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Day 1 February 22

1 Amy and Bob play the game. At the beginning, Amy writes down a positive integer on the board. Then the players take moves in turn, Bob moves first. On any move of his, Bob replaces the number n on the blackboard with a number of the form $n-a^2$, where a is a positive integer. On any move of hers, Amy replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bob wins if the number on the board becomes zero.
Can Amy prevent Bob's win?

Maxim Didin, Russia

2 Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

Jakob Jurij Snoj, Slovenia

3 Given any positive real number ε, prove that, for all but finitely many positive integers v, any graph on v vertices with at least $(1+\varepsilon)v$ edges has two distinct simple cycles of equal lengths. (Recall that the notion of a simple cycle does not allow repetition of vertices in a cycle.)

Fedor Petrov, Russia

Day 2 February 23

4 Prove that for every positive integer n there exists a (not necessarily convex) polygon with no three collinear vertices, which admits exactly n diffferent triangulations.

(A *triangulation* is a dissection of the polygon into triangles by interior diagonals which have no common interior points with each other nor with the sides of the polygon)

5 Determine all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x+yf(x))+f(xy)=f(x)+f(2019y),$$

for all real numbers x and y.

6 Find all pairs of integers (c, d), both greater than 1, such that the following holds:

For any monic polynomial Q of degree d with integer coefficients and for any prime $p > c(2c+1)$, there exists a set S of at most $\left(\frac{2c-1}{2c+1}\right)p$ integers, such that

$$\bigcup_{s\in S}\{s,\ Q(s),\ Q(Q(s)),\ Q(Q(Q(s))),\ \dots\}$$

contains a complete residue system modulo p (i.e., intersects with every residue class modulo p).