Functional Equations in Mathematical Competitions: Problems and Solutions

Mohammad Mahdi Taheri

July 3, 2015

Abstract

In this paper I've gathered almost all of Functional Equation problems from recent years.

contact info:

mohammadmahdit@gmail.com

www.hamsaze.com

1 Problems

1. (Canada 2015) Find all functions $f : \mathbf{N} \to \mathbf{N}$ such that for all $n \in \mathbf{N}$
$$(n-1)^2 < f(n)f(f(n)) < n^2 + n$$

2. (APMO 2015) Let $S = \{2, 3, 4, ...\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that for all $a, b \in S$ with $a \neq b$
$$f(a)f(b) = f(a^2b^2)$$

3. (India 2015) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$
$$f(x^2 + yf(x)) = xf(x + y)$$

4. (Zhautykov Olympiad 2015) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$
$$f(x^3 + y^3 + xy) = x^2f(x) + y^2f(y) + f(xy)$$

5. (ISI Entrance 2015) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$
$$|f(x) - f(y)| = 2|x - y|$$

6. (Moldava TST 2015) Find all functions $f : \mathbf{N} \to \mathbf{N}$ that for all $m, n \in \mathbf{N}$
$$f(mf(n)) = n + f(2015m)$$

7. (Turkey TST 2015) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$
$$f(x^2) + 4y^2f(y) = (f(x - y) + y^2)(f(x + y) + f(y))$$

8. (USAJMO 2015) Find all functions $f : \mathbf{Q} \to \mathbf{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression. ($\mathbf{Q}$ is the set of all rational numbers.)

9. (Baltic Way 2014) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(f(y)) + f(x - y) = f(xf(y) - x)$$

10. (Albania TST 2014) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(x)f(y) = f(x + y) + xy$$

11. (Bulgaria 2014) Find all functions $f : \mathbf{Q}^+ \to \mathbf{R}^+$ such that for all $x, y \in \mathbf{Q}^+$

$$f(xy) = f(x + y)(f(x) + f(y))$$

12. (European Girls' Mathematical Olympiad 2014) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(y^2 + 2xf(y) + f(x)^2) = (y + f(x))(x + f(y))$$

13. (Romanian District Olympiad 2014) Find all functions $f : \mathbf{N} \to \mathbf{N}$ such that for all $m, n \in \mathbf{N}$

$$f(m + n) - 1 | f(m) + f(n)$$

and $n^2 - f(n)$ is a square.

14. (Romanian District Olympiad 2014) Find all functions $f : \mathbf{Q} \to \mathbf{Q}$ such that for all $x, y \in \mathbf{Q}$

$$f(x + 3f(y)) = f(x) + f(y) + 2y$$

15. (ELMO 2014) Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x + f(y)) &= g(x) + h(y) \\ g(x + g(y)) &= h(x) + f(y) \\ h(x + h(y)) &= f(x) + g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is injective if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

16. (ELMO 2014 Shortlist) Let $\mathbf{R}^*$ denote the set of nonzero reals. Find all functions $f : \mathbf{R}^* \to \mathbf{R}^*$ such that for all $x, y \in \mathbf{R}^*$ with $x^2 + y \neq 0$

$$f(x^2 + y) + 1 = f(x^2 + 1) + \frac{f(xy)}{f(x)}$$

17. (Britain 2014) Find all functions $f:\mathbf{N}\to\mathbf{N}$ such that for all $n\in\mathbf{N}$

$$f(n)+f(n+1)=f(n+2)f(n+3)-2010$$

18. (IMO Shortlist 2013) Find all functions $f:\mathbf{N}\to\mathbf{N}$ such that for all $m,n\in\mathbf{N}$

$$m^2+f(n)\mid mf(m)+n$$

19. (Zhautykov Olympiad 2014) Does there exist a function $f:\mathbf{R}\to\mathbf{R}$ satisfying the following conditions: for each real y there is a real x such that $f(x)=y$, and

$$f(f(x))=(x-1)f(x)+2$$

for all real x ?

20. (Iran 2014) Find all continuous function $f:\mathbf{R}^{\geq 0}\to\mathbf{R}^{\geq 0}$ such that for all $x,y\in\mathbf{R}^{\geq 0}$

$$f(xf(y))+f(f(y))=f(x)f(y)+2$$

21. (Iran TST 2014) Does there exist a function $f:\mathbf{N}\to\mathbf{N}$ satisfying the following conditions: (i)

$$\exists n\in N: f(n)\neq n$$

(ii) the number of divisors of m is $f(n)$ if and only if the number of divisors of $f(m)$ is n

22. (Iran TST 2014) Find all functions $f:\mathbf{R}^+\to\mathbf{R}^+$ such that for all $x,y\in\mathbf{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right)+f\left(\frac{x+1}{xf(y)}\right)=f(y)$$

23. (Kazakhstan 2014) Find all functions $f:\mathbf{Q}\times\mathbf{Q}\to\mathbf{Q}$ such that for all $x,y,z\in\mathbf{Q}$

$$f(x,y)+f(y,z)+f(z,x)=f(0,x+y+z)$$

24. (Korea 2014) Find all functions $f:\mathbf{R}\to\mathbf{R}$ such that for all $x,y\in\mathbf{R}$

$$f(xf(x)+f(x)f(y)+y-1)=f(xf(x)+xy)+y-1$$

25. (Middle European Mathematical Olympiad 2014) Find all functions $f:\mathbf{R}\to\mathbf{R}$ such that for all $x,y\in\mathbf{R}$

$$xf(y)+f(xf(y))-xf(f(y))-f(xy)=2x+f(y)-f(x+y)$$

26. (Middle European Mathematical Olympiad 2014) Find all functions $f:\mathbf{R}\to\mathbf{R}$ such that for all $x,y\in\mathbf{R}$

$$xf(xy)+xyf(x)\geq f(x^2)f(y)+x^2y$$

27. (Moldava TST 2014) Find all functions $f:\mathbf{R}\to\mathbf{R}$ such that for all $x,y\in\mathbf{R}$

$$f(xf(y)+y)+f(xy+x)=f(x+y)+2xy$$

28. (Turkey TST 2014) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(f(y) + x^2 + 1) + 2x = y + (f(x+1))^2$$

29. (USAMO 2014) Find all functions $f : \mathbf{Z} \to \mathbf{Z}$ such that for all $x, y \in \mathbf{Z}$

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

30. (Uzbekistan 2014) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(x^3) + f(y^3) = (x+y)(f(x^2) + f(y^2) - f(xy))$$

31. (Balkan 2013) Let S be the set of positive real numbers. Find all functions $f\colon S^3 \to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:

    (a) $xf(x, y, z) = zf(z, y, x)$,

    (b) $f(x, ky, k^2 z) = kf(x, y, z)$,

    (c) $f(1, k, k+1) = k+1$.

32. (Benelux MO 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(x+y) + y \leq f(f(f(x)))$$

33. (Baltic Way 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(xf(y) + y) + f(-f(x)) = f(yf(x) - y) + y$$

34. (Brazil 2013) Find all injective functions $f : \mathbf{R}^* \to \mathbf{R}^*$ such that for all $x, y \in \mathbf{R}^*$

$$f(x+y)\,(f(x) + f(y)) = f(xy)$$

35. (ELMO Shortlist 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(x) + f(y) = f(x+y)$$

    and

$$f(x^{2013}) = f(x)^{2013}$$

36. (Austrian Federal Competition 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ satisfying following conditions:

    (a) $f(x) \geq 0$ for all $x \in \mathbf{R}$.

    (b) For $a, b, c, d \in \mathbf{R}$ with $ab + bc + cd = 0$, equality $f(a-b) + f(c-d) = f(a) + f(b+c) + f(d)$ holds.

37. (Austrian Federal Competition 2013) Let k be an integer. Determine all functions $f\mathbf{R} \to \mathbf{R}$ with $f(0) = 0$ and

$$f(x^k y^k) = xyf(x)f(y) \qquad \text{for } x, y \neq 0.$$

38. (IMO 2013) Let $\mathbf{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbf{Q}_{>0} \to \mathbf{R}$ be a function satisfying the following three conditions:

    (i) for all $x, y \in \mathbf{Q}_{>0}$, $f(x)f(y) \geq f(xy)$;

    (ii) for all $x, y \in \mathbf{Q}_{>0}$, $f(x+y) \geq f(x) + f(y)$;

    (iii) there exists a rational number $a > 1$ such that $f(a) = a$.

    Prove that $f(x) = x$ for all $x \in \mathbf{Q}_{>0}$

39. (IMO Shortlist 2013) Let $\mathbf{Z}_{\geq 0}$ be the set of all non-negative integers. Find all the functions $f : \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

    for all $n \in \mathbf{Z}_{\geq 0}$

40. (IMO Shortlist 2013) Determine all functions $f : \mathbf{Q} \to \mathbf{Z}$ satisfying

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right)$$

    for all $x \in \mathbf{Q}$, $a \in \mathbf{Z}$, and $b \in \mathbf{Z}_{>0}$. (Here, $\mathbf{Z}_{>0}$ denotes the set of positive integers.)

41. (India TST 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(x(1+y)) = f(x)(1 + f(y))$$

42. (Iran 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that $f(0) \in \mathbf{Q}$ and

$$f(x + f(y)^2) = f(x+y)^2.$$

43. (Iran TST 2013) find all functions $f, g : \mathbf{R}^+ \to \mathbf{R}^+$ such that f is increasing and also:

    (i) $\quad$ f$(f(x) + 2g(x) + 3f(y)) = g(x) + 2f(x) + 3g(y)$

    (ii) $\quad$ g$(f(x) + y + g(y)) = 2x - g(x) + f(y) + y$

44. (Japan 2013) Find all functions $f : \mathbf{Z} \to \mathbf{R}$ such that the equality

$$f(m) + f(n) = f(mn) + f(m + n + mn)$$

    holds for all $m, n \in \mathbf{Z}$

45. (Korea 2013) Find all functions $f : \mathbf{N} \to \mathbf{N}$ such that for all $m, n \in \mathbf{N}$

$$f(mn) = \operatorname{lcm}(m, n) \cdot \gcd(f(m), f(n))$$

46. (Middle European Mathematical Olympiad 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in \mathbf{R}$

$$f(xf(x) + 2y) = f(x^2) + f(y) + x + y - 1$$

47. (Romania 2013) Given $f : \mathbf{R} \to \mathbf{R}$ an arbitrary function and $g : \mathbf{R} \to \mathbf{R}$ a function of the second degree, with the property: for any real numbers m and n equation $f(x) = mx + n$ has solutions if and only if the equation $g(x) = mx + n$ has solutions Show that the functions f and g are equal.

48. (Romania 2013) Find all injective functions$f : \mathbf{Z} \to \mathbf{Z}$ that satisfy:

$$|f(x) - f(y)| \leq |x - y|$$

for any $x, y \in \mathbf{Z}$.

49. (Romania 2013) Determine continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$\left(a^2 + ab + b^2\right) \int_a^b f(x)\,dx = 3 \int_a^b x^2 f(x)\,dx,$$

for every $a, b \in \mathbf{R}$

50. (Romania TST 2013) Determine all injective functions defined on the set of positive integers into itself satisfying the following condition: If S is a finite set of positive integers such that $\sum\limits_{s \in S} \frac{1}{s}$ is an integer, then $\sum\limits_{s \in S} \frac{1}{f(s)}$ is also an integer.

51. (Pan African 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x, y \in R$

$$f(x)f(y) + f(x + y) = xy$$

52. (Romanian Masters in Mathematics 2013) Does there exist a pair (g, h) of functions $g, h : \mathbf{R} \to \mathbf{R}$ such that the only function $f : \mathbf{R} \to \mathbf{R}$ satisfying $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbf{R}$ is identity function $f(x) \equiv x$?

53. (Stars of Mathematics 2013) Given a (fixed) positive integer N, solve the functional equation

$$f\colon \mathbf{Z} \to \mathbf{R},\ f(2k) = 2f(k) \text{ and } f(N - k) = f(k), \text{ for all } k \in \mathbf{Z}.$$

54. (USA TSTST 2013) Find all functions $f : \mathbf{N} \to \mathbf{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$. (Here $f^1(n) = f(n)$ and $f^k(n) = f(f^{k-1}(n))$ for every integer k greater than 1.)

55. (Turkey TST 2013) Determine all functions $f : \mathbf{R} \to \mathbf{R}^+$ such that for all real numbers x, y the following conditions hold:
    i. $f(x^2) = f(x)^2 - 2xf(x)$
    ii. $f(-x) = f(x - 1)$
    iii. $1 < x < y \Longrightarrow f(x) < f(y)$.

56. (Vietnam 2013) Find all functions $f : \mathbf{R} \to \mathbf{R}$ that satisfies $f(0) = 0, f(1) = 2013$ and

$$(x-y)(f(f^2(x)) - f(f^2(y))) = (f(x) - f(y))(f^2(x) - f^2(y))$$

Note: $f^2(x) = (f(x))^2$

57. (Uzbekistan 2013) Find all functions $f : \mathbf{Q} \to \mathbf{Q}$ such that

$$f(x+y)+f(y+z)+f(z+t)+f(t+x)+f(x+z)+f(y+t) \geq 6f(x-3y+5z+7t)$$

for all $x, y, z, t \in \mathbf{Q}$.

58. (Albania 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x^3) + f(y^3) = (x+y)f(x^2) + f(y^2) - f(xy)$$

for all $x \in \mathbf{R}$.

59. (Albania TST 2012) Let $f : \mathbf{R}^+ \to \mathbf{R}^+$ be a function such that:

$$x, y > 0 \qquad f(x + f(y)) = yf(xy+1).$$

a) Show that $(y-1)(f(y)-1) \leq 0$ for $y > 0$.

b) Find all such functions that require the given condition.

60. (Balkan 2012) Let $\mathbf{Z}^+$ be the set of positive integers. Find all functions $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ such that the following conditions both hold:

(i) $f(n!) = f(n)!$ for every positive integer n,

(ii) $m - n$ divides $f(m) - f(n)$ whenever m and n are different positive integers.

61. (Baltic Way 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ for which

$$f(x+y) = f(x-y) + f(f(1-xy))$$

holds for all real numbers x and y

62. (Brazil 2012) Find all surjective functions $f\colon (0, +\infty) \to (0, +\infty)$ such that

$$2xf(f(x)) = f(x)(x + f(f(x)))$$

for all $x > 0$

63. (China TST 2012) n being a given integer, find all functions $f\colon \mathbf{Z} \to \mathbf{Z}$, such that for all integers x, y we have

$$f\left(x + y + f(y)\right) = f(x) + ny$$

.

64. (Czech-Polish-Slovak 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ satisfying

$$f(x + f(y)) - f(x) = (x + f(y))^4 - x^4$$

for all $x, y \in \mathbf{R}$

65. (European Girls' Mathematical Olympiad 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f\left(yf(x+y)+f(x)\right)=4x+2yf(x+y)$$

for all $x, y \in \mathbf{R}$

66. (ELMO Shortlist 2012) Find all functions $f : \mathbf{Q} \to \mathbf{R}$ such that

$$f(x)f(y)f(x+y)=f(xy)(f(x)+f(y))$$

for all $x, y \in \mathbf{Q}$

67. (Austrian Federal Competition for Advanced Students 2012) Determine all functions $f : \mathbf{Z} \to \mathbf{Z}$ satisfying the following property:

    For each pair of integers m and n (not necessarily distinct), $\gcd(m, n)$ divides $f(m) + f(n)$.

    Note: If $n \in \mathbf{Z}$, $\gcd(m, n) = \gcd(|m|, |n|)$ and $\gcd(n, 0) = n$

68. (IMO 2012) Find all functions $f : \mathbf{Z} \to \mathbf{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2+f(b)^2+f(c)^2=2f(a)f(b)+2f(b)f(c)+2f(c)f(a).$$

(Here $\mathbf{Z}$ denotes the set of integers.)

69. (IMO Shortlist 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ that satisfy the conditions

$$f(1+xy)-f(x+y)=f(x)f(y) \quad \text{for all } x, y \in \mathbf{R},$$

and $f(-1) \neq 0$.

70. (India 2012) Let $f : \mathbf{Z} \to \mathbf{Z}$ be a function satisfying $f(0) \neq 0$, $f(1) = 0$ and

    $(i) f(xy) + f(x)f(y) = f(x) + f(y)$

    $(ii) \left(f(x-y) - f(0)\right) f(x)f(y) = 0$

    for all $x, y \in \mathbf{Z}$, simultaneously.

    (a) Find the set of all possible values of the function f.

    (b) If $f(10) \neq 0$ and $f(2) = 0$, find the set of all integers n such that $f(n) \neq 0$.

71. (Indonesia 2012) Let $\mathbf{R}^+$ be the set of all positive real numbers. Show that there is no function $f : \mathbf{R}^+ \to \mathbf{R}^+$ satisfying

$$f(x+y)=f(x)+f(y)+\frac{1}{2012}$$

for all positive real numbers x and y.

72. (IMO Training Camp 2012) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function such that

$$f(x+y+xy) = f(x) + f(y) + f(xy)$$

for all $x, y \in \mathbf{R}$. Prove that f satisfies

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbf{R}$.

73. (IMO Training Camp 2012) Let $\mathbf{R}^+$ denote the set of all positive real numbers. Find all functions $f : \mathbf{R}^+ \longrightarrow \mathbf{R}$ satisfying

$$f(x) + f(y) \leq \frac{f(x+y)}{2}, \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y},$$

for all $x, y \in \mathbf{R}^+$.

74. (Iran TST 2012) The function $f : \mathbf{R}^{\geq 0} \longrightarrow \mathbf{R}^{\geq 0}$ satisfies the following properties for all $a, b \in \mathbf{R}^{\geq 0}$:

    a) $f(a) = 0 \Leftrightarrow a = 0$

    b) $f(ab) = f(a)f(b)$

    c) $f(a+b) \leq 2\max\{f(a), f(b)\}$.

    Prove that for all $a, b \in \mathbf{R}^{\geq 0}$ we have $f(a+b) \leq f(a) + f(b)$.

75. (Iran TST 2012) Let $g(x)$ be a polynomial of degree at least 2 with all of its coefficients positive. Find all functions $f : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ such that

$$f(f(x) + g(x) + 2y) = f(x) + g(x) + 2f(y) \quad \forall x, y \in \mathbf{R}^+.$$

76. (Japan 2012) Find all functions $f : \mathbf{R} \mapsto \mathbf{R}$ such that

$$f(f(x+y)f(x-y)) = x^2 - yf(y)$$

for all $x, y \in \mathbf{R}$.

77. (Kazakhstan 2012) Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that

$$f(xf(y)) = yf(x)$$

for any x, y are real numbers. Prove that

$$f(-x) = -f(x)$$

for all real numbers x.

78. (Kyrgyzstan 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that $f(f(x)^2 + f(y)) = xf(x) + y, \forall x, y \in \mathbf{R}$.

79. (Macedonia 2012) Find all functions $f : \mathbf{R} \to \mathbf{Z}$ which satisfy the conditions:

    $f(x+y) < f(x) + f(y)$

    $f(f(x)) = \lfloor x \rfloor + 2$

80. (Middle European Mathematical Olympiad 2012) Let $\mathbf{R}^+$ denote the set of all positive real numbers. Find all functions $\mathbf{R}^+ \to \mathbf{R}^+$ such that

$$f(x + f(y)) = yf(xy + 1)$$

holds for all $x, y \in \mathbf{R}^+$.

81. (Pan African 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x^2 - y^2) = (x + y)(f(x) - f(y))$$

for all real numbers x and y.

82. (Puerto Rico TST 2012) Let f be a function with the following properties:

1) $f(n)$ is defined for every positive integer n;

2) $f(n)$ is an integer;

3) $f(2) = 2$;

4) $f(mn) = f(m)f(n)$ for all m and n;

5) $f(m) > f(n)$ whenever $m > n$.

Prove that $f(n) = n$.

83. (Romania 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ with the following property: for any open bounded interval I, the set $f(I)$ is an open interval having the same length with I

84. (Romania 2012) Find all differentiable functions $f\colon [0, \infty) \to [0, \infty)$ for which $f(0) = 0$ and $f'(x^2) = f(x)$ for any $x \in [0, \infty)$

85. (Poland 2012) Find all functions $f, g : \mathbf{R} \to \mathbf{R}$ satisfying $\forall x, y \in \mathbf{R}$:

$$g(f(x) - y) = f(g(y)) + x.$$

86. (Singapore 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$(x + y)(f(x) - f(y)) = (x - y)f(x + y)$$

for all x, y that belongs to $\mathbf{R}$.

87. (South Africa 2012) Find all functions $f : \mathbf{N} \to \mathbf{R}$ such that

$$f(km) + f(kn) - f(k)f(mn) \geq 1$$

for all $k, m, n \in \mathbf{N}$.

88. (Spain 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$(x - 2)f(y) + f(y + 2f(x)) = f(x + yf(x))$$

for all $x, y \in \mathbf{R}$.

89. (Turkey 2012) Find all non-decreasing functions from real numbers to itself such that for all real numbers x, y

$$f(f(x^2) + y + f(y)) = x^2 + 2f(y)$$

holds

90. (USA TST 2012) Determine all functions $f : \mathbf{R} \to \mathbf{R}$ such that for every pair of real numbers x and y,

$$f(x + y^2) = f(x) + |yf(y)|.$$

91. (USAMO 2012) Find all functions $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ (where $\mathbf{Z}^+$ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

92. (Vietnam 2012) Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that:

    (a) For every real number a there exist real number b:$f(b) = a$

    (b) If $x > y$ then $f(x) > f(y)$

    (c) $f(f(x)) = f(x) + 12x$.

2 Solutions

1. http://www.artofproblemsolving.com/community/c6h1081576p4756433
2. http://www.artofproblemsolving.com/community/c6h1071764p4663883
3. http://www.artofproblemsolving.com/community/c6h623452p3730730
4. http://www.artofproblemsolving.com/community/c6h620959p3710932
5. http://www.artofproblemsolving.com/community/c6h1087204p4812908
6. http://www.artofproblemsolving.com/community/c6h1072326p4667948
7. http://www.artofproblemsolving.com/community/c6h1072726p4670811
8. http://www.artofproblemsolving.com/community/c5h1083475p4774049
9. http://www.artofproblemsolving.com/community/c6h613426p3649203
10. http://www.artofproblemsolving.com/community/c6h586579p3470488
11. http://www.artofproblemsolving.com/community/c6h591459p3504652
12. http://www.artofproblemsolving.com/community/c6h585191p3460735
13. http://www.artofproblemsolving.com/community/c6h593728p3520986
14. http://www.artofproblemsolving.com/community/c6h593734p3521010
15. http://www.artofproblemsolving.com/community/c6h595782p3534942
16. http://www.artofproblemsolving.com/community/c6h599341p3557425
17. http://www.artofproblemsolving.com/community/c6h589266p3489256
18. http://www.artofproblemsolving.com/community/c6h597243p3544096
19. http://www.artofproblemsolving.com/community/c6h571163p3355255
20. http://www.artofproblemsolving.com/community/c6h604617p3590692

21. http://www.artofproblemsolving.com/community/c6h590692p3498505
22. http://www.artofproblemsolving.com/community/c6h590564p3497449
23. http://www.artofproblemsolving.com/community/c6h581896p3438510
24. http://www.artofproblemsolving.com/community/c6h621847p3717630
25. http://www.artofproblemsolving.com/community/c6h606969p3606608
26. http://www.artofproblemsolving.com/community/c6h607043p3607073
27. http://www.artofproblemsolving.com/community/c6h583295p3448269
28. http://www.artofproblemsolving.com/community/c6h580308p3426129
29. http://www.artofproblemsolving.com/community/c5h587520p3477690
30. http://www.artofproblemsolving.com/community/c6h592762p3514518
31. http://www.artofproblemsolving.com/community/c6h541453p3119999
32. http://www.artofproblemsolving.com/community/c6h532050p3040367
33. http://www.artofproblemsolving.com/community/c6h569071p3338897
34. http://www.artofproblemsolving.com/community/c6h545068p3151936
35. http://www.artofproblemsolving.com/community/c6h545068p3151936
36. http://www.artofproblemsolving.com/community/c6h526259p2983426
37. http://www.artofproblemsolving.com/community/c6h539381p3103184
38. http://www.artofproblemsolving.com/community/c6h545437p3154373
39. http://www.artofproblemsolving.com/community/c6h597123p3543371
40. http://www.artofproblemsolving.com/community/c6h597248p3544109
41. http://www.artofproblemsolving.com/community/c6h546366p3162056
42. http://www.artofproblemsolving.com/community/c6h553562p3215983
43. http://www.artofproblemsolving.com/community/c6h531087p3032850
44. http://www.artofproblemsolving.com/community/c6h520421p2931202
45. http://www.artofproblemsolving.com/community/c6h561919p3275970
46. http://www.artofproblemsolving.com/community/c6h589872p3493136
47. http://www.artofproblemsolving.com/community/c6h529910p3022740
48. http://www.artofproblemsolving.com/community/c6h527864p3000220
49. http://www.artofproblemsolving.com/community/c7h529921p3022771
50. http://www.artofproblemsolving.com/community/c6h528322p3005122

51. http://www.artofproblemsolving.com/community/c6h541491p3120210
52. http://www.artofproblemsolving.com/community/c6h523071p2951250
53. http://www.artofproblemsolving.com/community/c6h559025p3251902
54. http://www.artofproblemsolving.com/community/c6h548636p3181484
55. http://www.artofproblemsolving.com/community/c6h527852p3000070
56. http://www.artofproblemsolving.com/community/c6h516059p2902445
57. http://www.artofproblemsolving.com/community/c6h535870p3076137
58. http://www.artofproblemsolving.com/community/c6h472779p2647093
59. http://www.artofproblemsolving.com/community/c6h534834p3066172
60. http://www.artofproblemsolving.com/community/c6h477235p2672171
61. http://www.artofproblemsolving.com/community/c6h508345p2855945
62. http://www.artofproblemsolving.com/community/c6h509222p2861814
63. http://www.artofproblemsolving.com/community/c6h469660p2629262
64. http://www.artofproblemsolving.com/community/c6h529313p3016148
65. http://www.artofproblemsolving.com/community/c6h474746p2658967
66. http://www.artofproblemsolving.com/community/c6h486922p2728445
67. http://www.artofproblemsolving.com/community/c6h480871p2693139
68. http://www.artofproblemsolving.com/community/c6h488500p2737336
69. http://www.artofproblemsolving.com/community/c6h546165p3160554
70. http://www.artofproblemsolving.com/community/c6h462176p2592052
71. http://www.artofproblemsolving.com/community/c6h497185p2792802
72. http://www.artofproblemsolving.com/community/c6h489338p2743107
73. http://www.artofproblemsolving.com/community/c6h490299p2749512
74. http://www.artofproblemsolving.com/community/c6h476663p2668795
75. http://www.artofproblemsolving.com/community/c6h479267p2683530
76. http://www.artofproblemsolving.com/community/c6h463346p2597733
77. http://www.artofproblemsolving.com/community/c6h480485p2690907
78. http://www.artofproblemsolving.com/community/c6h532524p3045224
79. http://www.artofproblemsolving.com/community/c6h473827p2653068
80. http://www.artofproblemsolving.com/community/c6h498383p2800435

81. http://www.artofproblemsolving.com/community/c6h502357p2822619
82. http://www.artofproblemsolving.com/community/c6h87616p511005
83. http://www.artofproblemsolving.com/community/c7h473474p2650922
84. http://www.artofproblemsolving.com/community/c7h473484p2650932
85. http://www.artofproblemsolving.com/community/c6h465000p2604855
86. http://www.artofproblemsolving.com/community/c6h486615p2726723
87. http://www.artofproblemsolving.com/community/c6h497353p2793689
88. http://www.artofproblemsolving.com/community/c6h482804p2705223
89. http://www.artofproblemsolving.com/community/c6h508593p2857941
90. http://www.artofproblemsolving.com/community/c6h550607p3195788
91. http://www.artofproblemsolving.com/community/c5h476852p2669997
92. http://www.artofproblemsolving.com/community/c6h457745p2570340