14th RMM 2023

www.artofproblemsolving.com/community/c3303738

by oVlad, Taco12, sqing

Day 1 March 1

1 Determine all prime numbers p and all positive integers x and y satisfying

$$x^3+y^3=p(xy+p).$$

2 Fix an integer $n\geq 3$. Let $\mathcal{S}$ be a set of n points in the plane, no three of which are collinear. Given different points A,B,C in $\mathcal{S}$, the triangle ABC is *nice* for AB if $[ABC]\leq [ABX]$ for all X in $\mathcal{S}$ different from A and B. (Note that for a segment AB there could be several nice triangles). A triangle is *beautiful* if its vertices are all in $\mathcal{S}$ and is nice for at least two of its sides.

Prove that there are at least $\frac{1}{2}(n-1)$ beautiful triangles.

3 Let $n\geq 2$ be an integer and let f be a $4n$-variable polynomial with real coefficients. Assume that, for any $2n$ points $(x_1,y_1),\dots,(x_{2n},y_{2n})$ in the Cartesian plane, $f(x_1,y_1,\dots,x_{2n},y_{2n})=0$ if and only if the points form the vertices of a regular $2n$-gon in some order, or are all equal.

Determine the smallest possible degree of f.

(Note, for example, that the degree of the polynomial

$$g(x,y)=4x^3y^4+yx+x-2$$

is 7 because $7=3+4$.)

Ankan Bhattacharya

Day 2 March 2

4 An acute triangle ABC is given and H and O be its orthocenter and circumcenter respectively. Let K be the midpoint of AH and ℓ be a line through O. Let P and Q be the projections of B and C on ℓ. Prove that

$$KP+KQ\geq BC$$

5 Let P,Q,R,S be non constant polynomials with real coefficients, such that $P(Q(x))=R(S(x))$ and the degree of P is multiple of the degree of R. Prove that there exists a polynomial T with real coefficients such that

$$P(x)=R(T(x))$$

6 Let r, g, b be non negative integers and Γ be a connected graph with $r+g+b+1$ vertices. Its edges are colored in red green and blue. It turned out that Γ contains

A spanning tree with exactly r red edges.
A spanning tree with exactly g green edges.
A spanning tree with exactly b blue edges.

Prove that Γ contains a spanning tree with exactly r red edges, g green edges and b blue edges.