RMM 2021 Shortlist

www.artofproblemsolving.com/community/c3560410

by augustin_p, Assassino9931

– Algebra

A1 Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xy+f(x))+f(y)=xf(y)+f(x+y)$$

for all real numbers x and y.

A2 Let n be a positive integer and let $x_1,\dots,x_n,y_1,\dots,y_n$ be integers satisfying the following condition: the numbers $x_1,\dots,x_n$ are pairwise distinct and for every positive integer m there exists a polynomial P_m with integer coefficients such that $P_m(x_i)-y_i$, $i=1,\dots,n$, are all divisible by m. Prove that there exists a polynomial P with integer coefficients such that $P(x_i)=y_i$ for all $i=1,\dots,n$.

A3 A *tile* T is a union of finitely many pairwise disjoint arcs of a unit circle K. The *size* of T, denoted by $|T|$, is the sum of the lengths of the arcs T consists of, divided by 2π. A *copy* of T is a tile T' obtained by rotating T about the centre of K through some angle. Given a positive real number $\varepsilon<1$, does there exist an infinite sequence of tiles $T_1,T_2,\dots,T_n,\dots$ satisfying the following two conditions simultaneously:
1) $|T_n|>1-\varepsilon$ for all n;
2) The union of all T'_n (as n runs through the positive integers) is a proper subset of K for any choice of the copies T'_1, T'_2, $\dots$, $T'_n,\dots$?

In the extralist the problem statement had the clause "three conditions" rather than two, but only two are presented, the ones you see. I am quite confident this is a typo or that the problem might have been reformulated after submission.

A4 Let $f:\mathbb{R}\to\mathbb{R}$ be a non-decreasing function such that $f(y)-f(x)<y-x$ for all real numbers x and $y>x$. The sequence $u_1,u_2,\dots$ of real numbers is such that $u_{n+2}=f(u_{n+1})-f(u_n)$ for all $n\ge 1$. Prove that for any $\varepsilon>0$ there exists a positive integer N such that $|u_n|<\varepsilon$ for all $n\ge N$.

– Combinatorics

C1 Determine the largest integer $n\ge 3$ for which the edges of the complete graph on n vertices can be assigned pairwise distinct non-negative integers such that the edges of every triangle have numbers which form an arithmetic progression.

C2 Fix a positive integer n and a fi nite graph with at least one edge; the endpoints of each edge are distinct, and any two vertices are joined by at most one edge. Vertices and edges are assigned (not necessarily distinct) numbers in the range from 0 to $n-1$, one number each. A vertex assignment and an edge assignment are *compatible* if the following condition is satisfi ed
at each vertex v: The number assigned to v is congruent modulo n to the sum of the numbers assigned to the edges incident to v. Fix a vertex assignment and let N be the total number of compatible edge assignments; compatibility refers, of course, to the fixed vertex assignment. Prove that, if $N \neq 0$, then the prime divisors of N are all at most n.

– Geometry

G1 Let $ABCD$ be a parallelogram. A line through C crosses the side AB at an interior point X, and the line AD at Y. The tangents of the circle AXY at X and Y, respectively, cross at T. Prove that the circumcircles of triangles ABD and TXY intersect at two points, one lying on the line AT and the other one lying on the line CT.

G2 Let ABC be a triangle with incenter I. The line through I, perpendicular to AI, intersects the circumcircle of ABC at points P and Q. It turns out there exists a point T on the side BC such that $AB + BT = AC + CT$ and $AT^2 = AB \cdot AC$. Determine all possible values of the ratio IP/IQ.

G3 Let Ω be the circumcircle of a triangle ABC with $\angle BAC > 90^\circ$ and $AB > AC$. The tangents of Ω at B and C cross at D and the tangent of Ω at A crosses the line BC at E. The line through D, parallel to AE, crosses the line BC at F. The circle with diameter EF meets the line AB at P and Q and the line AC at X and Y. Prove that one of the angles $\angle AEB$, $\angle PEQ$, $\angle XEY$ is equal to the sum of the other two.

G4 Let ABC be an acute triangle, let H and O be its orthocentre and circumcentre, respectively, and let S and T be the feet of the altitudes from B to AC and from C to AB, respectively.
Let M be the midpoint of the segment ST, and let N be the midpoint of the segment AH. The line
through O, parallel to BC, crosses the sides AC and AB at F and G, respectively. The line NG meets the circle BGO again at K, and the line NF meets the circle CFO again at L. Prove that the triangles BCM and KLN are similar.

– Number Theory

N1 Given a positive integer N, determine all positive integers n, satisfying the following condition: for any list $d_1, d_2, \ldots, d_k$ of (not necessarily distinct) divisors of n such that $\frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_k} > N$, some of the fractions $\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_k}$ add up to exactly N.

N2 We call a set of positive integers *suitable* if none of its elements is coprime to the sum of all elements of that set. Given a real number $\varepsilon \in (0,1)$, prove that, for all large enough positive integers N, there exists a suitable set of size at least εN, each element of which is at most N.