15th RMM 2024

www.artofproblemsolving.com/community/c3771372

by Tintarn, Assassino9931

Day 1 February 28

1 Let n be a positive integer. Initially, a bishop is placed in each square of the top row of a $2^n \times 2^n$ chessboard; those bishops are numbered from 1 to 2^n from left to right. A *jump* is a simultaneous move made by all bishops such that each bishop moves diagonally, in a straight line, some number of squares, and at the end of the jump, the bishops all stand in different squares of the same row.

Find the total number of permutations σ of the numbers $1, 2, \ldots, 2^n$ with the following property: There exists a sequence of jumps such that all bishops end up on the bottom row arranged in the order $\sigma(1), \sigma(2), \ldots, \sigma(2^n)$, from left to right.

Israel

2 Consider an odd prime p and a positive integer $N < 50p$. Let $a_1, a_2, \ldots, a_N$ be a list of positive integers less than p such that any specific value occurs at most $\frac{51}{100}N$ times and $a_1+a_2+\cdots+a_N$ is not divisible by p. Prove that there exists a permutation $b_1, b_2, \ldots, b_N$ of the a_i such that, for all $k = 1, 2, \ldots, N$, the sum $b_1 + b_2 + \cdots + b_k$ is not divisible by p.

Will Steinberg, United Kingdom

3 Given a positive integer n, a collection $\mathcal{S}$ of $n-2$ unordered triples of integers in $\{1, 2, \ldots, n\}$ is [i]n-admissible[/i] if for each $1 \le k \le n-2$ and each choice of k distinct $A_1, A_2, \ldots, A_k \in \mathcal{S}$ we have

$$|A_1 \cup A_2 \cup \cdots A_k| \ge k+2.$$

Is it true that for all $n > 3$ and for each n-admissible collection $\mathcal{S}$, there exist pairwise distinct points $P_1, \ldots, P_n$ in the plane such that the angles of the triangle $P_iP_jP_k$ are all less than 61° for any triple $\{i, j, k\}$ in $\mathcal{S}$?

Ivan Frolov, Russia

Day 2 February 29

4 Fix integers a and b greater than 1. For any positive integer n, let r_n be the (non-negative) remainder that b^n leaves upon division by a^n. Assume there exists a positive integer N such that $r_n < \frac{2^n}{n}$ for all integers $n \ge N$. Prove that a divides b.

Pouria Mahmoudkhan Shirazi, Iran

5 Let BC be a fixed segment in the plane, and let A be a variable point in the plane not on the line BC. Distinct points X and Y are chosen on the rays $CA^{\rightarrow}$ and $BA^{\rightarrow}$, respectively, such that $\angle CBX = \angle YCB = \angle BAC$. Assume that the tangents to the circumcircle of ABC at B and C meet line XY at P and Q, respectively, such that the points X, P, Y and Q are pairwise distinct and lie on the same side of BC. Let Ω_1 be the circle through X and P centred on BC. Similarly, let Ω_2 be the circle through Y and Q centred on BC. Prove that Ω_1 and Ω_2 intersect at two fixed points as A varies.

Daniel Pham Nguyen, Denmark

---

6 A polynomial P with integer coefficients is *square-free* if it is not expressible in the form $P = Q^2R$, where Q and R are polynomials with integer coefficients and Q is not constant. For a positive integer n, let P_n be the set of polynomials of the form

$$1 + a_1x + a_2x^2 + \cdots + a_nx^n$$

with $a_1, a_2, \ldots, a_n \in \{0, 1\}$. Prove that there exists an integer N such that for all integers $n \geq N$, more than 99% of the polynomials in P_n are square-free.

Navid Safaei, Iran
