempty subset I of $\{1, 2 \ldots, m\}$,

$$\left| \bigcup_{i \in I} A_i \right| \geq |I| + 1.$$

Show that the elements of A can be colored black and white, so that each of $A_1, A_2, \ldots, A_m$ contains both black and white elements.

Problem (China TST, P1 Test 4, 2022)

Initially, each unit square of an $n \times n$ grid is colored red, yellow or blue. In each round, perform the following operation for every unit square simultaneously:

1. For a red square, if there is a yellow square that has a common edge with it, then color it yellow.
2. For a yellow square, if there is a blue square that has a common edge with it, then color it blue.
3. For a blue square, if there is a red square that has a common edge with it, then color it red.

It is known that after several rounds, all unit squares of this $n \times n$ grid have the same color. Prove that the grid has became monochromatic no later than the end of the $(2n-2)$-th round.

Problem (Argentina MO, National level 1 2021 P2)

On each OMA lottery ticket there is a 9-digit number that only uses the digits $1, 2$ and 3 (not necessarily all three). Each ticket has one of the three colors red, blue or green. It is known that if two banknotes do not match in any of the 9 figures, then they are of different colors. Bill 122222222 is red, 222222222 is green, what color is bill 123123123?

Problem (Iran 2022 round 2, P3)

Take a $n \times n$ chess page.Determine the n such that we can put the numbers $1, 2, 3, \ldots, n$ in the squares of the page such that we know the following two conditions are true:

1. For each row we know all the numbers $1, 2, 3, \ldots, n$ have appeared on it and the numbers that are in the black squares of that row have the same sum as the sum of the numbers in the white squares of that row.
2. For each column we know all the numbers $1, 2, 3, \ldots, n$ have appeared on it and the numbers that are in the black squares in that column have the same sum as the sum of the numbers in the white squares of that column.

Problem (USAMO 2019, P4)

Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, for integers i, j with $0 \leq i, j \leq n$, such that:

1. for all $0 \leq i, j \leq n$, the set $S_{i,j}$ has $i + j$ elements; and

2. $S_{i,j} \subseteq S_{k,l}$ whenever $0 \leq i \leq k \leq n$ and $0 \leq j \leq l \leq n$.

Problem (Japan MO 1997)

Prove that among any ten points located inside a circle with diameter 5, there exist at least two at a distance less than 2 from each other.

Problem (USAMTS Year 18 - Round 1 - Problem 4)

Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.

Problem (JBMOSL 2019 C4)

We have a group of n kids. For each pair of kids, at least one has sent a message to the other one. For each kid A, among the kids to whom A has sent a message, exactly 25% have sent a message to A. How many possible two-digit values of n are there?

Problem (Iran MO 2nd round 2022 , P4)

There is an $n * n$ table with some unit cells colored black and the others are white. In each step , Amin takes a *row* with exactly one black cell in it , and color all cells in that black cell's *column* red. While Ali , takes a *column* with exactly one black cell in it , and color all cells in that black cell's *row* red. Prove that Amin can color all the cells red , iff Ali can do so.

Problem (Romania JBMO TST 2022)

Find how many positive integers $k \in \{1, 2, \ldots, 2022\}$ have the following property: if 2022 real numbers are written on a circle so that the sum of any k consecutive numbers is equal to 2022 then all of the 2022 numbers are equal.

Problem (BMO Shortlist 2021 C6)

There is a population P of 10000 bacteria, some of which are friends (friendship is mutual), so that each bacterion has at least one friend and if we wish to assign to each bacterion a coloured membrane so that no two friends have the same colour, then there is a way to do it with 2021 colours, but not with 2020 or less. Two friends A and B can decide to merge in which case they become a single bacterion whose friends are precisely the union of friends of A and B. (Merging is not allowed if A and B are not friends.) It turns out that no matter how we perform one merge or two consecutive merges, in the resulting population it would be possible to assign 2020 colours or less so that no two friends have the same colour. Is it true that in any such population P every bacterium has at least 2021 friends?

Problem (Azerbaijan 2022 Junior National Olympiad)

There is a 8×8 board and the numbers $1, 2, 3, 4, ..., 63, 64$. In all the unit squares of the board, these numbers are places such that only 1 numbers goes to only one unit square. Prove that there is atleast 4 2×2 squares such that the sum of the numbers in $2 * 2$ is greater than 100.

Problem (2022 Bulgarian Spring Math Competition, P8.4)

Let $p = (a_1, a_2, \ldots, a_{12})$ be a permutation of $1, 2, \ldots, 12$. We will denote

$$S_p = |a_1 - a_2| + |a_2 - a_3| + \ldots + |a_{11} - a_{12}|$$

We'll call p *optimistic* if $a_i > \min(a_{i-1}, a_{i+1})$ $\forall i = 2, \ldots, 11$.

1. What is the maximum possible value of S_p. How many permutations p achieve this maximum?
2. What is the number of *optimistic* permtations p?
3. What is the maximum possible value of S_p for an *optimistic* p? How many *optimistic* permutations p achieve this maximum?

Problem (FKMO 2022 P2)

There are n boxes $A_1, ..., A_n$ with non-negative number of pebbles inside it(so it can be empty). Let a_n be the number of pebbles in the box A_n. There are total $3n$ pebbles in the boxes. From now on, Alice plays the following operation.

In each operation, Alice choose one of these boxes which is non-empty. Then she divide this pebbles into n group such that difference of number of pebbles in any two group is at most 1, and put these n group of pebbles into n boxes one by one. This continues until only one box has all the pebbles, and the rest of them are empty. And when it's over, define *Length* as the total number of operations done by Alice.

Let $f(a_1, ..., a_n)$ be the smallest value of *Length* among all the possible operations on $(a_1, ..., a_n)$. Find the maximum possible value of $f(a_1, ..., a_n)$ among all the ordered pair $(a_1, ..., a_n)$, and find all the ordered pair $(a_1, ..., a_n)$ that equality holds.

Problem (Serbian MO 2022 P5)

On the board are written n natural numbers, $n \in \mathbb{N}$. In one move it is possible to choose two equal written numbers and increase one by 1 and decrease the other by 1. Prove that in this the game cannot be played more than $\frac{n^3}{6}$ moves.

Problem (CMO 2022 P3)

Vishal starts with n copies of the number 1 written on the board. Every minute, he takes two numbers a, b and replaces them with either $a + b$ or $\min(a^2, b^2)$. After $n - 1$ there is 1 number on the board. Let the maximal possible value of this number be $f(n)$. Prove $2^{n/3} < f(n) \leq 3^{n/3}$

Problem (BMOSL 2021 C1, Greece NMO 2022 P4)

Let $\mathcal{A}_n$ be the set of n-tuples $x = (x_1, ..., x_n)$ with $x_i \in \{0, 1, 2\}$. A triple x, y, z of distinct elements of $\mathcal{A}_n$ is called good if there is some i such that $\{x_i, y_i, z_i\} = \{0, 1, 2\}$. A subset A of $\mathcal{A}_n$ is called good if every three distinct elements of A form a good triple. Prove that every good subset of $\mathcal{A}_n$ has at most $2(\frac{3}{2})^n$ elements.

Problem (Greece JBMO TST 2019 P4)

Consider a 8×8 chessboard where all 64 unit squares are at the start white. Prove that, if any 12 of the 64 unit square get painted black, then we can find 4 lines and 4 rows that have all these 12 unit squares.

Problem (Malaysia TST 2022 P2)

Let $\mathcal{S}$ be a set of 2023 points in a plane, and it is known that the distances of any two different points in S are all distinct. Ivan colors the points with k colors such that for every point $P \in \mathcal{S}$, the closest and the furthest point from $P \in \mathcal{S}$ also have the same color as P.

What is the maximum possible value of k?

Problem (Coatia TST 2009)

Every natural number is coloured in one of the k colors. Prove that there exist four distinct natural numbers a, b, c, d, all coloured in the same colour, such that $ad = bc$, $\frac{b}{a}$ is power of 2 and $\frac{c}{a}$ is power of 3.

Problem (Balkan MO 2022 P4)

Consider an $n \times n$ grid consisting of n^2 until cells, where $n \geq 3$ is a given odd positive integer. First, Dionysus colours each cell either red or blue. It is known that a frog can hop from one cell to another if and only if these cells have the same colour and share at least one vertex. Then, Xanthias views the colouring and next places k frogs on the cells so that each of the n^2 cells can be reached by a frog in a finite number (possible zero) of hops. Find the least value of k for which this is always possible regardless of the colouring chosen by Dionysus.

Problem (MEMO TST 2009)

In each field of 2009×2009 table you can write either 1 or -1. Denote A_k multiple of all numbers in k-th row and B_j the multiple of all numbers in j-th column. Is it possible to write the numbers in such a way that

$$\sum_{i=1}^{2009} A_i + \sum_{i=1}^{2009} B_i = 0$$

?

Problem (MEMO TST 2009 P2)

On sport games there was 1991 participant from which every participant knows at least n other participants(friendship is mutual). Determine the lowest possible n for which we can be sure that there are 6 participants between which any two participants know each other.

Problem (Thailand MO 2016 P4)

Each point on the plane is colored either red, green, or blue. Prove that there exists an isosceles triangle whose vertices all have the same color.

Problem (Bulgaria JBMO 2022 TST P3)

For a positive integer n let t_n be the number of unordered triples of non-empty and pairwise disjoint subsets of a given set with n elements. For example, $t_3 = 1$. Find a closed form formula for t_n and determine the last digit of t_{2022}.

Problem (RMMSL 2019 C1)

Let k and N be integers such that $k > 1$ and $N > 2k + 1$. A number of N persons sit around the Round Table, equally spaced. Each person is either a knight (always telling the truth) or a liar (who always lies). Each person sees the nearest k persons clockwise, and the nearest k persons anticlockwise. Each person says: "I see equally many knights to my left and to my right." Establish, in terms of k and N, whether the persons around the Table are necessarily all knights

Problem (BMOSL 2021 C4)

A sequence of $2n + 1$ non-negative integers $a_1, a_2, ..., a_{2n+1}$ is given. There's also a sequence of $2n + 1$ consecutive cells enumerated from 1 to $2n + 1$ from left to right, such that initially the number a_i is written on the i-th cell, for $i = 1, 2, ..., 2n + 1$. Starting from this initial position, we repeat the following sequence of steps, as long as it's possible:

Step 1: Add up the numbers written on all the cells, denote the sum as s.

Step 2: If s is equal to 0 or if it is larger than the current number of cells, the process terminates. Otherwise, remove the s-th cell, and shift shift all cells that are to the right of it one position to the left. Then go to Step 1.

Example: $(1, 0, 1, \underline{2}, 0) \to (1, \underline{0}, 1, 0) \to (1, \underline{1}, 0) \to (\underline{1}, 0) \to (0)$.

A sequence $a_1, a_2, ..., a_{2n+1}$ of non-negative integers is called balanced, if at the end of this process there's exactly one cell left, and it's the cell that was initially enumerated by $(n + 1)$, i.e. the cell that was initially in the middle.

Find the total number of balanced sequences as a function of n.

Problem (BMOSL 2021 C3, Azerbaijan TST 2022 P3)
In an exotic country, the National Bank issues coins that can take any value in the interval $[0, 1]$. Find the smallest constant $c > 0$ such that the following holds, no matter the situation in that country:

Any citizen of the exotic country that has a finite number of coins, with a total value of no more than 1000, can split those coins into 100 boxes, such that the total value inside each box is at most c.

Problem (BMOSL 2021 C5)
Angel has a warehouse, which initially contains 100 piles of 100 pieces of rubbish each. Each morning, Angel either clears every piece of rubbish from a single pile, or one piece of rubbish from each pile. However, every evening, a demon sneaks into the warehouse and adds one piece of rubbish to each non-empty pile, or creates a new pile with one piece. What is the first morning when Angel can guarantee to have cleared all the rubbish from the warehouse?

Problem (Iranian TST 2022 P11)
Consider a table with n rows and $2n$ columns. we put some blocks in some of the cells. After putting blocks in the table we put a robot on a cell and it starts moving in one of the directions right, left, down or up. It can change the direction only when it reaches a block or border. Find the smallest number m such that we can put m blocks on the table and choose a starting point for the robot so it can visit all of the unblocked cells. (the robot can't enter the blocked cells.)

Problem (Spain MO 2022 P5)
Given is a simple graph G with 2022 vertices, such that for any subset S of vertices (including the set of all vertices), there is a vertex v with $deg_S(v) \leq 100$. Find $\chi(G)$ and the maximal number of edges G can have.

Problem (Philippine TST P1 2019)
Let n and ℓ be integers such that $n \geq 3$ and $1 < \ell < n$. A country has n cities. Between any two cities A and B, either there is no flight from A to B and also none from B to A, or there is a unique two-way trip between them. A two-way trip is a flight from A to B and a flight from B to A. There exist two cities such that the least possible number of flights required to travel from one of them to the other is ℓ. Find the maximum number of two-way trips among the n cities.

Problem (CJMO 2022 P2)

You have an infinite stack of T-shaped tetrominoes (composed of four squares of side length 1), and an n × n board. You are allowed to place some tetrominoes on the board, possibly rotated, as long as no two tetrominoes overlap and no tetrominoes extend off the board. For which values of n can you cover the entire board?

Problem (Israel 2021 TST P2)

Given 10 light switches, each can be in two states: on and off. For each pair of switches there is a light bulb which is on if and only if when both switches are on (45 bulbs in total). The bulbs and the switches are unmarked so it is unclear which switches correspond to which bulb. In the beginning all switches are off. How many flips are needed to find out regarding all bulbs which switches are connected to it? On each step you can flip precisely one switch

Problem (Israel 2021 TST P2)

A game is played on a $n \times n$ chessboard. In the beginning Bars the cat occupies any cell according to his choice. The d sparrows land on certain cells according to their choice (several sparrows may land in the same cell). Bars and the sparrows play in turns. In each turn of Bars, he moves to a cell adjacent by a side or a vertex (like a king in chess). In each turn of the sparrows, precisely one of the sparrows flies from its current cell to any other cell of his choice. The goal of Bars is to get to a cell containing a sparrow. Can Bars achieve his goal

1. if $d = \lfloor \frac{3 \cdot n^2}{25} \rfloor$, assuming n is large enough?

2. if $d = \lfloor \frac{3 \cdot n^2}{19} \rfloor$, assuming n is large enough?

3. if $d = \lfloor \frac{3 \cdot n^2}{14} \rfloor$, assuming n is large enough?

Problem (All-Russian olympiad 2010 Grade 10 P8)

In the county some pairs of towns connected by two-way non-stop flight. From any town we can flight to any other (may be not on one flight). Gives, that if we consider any cyclic (i.e. beginning and finish towns match) route, consisting odd number of flights, and close all flights of this route, then we can found two towns, such that we can't fly from one to other. Proved, that we can divided all country on 4 regions, such that any flight connected towns from other regions.

Problem (Indonesian TST 2021 P4)
For every positive integer n, let $p(n)$ denote the number of sets $\{x_1, x_2, \ldots, x_k\}$ of integers with $x_1 > x_2 > \cdots > x_k > 0$ and $n = x_1 + x_3 + x_5 + \ldots$ (the right hand side here means the sum of all odd-indexed elements). As an example, $p(6) = 11$ because all satisfying sets are as follows:

$$\{6\}, \{6,5\}, \{6,4\}, \{6,3\}, \{6,2\}, \{6,1\}, \{5,4,1\}, \{5,3,1\}, \{5,2,1\}, \{4,3,2\}, \{4,3,2,1\}.$$

Show that $p(n)$ equals to the number of partitions of n for every positive integer n.

Problem (Azerbaijan Junior/Senior National Olympiad 2017)
A student firstly wrote $x = 3$ on the board. For each procces, the stutent deletes the number x and replaces it with either $(2x+4)$ or $(3x+8)$ or (x^2+5x). Is this possible to make the number $(20^{17}+2016)$ on the board?

Problem (BxMO 2022 P2)
Let n be a positive integer. There are n ants walking along a line at constant nonzero speeds. Different ants need not walk at the same speed or walk in the same direction. Whenever two or more ants collide, all the ants involved in this collision instantly change directions. (Different ants need not be moving in opposite directions when they collide, since a faster ant may catch up with a slower one that is moving in the same direction.) The ants keep walking indefinitely. Assuming that the total number of collisions is finite, determine the largest possible number of collisions in terms of n.

Problem (St. Petersburg MO 2017 Grade-9 P7)
Divide the upper right quadrant of the plane into square cells with side length 1. In this quadrant, n^2 cells are colored, show that there're at least n^2+n cells (possibly including the colored ones) that at least one of its neighbors are colored.

Problem (St. Petersburg MO 2017 Grade-9 P7)
Five people are gathered in a meeting. Some pairs of people shakes hands. An ordered triple of people (A, B, C) is a trio if one of the following is true:

1. A shakes hands with B, and B shakes hands with C, or
2. A doesn't shake hands with B, and B doesn't shake hands with C.

If we consider (A, B, C) and (C, B, A) as the same trio, find the minimum possible number of trios.

Problem (Italy TST 2018 P1)

The lengths of the sides of a rectangle R are expressed by odd positive integers. The rectangle is subdivided into smaller rectangles, all with sides of integer length. Prove that there exists at least one small rectangle such that the distances of its sides from the corresponding sides of R are all even or all odd.

Problem (Russia MO 2003)

There are 100 cities in a country, some of them being joined by roads. Any four cities are connected to each other by at least two roads. Assume that there is no path passing through every city exactly once. Prove that there are two cities such that every other city is connected to at least one of them.

Problem (JBMO 2012 P3)

On a board there are n nails, each two connected by a rope. Each rope is colored in one of n given distinct colors. For each three distinct colors, there exist three nails connected with ropes of these three colors.

1. Can n be 6 ?

2. Can n be 7 ?

Problem (USAMO 2005 P1)

Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem (INMO 2006 P2)

Find all natural numbers n for which there is a permutation σ of $\{1, 2, \ldots, n\}$ that satisfies:

$$\sum_{i=1}^{n} \sigma(i)(-2)^{i-1} = 0$$

Problem (China TST 2022 P4)

Find all positive integer k such that one can find a number of triangles in the Cartesian plane, the centroid of each triangle is a lattice point, the union of these triangles is a square of side length k (the sides of the square are not necessarily parallel to the axis, the vertices of the square are not necessarily lattice points), and the intersection of any two triangles is an empty-set, a common point or a common edge.

Problem (Mexican MO 1988 P1)

In how many ways can one arrange seven white and five black balls in a line in such a way that there are no two neighboring black balls?

Problem (Mexican MO 1988 P4)

In how many ways can one select eight integers $a_1, a_2, ..., a_8$, not necessarily distinct, such that $1 \leq a_1 \leq ... \leq a_8 \leq 8$?

Problem (Korean MO 2021 P4)

For a positive integer n, there are two countries A and B with n airports each and $n^2 - 2n + 2$ airlines operating between the two countries. Each airline operates at least one flight. Exactly one flight by one of the airlines operates between each airport in A and each airport in B, and that flight operates in both directions. Also, there are no flights between two airports in the same country. For two different airports P and Q, denote by "(P, Q)-travel route" the list of airports $T_0, T_1, \ldots, T_s$ satisfying the following conditions.

1. $T_0 = P,\ T_s = Q$
2. $T_0, T_1, \ldots, T_s$ are all distinct.
3. There exists an airline that operates between the airports T_i and T_{i+1} for all $i = 0, 1, \ldots, s-1$.

Prove that there exist two airports P, Q such that there is no or exactly one (P, Q)-travel route.

Problem (Romania Junior TST P3 Day 2)

Decompose a 6×6 square into unit squares and consider the 49 vertices of these unit squares. We call a square good if its vertices are among the 49 points and if its sides and diagonals do not lie on the gridlines of the 6×6 square.

1. Find the total number of good squares.
2. Prove that there exist two good disjoint squares such that the smallest distance between their vertices is $1/\sqrt{5}$.

Problem (Mongolian MO 2022)
A class has 30 students. Due to school capacity they will be divided into 3 classrooms. At the beginning of every month the professor rearrange their seats. Then find the minimum number of months required to arrange them so that every pair of students have sat together at least 1 month.

Problem (Mongolian MO 2022)
In a beautiful Gallery, there are $2n$ people standing in the circle with one paintings in each of them. Each of them sorted $2n$ paintings by their satisfaction. People do the following process:

If both of subsequent 2 people like the other one's painting more than theirs they will switch up the arts.

Find the maximum number of process they can do regardless of their standings and sorting.

Problem (Romania TST 2022 P5 Day 2)
Let $m, n \geq 2$ be positive integers and $S \subseteq [1, m] \times [1, n]$ be a set of lattice points. Prove that if

$$|S| \geq m + n + \left\lfloor \frac{m+n}{4} - \frac{1}{2} \right\rfloor$$

then there exists a circle which passes through at least four distinct points of S.

Problem (British TST)
Given a set L of lines in general position in the plane (no two lines in L are parallel, and no three lines are concurrent) and another line ℓ, show that the total number of edges of all faces in the corresponding arrangement, intersected by ℓ, is at most $6|L|$.

Problem (Latvia TST 2018 P8)

Let natural $n \geq 2$ be given. Let Laura be a student in a class of more than $n+2$ students, all of which participated in an olympiad and solved some problems. Additionally, it is known that:

1. for every pair of students there is exactly one problem that was solved by both students;
2. for every pair of problems there is exactly one student who solved both of them;
3. Laura and exactly n other students solved exactly one problem.

Determine the number of students in Laura's class.

Problem (Latvia TST 2018 P7)

Let $n \geq 3$ points be given in the plane, no three of which lie on the same line. Determine whether it is always possible to draw an n-gon whose vertices are the given points and whose sides do not intersect. Remark. The n-gon can be concave.

Problem (Brazil TST)

Let $n \geq 3$ be a positive integer. Draw the maximum number of diagonals without an intersection on the interior of the n-agon. Each diagonal is associed with the number of sides between your two points, considering the shortest way between them. What is the maximum value for the sum of the associated numbers with the diagonals?

Problem (China TST, 1987 P6)

Let G be a simple graph with $2n$ vertices and n^2+1 edges. Show that this graph G contains a $K_4 - 1$ edge,that is, two triangles with a common edge.

Problem (British MO)

Given a set L of lines in general position in the plane (no two lines in L are parallel, and no three lines are concurrent) and another line ℓ, show that the total number of edges of all faces in the corresponding arrangement, intersected by ℓ, is at most $6|L|$.

Problem (Croatia TST)

Every natural number is coloured in one of the k colors. Prove that there exist four distinct natural numbers a, b, c, d, all coloured in the same colour, such that $ad = bc$, $\frac{b}{a}$ is power of 2 and $\frac{c}{a}$ is power of 3.

Problem (China TST, 1989 P7)

1989 equal circles are arbitrarily placed on the table without overlap. What is the least number of colors are needed such that all the circles can be painted with any two tangential circles colored differently.

Problem (Vietnam TST, 1994 P6)

Calculate

$$T = \frac{1}{n_1! \cdot n_2! \cdots n_{1994}! \cdot (n_2 + 2 \cdot n_3 + 3 \cdot n_4 \cdots 1993 \cdot n_{1994})!}$$

where the sum is taken over all 1994-tuples of the numbers $n_1, n_2, n_3, \cdots, n_{1994} \in \mathbb{N} \cup 0$ satisfying $n_1 + 2 \cdot n_2 + 3 \cdot n_3 + \cdots + 1994 \cdot n_{1994} = 1994$

§6 IMO Problems

Problem (IMO 2011 P4)

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

Problem (IMOSL 2005 C1, Australia MO 1990)

A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

Problem (IMO 2016 P1)

Each point on the plane is colored either red, green, or blue. Prove that there exists an isosceles triangle whose vertices all have the same color.

Problem (IMO 1987 P1)

Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ which have exactly k fixed points. Prove that $\sum_{k=0}^{n} kp_n(k) = n!$.

Problem (IMO 2020 P4)

There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem (IMOSL 2010 C3)

2500 chess kings have to be placed on a 100×100 chessboard so that

1. No king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
2. Each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Problem (IMO 2018 P3)

An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + 3 + \cdots + 2018$?

Problem (IMOSL 2001 C1)

Let $A = (a_1, a_2, \ldots, a_{2001})$ be a sequence of positive integers. Let m be the number of 3-element subsequences (a_i, a_j, a_k) with $1 \leq i < j < k \leq 2001$, such that $a_j = a_i + 1$ and $a_k = a_j + 1$. Considering all such sequences A, find the greatest value of m.

Problem (IMOSL 2001 C1)

A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order:

1. The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

2. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

3. The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

Problem (IMO 2006)

Let P be a regular 2006-gon. A diagonal is called good if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called good. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Problem (IMOSL 2018 C3)

Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n. Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

Problem (IMOSL 2019 C1)

The infinite sequence $a_0, a_1, a_2, \ldots$ of (not necessarily distinct) integers has the following properties: $0 \le a_i \le i$ for all integers $i \ge 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \ge 0$. Prove that all integers $N \ge 0$ occur in the sequence (that is, for all $N \ge 0$, there exists $i \ge 0$ with $a_i = N$).

§7 Other Competitions

Problem (TOT 2022 Spring Senior O level)

The fox and pinocchio have grown a tree on the field of miracles with 8 golden coins. It is known that exactly 3 of them are counterfeit. All the real coins weigh the same, the counterfeit coins also weigh the same but are lighter. The fox and pinocchio have collected the coins and wish to divide them. The fox is going to give 3 coins to pinocchio, but pinocchio wants to check whether they all are real. Can he check this using 2 weighings on a balance scale with no weights?

Problem (Tournament of Mathematical Battles "League of Winners", 2016)

Is there a finite set of points on the plane such that each point has nearest at least 4 ?

Problem (Indian Statistical Institute Entrance, 2022)

Consider a board having 2 rows and n columns. Thus there are $2n$ cells in the board. Each cell is to be filled in by 0 or 1 .

1. In how many ways can this be done such that each row sum and each column sum is even?
2. In how many ways can this be done such that each row sum and each column sum is odd?

Problem (Manhattan Mathematical Olympiad, 2004)

Seven line segments, with lengths no greater than 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.

Problem (Manhattan Mathematical Olympiad, 2005)

Prove that having 100 whole numbers, one can choose 15 of them so that the difference of any two is divisible by 7.

Problem (Manhattan Mathematical Olympiad, 2003)

Prove that from any set of one hundred whole numbers, one can choose either one number which is divisible by 100, or several numbers whose sum is divisible by 100.

Problem (Red MOP lecture 2006)

There are 51 senators in a senate. The senate needs to be divided into n committees such that each senator is on exactly one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does 'not' necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

Problem (AHSME 1968 P30)

Convex polygons P_1 and P_2 are drawn in the same plane with n_1 and n_2 sides, respectively, $n_1 \leq n_2$. If P_1 and P_2 do not have any line segment in common, then the maximum number of intersections of P_1 and P_2 is?

Problem (AHSME 1986 P17)

A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks and 40 black socks. A youngster selects socks one at a time from the drawer but is unable to see the color of the socks drawn. What is the smallest number of socks that must be selected to guarantee that the selection contains at least 10 pairs? (A pair of socks is two socks of the same color. No sock may be counted in more than one pair.)

Problem (AHSME 1986 P22)

Six distinct integers are picked at random from $\{1, 2, 3, \ldots, 10\}$. What is the probability that, among those selected, the second smallest is 3?

Problem (TOT 2022 Spring Senior O level)

Let us call a 1×3 rectangle a tromino. Alice and Bob go to different rooms, and each divides a 20 × 21 board into trominos. Then they compare the results, compute how many trominos are the same in both splittings, and Alice pays Bob that number of dollars. What is the maximal amount Bob may guarantee to himself no matter how Alice plays?

Problem (VJIMC 2006 P1.3)

Two players play the following game: Let n be a fixed integer greater than 1. Starting from number $k = 2$, each player has two possible moves: either replace the number k by $k + 1$ or by $2k$. The player who is forced to write a number greater than n loses the game. Which player has a winning strategy for which n?

Problem (Miklós schweitzer competition 1995)

Let A be a subset of the set $\{1, 2, ..., n\}$ with at least $100\sqrt{n}$ elements. Prove that there is a four-element arithmetic sequence in which each element is the sum of two different elements of the set A.

Problem (NIMO Summer Contest 2016)

Evan writes a computer program that randomly rearranges the digits $0, 2, 4, 6, 8$ to create a five-digit number with no leading zeroes. If he executes this program once, the probability the program outputs an integer divisible by 4 can be written in the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}^+$ and $(m, n) = 1$. What is $m + n$?

Problem (NIMO Summer Contest 2016)

The ARML Local staff have ranked each of the 10 individual problems in order of difficulty from 1 to 10. They wish to order the problems such that the problem with difficulty i is before the problem with difficulty $i + 2 \ \ \forall i$ such that $1 \leq i \leq 8$. Compute the number of orderings of the problems that satisfy this condition

Problem (Indonesia Monthly contest)

For A $7 - Tuple$ (a, b, c, d, e, f, g) multiply the adjacent number in the permutations ,then add up all the numbers . The new number will be called as $jumlahkali$ from (a, b, c, d, e, f, g).As example $jumlahkali$ from $(1, 2, 3, 4, 5, 6, 7)$

$1 * 2 + 2 * 3 + ... + 6 * 7 = 112$ .Genkun writes all the possible permutations $(1, 2, 3, 4, 5, 6, 7)$ and counts all the $jumlahkali$ of its perms. what is the sum of all $jumlahkali$?

Problem (IGMO 2020 Round 2)

People from 80 countries participated in the 1st round of IGMO. In order to ensure that participants from any of the 80 countries can travel to any one of the remaining 79 countries by at most 2 flights, the countries have an agreement such that among the 80 countries, each country's airport connects to at least n other countries' airport.

1. Find the minimum value of n, proving that it is the minimum

2. Prove that for this minimum n, if there isn't a direct flight between 2 countries, then there are at least 2 paths that require 2 flights between the countries.

Note: For two airports A and B to be connected, it must be possible to have a flight from A to B and a flight from B to A. This adds 1 to both the number of connections from the airports. Also note that for this problem, each country has only 1 airport.

Problem (IGMO Mock 2021)

Santa has almost finished decorating his giant gingerbread house for Christmas. The only thing left to do is to create a circular fence around it. For this purpose Santa wants to use $n \geq 2$ candy canes in 3 colors: Green, Red and White, but he doesn't want any two adjacent candy canes to have the same color. Find the number of possible arrangements of this fence in terms of n

Problem (IZhO 2020 P6)

Some squares of a $n \times n$ tabel ($n > 2$) are black, the rest are withe. In every white square we write the number of all the black squares having at least one common vertex with it. Find the maximum possible sum of all these numbers.

Problem (TOT 2022 Spring Senior P6)

The king assembled 300 wizards and gave them the following challenge. For this challenge, 25 colors can be used, and they are known to the wizards. Each of the wizards receives a hat of one of those 25 colors. If for each color the number of used hats would be written down then all these number would be different, and the wizards know this. Each wizard sees what hat was given to each other wizard but does not see his own hat. Simultaneously each wizard reports the color of his own hat. Is it possible for the wizards to coordinate their actions beforehand so that at least 150 of them would report correctly?

Problem (APMO 2004)

Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathcal{L}$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to colour the points of S with at most two colours, such that for any points p, q of S, the number of lines in $\mathcal{L}$ which separate p from q is odd if and only if p and q have the same colour.

Note: A line ℓ separates two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ.

Problem (Lusophon MO 2021 Problem 2)

Esmeralda has created a special knight to play on quadrilateral boards that are identical to chessboards. If a knight is in a square then it can move to another square by moving 1 square in one direction and 3 squares in a perpendicular direction (which is a diagonal of a 2×4 rectangle instead of 2×3 like in chess). In this movement, it doesn't land on the squares between the beginning square and the final square it lands on.

A trip of the length n of the knight is a sequence of n squares $C1, C2, ..., Cn$ which are all distinct such that the knight starts at the $C1$ square and for each i from 1 to $n-1$ it can use the movement described before to go from the Ci square to the $C(i+1)$.

Determine the greatest $N \in \mathbb{N}$ such that there exists a path of the knight with length N on a 5×5 board.

Problem (MOP 1997)

Let A and B be disjoint sets whose union is the set of natural numbers. Show that for any n, there exists distinct positive integers a and b such that $a, b, a+b$ all belong in the same set (as in A or B).

§8 Unknown Sources

Problem

Suppose that each point in the plane is colored yellow, green, red or blue. show that there are two points at a distance of 1 or $\sqrt{3}$ of the same color.

Problem

n numbers are written on a blackboard. In one step you may erase any two of the numbers, say a and b, and write, instead $\dfrac{a+b}{4}$. Repeating this step $(n-1)$ times, there is one number left. Prove that, intially, if there were n ones on the board, at the end, a number, which is not less than $\dfrac{1}{n}$ will remain.

Problem

Let n be an odd integer greater than 1. Find the number of permutations σ of the set $\{1,\cdots,n\}$ for which $|\sigma(1)-1|+|\sigma(2)-2|+\cdots+|\sigma(n)-n|=\frac{n^2-1}{2}$.

Problem

An antelope is a chess piece which moves similarly to the knight: two cells (x_1,y_1) and (x_2,y_2) are joined by an antelope move if and only if

$$\{|x_1-x_2|,|y_1-y_2|\}=\{3,4\}.$$

The numbers from 1 to 10^{12} are placed in the cells of a $10^6\times 10^6$ grid. Let D be the set of all absolute differences of the form $|a-b|$, where a and b are joined by an antelope move in the arrangement. How many arrangements are there such that D contains exactly four elements?

Problem

Given a square $ABCD$ with sides 14.Pick a point P randomly inside the square such that $\angle APB$ and $\angle CPD$ are acute angle . If the probability of choosing point P randomly could be written as $\frac{m}{n}$ where $\gcd(m,n)=1$,what is $m+n$

Problem

Given 12 letters which consist of $[A,A,A,B,B,B,C,C,C,DDD]$ How many ways to form these letters such that no 4 letters next to each other make up the word $ABCD$ Example : $AAAABBBBCCCCDDDD$ is valid but $ABCDAABBCCDD$ is not valid

Problem

Find all positive integers m and n such that it is possible that in a group of n people, everyone shakes hands exactly with m persons. Handshake is mutual.

Problem

If a Martian has an infinite number of red, blue, yellow, and black socks in a drawer, how many socks must the Martian pull out of the drawer to guarantee he has a pair?

Problem

Suppose S is a set of $n+1$ integers. Prove that there exist distinct $a, b \in S$ such that $a-b$ is a multiple of n.

Problem

Show that in any group of n people, there are two who have an identical number of friends within the group.

Problem

Show that for any irrational $x \in \mathbb{R}$ and positive integer n, there exists a rational number $\frac{p}{q}$ with $1 \leq q \leq n$ such that $\left|x - \frac{p}{q}\right| < \frac{1}{nq}$.

Problem

Let $Q = \{0, 1, 2, \cdots, 2^n - 1\}$. A and B are two set satisfied $A \cup B = Q$ and $A \cap B = \varnothing$. Let $S = \{(a, b) \in A \times B \mid$ a and b are only different in 1 digit in binary $\}$. Find the minimum value of $\frac{|Q||S|}{|A||B|}$

Problem

Several segments are selected on the straight line so that among the selected there are no pairs $[a, b]$ and $[c, d]$ such that $a < c < b < d$. Prove you can color all segments in three colors so that no two segments of the form $[a, b]$, $[b, c]$ $(a < b < c)$ were not colored in the same color.

Problem

n points in general position are marked on the plane, their convex shell is triangulated (without adding additional nodes). It turned out that for all triangles' triangulation, all angles are greater than 1°. Potato is a convex polygon made entirely of triangulation triangles. Prove that there are at most n^{400} distinct potatoes.

Problem

A cage is cut out of a chessboard. Consider all broken lines whose vertices are different and lie at the centers of uncut cells, and each edge has length either $\sqrt{17}$ or $\sqrt{65}$. In which case broken lines are more: when $b1$ is cut out, or when is $c6$?

Problem

All faces of a convex polyhedron are triangles. Prove that its edges can be colored with two colors so that for each of them it would be possible to get from any vertex to any other along the edges of this color.

Problem

Prove that for a connected graph G there exists a path of length at least $\min(2\delta(G), \mid G \mid -1)$, where $\mid G \mid$ is the number of vertices in the graph G and $\delta(G)$ is the minimal degree of vertices in the graph.

Problem

Robot "Mag-o-matic" manipulates 101 glasses, displaced in a row whose positions are numbered from 1 to 101. In each glass you can find a ball or not. Mag-o-matic only accepts elementary instructions of the form $(a; b, c)$, which it interprets as "consider the glass in position a: if it contains a ball, then switch the glasses in positions b and c (together with their own content), otherwise move on to the following instruction" (it means that a, b, c are integers between 1 and 101, with b and c different from each other but not necessarily different from a). A $\underline{programme}$ is a finite sequence of elementary instructions, assigned at the beginning, that Mag-o-matic does one by one. A subset $S \subseteq \{0, 1, 2, \ldots, 101\}$ is said to be $\underline{identifiable}$ if there exists a programme which, starting from any initial configuration, produces a final configuration in which the glass in position 1 contains a ball if and only if the number of glasses containing a ball is an element of S.

1. Prove that the subset S of the odd numbers is identifiable.

2. Determine all subsets S that are identifiable.

Problem

If two points M and M' are selected at random on a line segment of length l. What is the expected value and the variance of distance MM'?

Problem

A convex polygon in the plane is such that its image, under any parallel translation, contains a point with integer coordinates. Prove that the distance between some two vertices of the polygon is at least $\sqrt{2}$.

Problem

Find The Exact Value Of Below Expression

$$\frac{\sum_{k=1}^{16} \binom{16}{k} * k! * (52-k)!}{52!}$$

Problem

Ben has a deck of cards numbered from 1 to 52. He takes some of these cards and puts them in a box. He notices that for any two cards numbered x and y, the card numbered $x - y$ is not in the box. What is the maximum number of cards that could possibly be in the box?

Problem

Is it true that if $a > b$ and $x > y$ and $(a-b)b > (x-y)y$ then

$$\binom{b}{a} > \binom{y}{x}$$

Problem

Fibonacci sequence $\{F_n\}$: $F_1 = F_2 = 1, F_{n+2} = F_n + F_{n+1}$. Prove that for $\forall n \in \mathbb{Z}^+$, we have:

1. $$\sum_{i=1}^{n} \binom{n+i}{n-i} = \sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$$

2. $$\sum_{i=1}^{n} \binom{n+i}{n-i} (-5)^{i-1} = (-1)^{n-1} \sum_{i=1}^{n} F_n^2$$

Problem

Given heptagon, draw a diagonal which intersects even number of already drawn diagonals. Prove that in the end, the number of drawn diagonals will be even.

Problem

A stage course is attended by $n \geq 4$ students. The day before the final exam, each group of three students conspire against another student to throw him/her out of the exam. Prove that there is a student against whom there are at least $\sqrt[3]{(n-1)(n-2)}$conspirators.

Problem

Melanie's fruit garden contains m melons and a row of $n \leq m$ fruit boxes. A distribution of all the melons into the boxes is considered seasonal if there are no empty boxes and there exists a contiguous segment of boxes that contains exactly k melons in total. The garden is colorful if it is seasonal for any distribution of the melons without empty boxes remaining. For what values of m, n, k is the garden colorful?

For example, $(m, n, k) = (5, 4, 3)$ is a colorful garden, because the only distributions of 5 melons into a row of 4 boxes with no empty ones are $(2, 1, 1, 1)$, $(1, 2, 1, 1)$, $(1, 1, 2, 1)$, $(1, 1, 1, 2)$, and in each of these cases, we can find some contiguous sequence of boxes that contain $k = 3$ melons in total.

Problem

In the French game of scrabble there are 102 letters including 2 jokers. In the beginning the first player gets 7 letters and then the second player gets another 7. The question is if one of the two players has more chances to get a joker on the first movement, or if the two players have the same probability.

Problem

Railway line connect to station A to another station K, passing successively through stations B, C, D,E, F, G, H, I and J (assume that these eleven stations are aligned). The distance between station A and K is 56 km. The length of a path linking three successive stations never exceeds 12 km. The length of a path linking four successive stations is at least 17km. What is the distance between station s B and G ?

Problem

For a positive integer n, consider the figure P which consists of all the unit line segments required to draw the perimeter of an n*n square, as well as all the interior unit segments needed to divide the n*n square into n2 unit squares. Tim wants to draw the figure P by drawing only squares. The squares may have some segments in common and may be of various sizes. What is the minimum number of squares Tim must draw to recover all of P?

Problem

Given a circle of radius 1 and a straight line not passing through its center. The grasshopper jumps from a circle to a straight line and from a straight line to a circle, and all its jumps have length 1. Prove that the grasshopper can only visit at a finite number of points.

Problem

The Fibonacci numbers $F_0, F_1, F_2, ...$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Given an integer $n \geq 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, ..., n$ there exist some $x, y \in S$ such that $x - y = F_k$.

Problem

The Cosmic Mathematics Olympiad consists of 6 questions, each with 7 points. 8^6 people from different parts of the universe participate here, the point is not the same for all two problems. The total score in this Olympiad is calculated by multiplying all the points, not adding. What is the score of the 7^6 place winner?

Problem

Prove that each player has a strategy to not lose a 3×3 tic tac toe game.

Problem

A 2010×2010 board is divided into corner-shaped figures of three cells. Prove that it is possible to mark one cell in each figure such that each row and each column will have the same number of marked cells.

Problem

A horizontal board is a support for four vertical pegs. One of the pegs passes through holes in n disks of increasing size, the largest disk being at the bottom, the smallest one on the top. We may take disks off a peg, always one at a time, and put each disk on another peg, but never in such a way that a disk is put on top of a smaller disk. Prove that the minimum number of transfers one should make to transfer all the disks to another peg is at least $2^{\sqrt{n}-1}$ but less than $2^{2\sqrt{n}+2}$.

Problem

n marbles are to be measured with the help of a one-armed balance. We are allowed to measure one or two marbles at a time. It may happen, however, that one of the results is misread. Denote by $f(n)$ the minimum number of weighings neccessary to determine the weight of each marble. Prove that $n + log_3 n - 3 < f(n) < n + log_3 n + 3$.

Problem

A piece is moving on an $n \times n$ board, its moves are restricted to one horizontal or one vertical step at a time. It starts at the upper left corner of the board and visits each field of the board exactly once. Let $f(n)$ denote the number of different possible routes of the piece. Prove that $1,25^{n^2} < f(n) > 2^{n^2}$, if n is large enough. Is it possible to improve upon the upper bound?

Problem

The Vertices of a regular 72-gon are each colored red, green or blue in equal amounts. Show that we can always choose four red, four green or four blue vertices such that each monochromatic set form a congruent quadrilateral.

Problem

Let A be an arbitrary coloring of the natural numbers with 2 colors. Prove that there exist 3 numbers with same color $a, b, c \in \mathbb{N}$ that $b = \frac{a+c}{2}$.

Problem

In a football tournament, team Barcelona scores at least one goal in each of the 15 consecutive matches. If they score a total of 23 goals in these 15 matches, prove that there is a sequence of consecutive matches $i, i+1, ..., j : 1 \leq i \leq j \leq 15$, where Barcelona scores a total of exactly 6 goals.

Problem

Let N a positive integer.

In a spaceship there are $2 \cdot N$ people, and each two of them are friends or foes (both relationships are symmetric). Two aliens play a game as follows:

1. The first alien chooses any person as she wishes.
2. Thenceforth, alternately, each alien chooses one person not chosen before such that the person chosen on each turn be a friend of the person chosen on the previous turn.
3. The alien that can't play in her turn loses.

Prove that second player has a winning strategy if, and only if, the $2 \cdot N$ people can be divided in N pairs in such a way that two people in the same pair are friends.

Problem

Consider a graph $G_n : (V_n, E_n)$, where $V_n = \{0, 1, \ldots, 2^n - 1\}$, $E_n = \{(x, y) : x, y \in V_n, \log_2 |x - y| \in \mathbb{Z}_{\geq}\}$. Actually, G_n is an n-dimensional hypercube. It can be proved that the number of spanning trees of G_n is $T_n = \prod_{k=2}^{n} (2k)^{\binom{n}{k}}$. Let $f(n, d)$ be the number of spanning trees of G_n that the degree of vertex 0 is d. If $q = \dfrac{2^{n-1}}{n2^{n-1} - 2^n + 1}$, prove that $f(n, d) = \dfrac{T_n d q^{d-1} \binom{n}{d}}{n(1+q)^{n-1}}$.

Problem

Five girls and five boys randomly sit in ten seats that are equally spaced around a circle. The probability that there is at least one diameter of the circle with two girls sitting on opposite ends of the diameter is mn , where m and n are relatively prime positive integers. Find $m + n$.

Problem

Given are n real points P_i in the coordinate plane, no three collinear. For each of these points P_i, let x_i the number of subsets $\mathcal{S} \in \{P_1, P_2, \cdots, P_n\}$ with $|\mathcal{S}| \geq 3$ such that the convex hull $\operatorname{Conv}(\mathcal{S})$ of these points contains the point P_i (as a vertex). Prescribe an algorithm that works in $\mathcal{O}(n^2 \log n)$ which determines all the values of the x_i, assuming knowledge of the locations of the points P_i.

Problem

Prove that:

$$\sum_{i=1}^{n} \binom{2n}{i+n} 2i = n\binom{2n}{n}$$

Problem

Is it possible to mark some cells of a 2023×2023 table such that each cell (marked or not marked) has exactly a marked neighbouring cell?

Problem

Two players, A and B play the following game: Two piles are given, each one contains some stones (not necessarily the same amount). In the first round, B tells A a k positive integer and A takes k stones from one of the piles (if k is grater than the number of stones in A's chosen pile, then he takes the whole pile). In the second round, A tells B a positive integer and B takes stones from one of the piles, ect. The one who takes the last stone loses. Who has a winning strategy and what is it?

Problem

On an infinite checkered plane, a triangle is drawn with vertices at the nodes, the area of which is less than 2020. It is known that exactly m nodes lie on each side of this triangle, not counting the ends. At what maximum m is this possible?

Problem

Suppose there are $2n$ children in a class, each starting with a certain number of sweets, such that the total number is a multiple of n. On each day, the teacher chooses n students and gives $c \in \mathbb{N}$ sweets to each of them. Find the minimum number of days needed such that no matter how the sweets were distributed at the start, the teacher can leave all the students with an equal number of sweets. (Note: c can change on a daily basis)

Problem

There are limited number of goods, total weight M, each not exceeding 1 ton. There are two trucks, the first has a maximum load of x ton and the second has a maximum load of y. We know that in any case, it can always be carried all at once by these two trucks. Find the maximum value of M.

Problem

We have an army base with $n \geq 3$ soldiers and some of them are friends. For each group of 3 different soldiers, if one of them is a friend of the second one and of the third one, then we know that the second one and the third one are also friends. If there are exactly $[\frac{n}{2}]$ pairs of friends, then prove that we can take $[\frac{n+1}{2}]$ different soldiers such that every 2 of them are not friends, where $[x]$ is the integer part of x.

Problem

A, B, and C listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the boys, there is at least one song liked by those two boys but disliked by the third. If it is possible in $2^k + 4 different ways, then find k.$

Problem

how many ways can you place two knights-white and black-on a chessboard consisting of 16*16 squares so that they threaten each other?

Problem

A bag contains 4 red marbles and 3 blue marbles. If the marbles are drawn one-by-one without replacement, what is the probability that the 5^{th} marble is red?

Problem

How many $n \times n$ matrix $\mathcal{A}$ possible whose each entries belongs to $\{0, 1, 2\}$ and satisfy $A_{ii} > \sum_{j=1, i\neq j}^{n} A_{ij}, \forall 1, 2...., n$.

Problem

There are 3 cities, A, B and C. To go to city B from A you can use any of 5 buses or 3 trains. And to move to city C from city B, you can use any of 3 buses or 4 trains. Then in how many ways can you go to city C from city A?

Problem

In how many ways can you make a flag with 4 stripes if you have 10 stripes, each having a diferent colour? Note that the ordering of the stripes will be important, i.e., diferent ordering leads to diferent flags.

Problem

In how many ways can you choose 5 students from a class of 12 students and arrange them in a line?

Problem

In how many ways can you arrange n distinct objects in a line?

Problem

Suppose you went to a sweet shop where 4 types of ice-creams are available, say Vanilla, Chocolate, Strawberry, Mango. In how many ways can you choose 2 diferent ice-creams?

Problem

Suppose n people go to a party. If each of them shakes hand with each other, then find the total number handshakes.

Problem

Suppose you have a (convex) polygon with n sides. Find its number of diagonals.

Problem

There are 2 girls and 7 boys in a chess club. A team of four persons must be chosen for a tournament, and it must contain at least one girl. In how many ways can this be done?

Problem

An 8×8 chessboard is colored in the usual way, but that's boring, so you decide to fix this. You can take any row, column, or 2×2 square, and reverse the colors inside it, switching black to white and white to black.
Prove that it's impossible to end up with 63 white squares and 1 black square.

Problem

The numbers $1, 2, \cdots, 100$ are written on a blackboard. You may choose any two numbers a and b and erase them, replacing them with the single number $a + b1$. After 99 steps, only a single number will be left. What is it?

Problem

At a party, some pairs of people shake hands. We call a person odd who has shaken hands with an odd number of other guests. Prove that there is an even number of odd people at the party.

Problem

A room is initially empty. Every minute, either two people enter or one person leaves. After exactly $3^{3^{3^3}}$ minutes, could the room contain exactly $3^{3^3} + 1$ people

Problem

A herd of 100 cows is divided into four pens: 10 cows in the north pen, 20 cows in the east pen, 30 cows in the south pen, and 40 cows in the west pen.

The pens are connected through a gateway we can use to let three cows out of one pen and distribute them between the others. For instance, if we let three cows out of the south pen, we end up with 11 cows in the north pen, 21 cows in the east pen, 27 cows in the south pen, and 41 cows in the west pen.

Prove that we can never use this gateway to split the herd into four equal groups, with 25 cows in each of the four pens.