Advanced Lemmas in Geometry

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Abstract

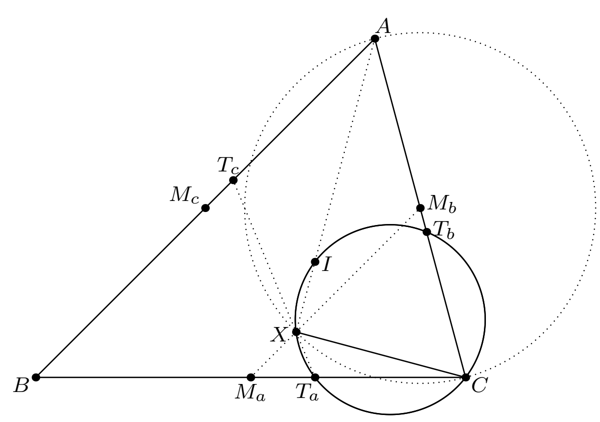
A good knowledge of lemmas makes difference between being able to solve easy (i.e IMO 1/4 level) and medium-hard (i.e. IMO 2/3/5/6 level) geometry problems. In this article I present several lemmas that can help you overcome this barrier.

1 Iran lemma

1.1 Main lemma

Let ABC be a triangle. Let I be the incenter, M_a, M_b, M_c be the midpoints of BC, CA, AB and let T_a, T_b, T_c be the points of tangency of the incircle with BC, CA, AB. Then AI, M_aM_b, T_aT_c and the circle with diameter AC concur.

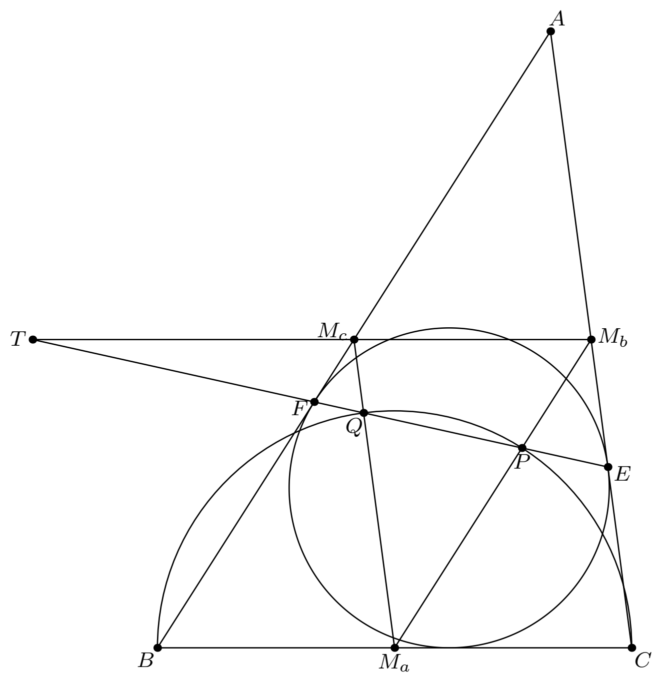
Proof. Let X be the projection of C onto AI. We'll show that X is the desired concurrency point; it clearly lies on AI and the circle with diameter AC. Note that $\angle CM_bX = 2\angle CAX = \angle CAB$, so $M_bX \parallel AB$ and X lies on M_aM_b. Also, X lies on the circle with diameter CI, so $\angle T_bT_aX = \angle T_bCX = 90^\circ - \frac{\angle A}{2} = \angle T_bT_aT_c$, so X lies on T_aT_c. $\square$



1.2 Example

(Sharygin 2019 9.7) Let the incircle ω of $\triangle ABC$ touch AC and AB at points E and F respectively. Points X, Y of ω are such that $\angle BXC = \angle BYC = 90^\circ$. Prove that EF and XY meet on the medial line of ABC.

Solution. Let M_a, M_b, M_c be the midpoints of BC, CA, AB and let $T = M_bM_c \cap EF$. It suffices to prove that T lies on the radical axis of ω and the circle with diameter BC. By Iran lemma, EF and the circle with diameter BC intersect at two points P and Q, lying on M_aM_b and M_aM_c, respectively. Then $M_bE \parallel M_cQ$ and $M_cF \parallel M_bP$, so $\dfrac{TE}{TQ} = \dfrac{TM_b}{TM_c} = \dfrac{TP}{TF} \implies TE\cdot TF = TP\cdot TQ$ and the conclusion follows.



1.3 Practice problems

Problem 1.1. (RMM 2020/1) Let ABC be a triangle with a right angle at C. Let I be the incentre of triangle ABC, and let D be the foot of the altitude from C to AB. The incircle ω of triangle ABC is tangent to sides BC, CA, and AB at A_1, B_1, and C_1, respectively. Let E and F be the reflections of C in lines C_1A_1 and C_1B_1, respectively. Let K and L be the reflections of D in lines C_1A_1 and C_1B_1, respectively.

Prove that the circumcircles of triangles A_1EI, B_1FI, and C_1KL have a common point.

Problem 1.2. (USA TST 2015/1) Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

Problem 1.3. (ISL 2000 G8) Let AH_1, BH_2, CH_3 be the altitudes of an acute angled triangle ABC. Its incircle touches the sides BC, AC and AB at T_1, T_2 and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3 and H_3H_1 with respect to the lines T_1T_2, T_2T_3 and T_3T_1. Prove that these images form a triangle whose vertices lie on the incircle of ABC.

Problem 1.4. (Iran TST 2009/9) In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $DP = MP$. If H is the orthocenter of BIC, prove that PH bisects EF.

Problem 1.5. (Sharygin 2015 9.8) The perpendicular bisector of side BC of triangle ABC meets lines AB and AC at points A_B and A_C respectively. Let O_a be the circumcenter of triangle AA_BA_C. Points O_b and O_c are defined similarly. Prove that the circumcircle of triangle $O_aO_bO_c$ touches the circumcircle of the original triangle.

Problem 1.6. Let ABC be a triangle. Line ℓ_a cuts segments equal to BC on rays AB and AC. ℓ_b and ℓ_c are defined similarly. Prove that the circumcircle of the triangle determined by ℓ_a, ℓ_b, ℓ_c is tangent to the circumcircle of $\triangle ABC$.

Problem 1.7. (ISL 2004 G7) For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

Problem 1.8. (ELMO 2016/6) Elmo is now learning olympiad geometry. In triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90^\circ$. Finally, let γ be the circumcircle of $\triangle AST$.

(a) Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

(b) Help Elmo show that γ is tangent to the incircle of $\triangle ABC$.

Problem 1.9. (Taiwan TST 2015 quiz 3/2) In a scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of triangle AEF at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter ST is orthogonal to the nine-point circle of triangle BIC.

2 Isogonal conjugation in polygons

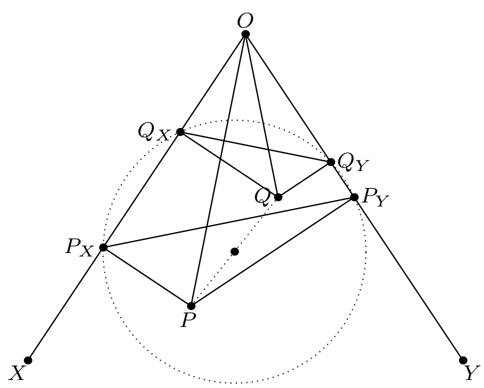
2.1 Main lemma

Let $\mathcal{A} = A_1 A_2 \ldots A_n$ be a convex polygon and P be a point in its interior. Then P has an isogonal conjugate with respect to $\mathcal{A}$ if and only if the projections of P onto the sides of $\mathcal{A}$ are concyclic.

To prove the main lemma, we'll need the following additional claim.

Claim. Rays OP and OQ are isogonal in angle XOY if and only if the four projections of P and Q onto OX and OY lie on a circle; moreover, the center of the circle is the midpoint of PQ.

Proof. Let P_X and Q_X be the projections of P and Q onto OX, and let P_Y and Q_Y be their projections onto OY. Then OP and OQ are isgonal $\iff \angle XOP = \angle YOQ \iff \angle OPP_X = \angle OQQ_Y \iff \angle OP_Y P_X = \angle OQ_X Q_Y \iff P_X$, P_Y, Q_X, Q_Y are concyclic. Moreover, the perpendicular bisectors of $P_X Q_X$ and $P_Y Q_Y$ are midlines of right trapezoids $P_X Q_X QP$ and $P_Y Q_Y QP$, respectively, so the circle has to be centered at the midpoint of PQ. □



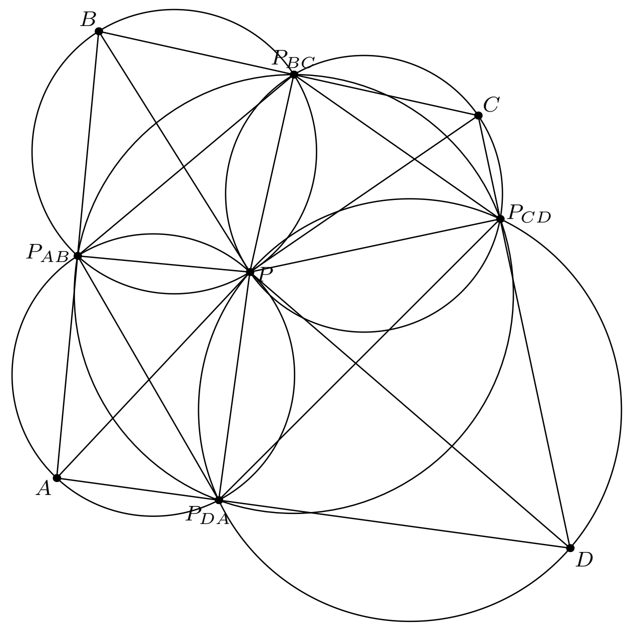
Now we are ready to prove the lemma itself.

Proof. If P and Q are isogonal conjugates, the claim implies that the projections of P and Q onto any pair of neighboring sides of $\mathcal{A}$ lie on the circle centered at the midpoint of PQ, so it easily follows that all of them lie on the same circle. Similarly, if the projections of P onto the sides of $\mathcal{A}$ lie on a circle, we define Q as the reflection of P about the center of the circle. Then the projections of Q onto the sides of $\mathcal{A}$ lie on the same circle, and we're done again by the lemma. □

Usually we have to deal with the case $n = 4$. For quadrilaterals, there also is the following property.

Claim. Point P has an isogonal conjugate with respect to the quadrilateral $ABCD$ if and only if $\angle APB + \angle CPD = 180°$.

Proof. let P_{AB}, P_{BC}, P_{CD}, P_{DA} be the projections of P onto the sides of $ABCD$. Then we have $\angle P_{DA}P_{AB}P_{BC} + \angle P_{BC}P_{CD}P_{DA} = \angle P_{DA}P_{AB}P + \angle PP_{AB}P_{BC} + \angle P_{BC}P_{CD}P + \angle PP_{CD}P_{DA} = \angle P_{DA}AP + \angle PBP_{BC} + \angle P_{BC}CP + \angle PDP_{DA} = 360^\circ - \angle PAB - \angle ABP - \angle PCD - \angle CDP = \angle APB + \angle CPD$, so $P_{AB}P_{BC}P_{CD}P_{DA}$ is cyclic $\iff \angle P_{DA}P_{AB}P_{BC} + \angle P_{BC}P_{CD}P_{DA} = 180^\circ \iff \angle APB + \angle CPD = 180^\circ$. $\square$



2.2 Example

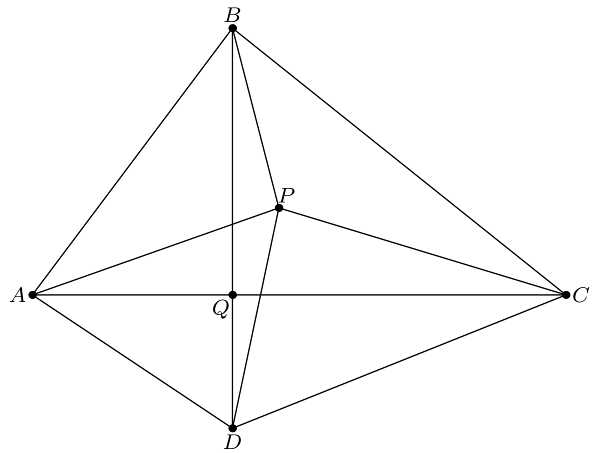
(ISL 2008 G6) There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

if and only if the diagonals AC and BD are perpendicular.

Solution. If such P exists, then the angle conditions implies that $\angle APB + \angle CPD = 180^\circ$, so P has an isogonal conjuate Q. By the angle condition again, it must satisfy $\angle AQB = \angle BQC = \angle CQD = \angle DQC = 90^\circ$, so Q is the point of intersection of perpendicular diagonals of $ABCD$.

If $ABCD$ has perpendicular diagonals intersecting at Q, then the isogonal conjugate of Q with respect to $ABCD$ satisfies the conditions.



2.3 Practice problems

Problem 2.1. (EGMO 2019/1) Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Problem 2.2. (Sharygin 2015 8.8) Points C_1, B_1 on sides AB, AC respectively of triangle ABC are such that $BB_1 \perp CC_1$. Point X lying inside the triangle is such that $\angle XBC = \angle B_1BA, \angle XCB = \angle C_1CA$. Prove that $\angle B_1XC_1 = 90^o - \angle A$.

Problem 2.3. (All-Russian 2017 11.8) Given a convex quadrilateral $ABCD$. We denote by I_A, I_B, I_C and I_D centers of ω_A, ω_B, ω_C and ω_D, inscribed in the triangles DAB, ABC, BCD and CDA, respectively. It turned out that $\angle BI_AA + \angle I_CI_AI_D = 180°$. Prove that $\angle BI_BA + \angle I_CI_BI_D = 180°$.

Problem 2.4. (IMO 2018/6) A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

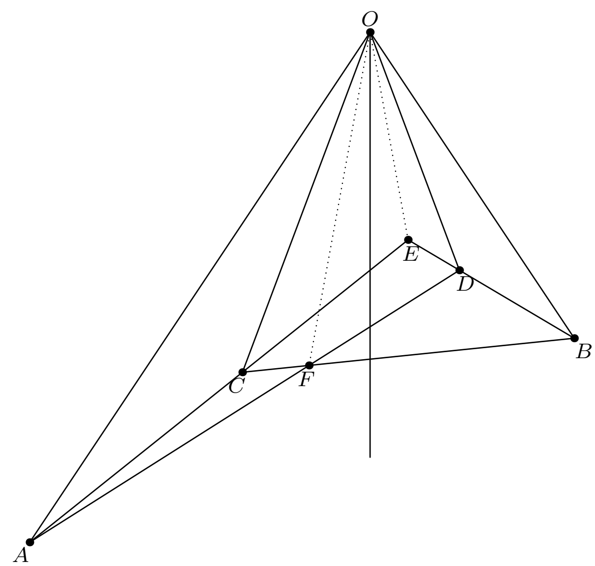
Prove that $\angle BXA + \angle DXC = 180°$.

3 Isogonal lemma

3.1 Main lemma

Suppose that $\angle AOB$ and $\angle COD$ have the same angle bisector ℓ. If $E = AC \cap BD$ and $F = AD \cap BC$, then ℓ is also the angle bisector of $\angle EOF$.

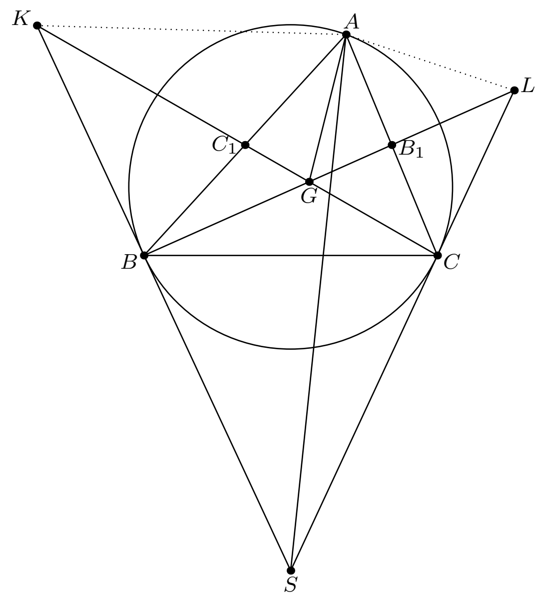
Proof. We use trigonometric Ceva's theorem in $\triangle OAB$. $\dfrac{\sin AOE}{\sin EOB}\dfrac{\sin AOF}{\sin FOB} = \dfrac{\sin OAE}{\sin EAB}\dfrac{\sin ABE}{\sin EBO}\dfrac{\sin OAF}{\sin FAB}\dfrac{\sin ABF}{\sin FBO} = \dfrac{\sin OAC}{\sin OBC}\dfrac{\sin ABC}{\sin BAC}\dfrac{\sin OAD}{\sin OBD}\dfrac{\sin ABD}{\sin BAD} = \dfrac{\sin AOC}{\sin COB}\dfrac{\sin AOD}{\sin DOA} = 1$ and the lemma follows. $\square$



3.2 Example

(Sharygin 2018 9.7) Let B_1, C_1 be the midpoints of sides AC, AB of a triangle ABC respectively. The tangents to the circumcircle at B and C meet the rays CC_1, BB_1 at points K and L respectively. Prove that $\angle BAK = \angle CAL$.

Solution. Let $G = BB_1 \cap CC_1$ and $S = BK \cap CL$. Since AS is the symmedian in $\triangle ABC$, AG and AS are isogonal in $\angle BAC$. Now we're done by Isogonal lemma.



3.3 Practice problems

Problem 3.1. (Folklore) Cevians AA_1, BB_1, CC_1 of $\triangle ABC$ concur. Prove that $\angle B_1A_1A = \angle AA_1C_1 \iff AA_1 \perp BC$.

Problem 3.2. (Sharygin 2013/20, correspondence round) Let C_1 be an arbitrary point on the side AB of triangle ABC. Points A_1 and B_1 on the rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

Problem 3.3. (ISL 2006 G3) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Problem 3.4. (ISL 2007 G3) The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

Problem 3.5. (RMM 2016/1) Let ABC be a triangle and let D be a point on the segment $BC, D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E. The circle ACD meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

Problem 3.6. (ISL 2011 G4) Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Problem 3.7. (ELMO SL 2018 G5) Let scalene triangle ABC have altitudes AD, BE, CF and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines PE at $X \neq P$ and PF at $Y \neq P$. Prove that $XY \parallel BC$.

4 Linearity of PoP

4.1 Main lemma

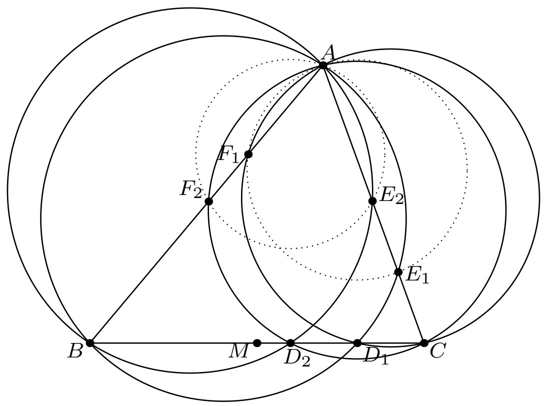
Let $P(X, \omega)$ denote the power of X with respect to ω. Then $P(X, \omega_1) - P(X, \omega_2)$ is a linear function of X.

Proof. Let $O_1 = (x_1, y_1)$ and $O_2 = (x_2, y_2)$ be the centers of ω_1, ω_2 and let r_1, r_2 be their radii. If $X = (x, y)$, then $P(X, \omega_1) - P(X, \omega_2) = PO_1^2 - r_1^2 - PO_2^2 + r_2^2 = (x - x_1)^2 + (y - y_1)^2 - (x - x_2)^2 - (y - y_2)^2 + r_2^2 - r_1^2 = x(-2x_1 - 2x_2) + y(-2y_1 - 2y_2) + x_1^2 - x_2^2 + y_1^2 - y_2^2 + r_2^2 - r_1^2$, which is a linear function of x and y. □

4.2 Example

(ELMO SL 2013 G3) In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Solution. Let D_1 and D_2 be two different points on BC and let E_1, F_1, E_2, F_2 be the corresponding intersection points. It suffices to prove that M, the midpoint of BC, lies on the radical axis of (AE_1F_1) and (AE_2F_2). By Linearity of PoP, $P(M, (AE_1F_1)) - P(M, (AE_2F_2)) = \frac{1}{2}(P(B, (AE_1F_1)) + P(C, (AE_1F_1)) - P(B, (AE_2F_2)) - P(C, (AE_2F_2))) = \frac{1}{2}(BA \cdot BF_1 + CA \cdot CE_1 - BA \cdot BF_2 - CA \cdot CE_1) = \frac{1}{2}(BC \cdot BD_1 + CB \cdot CD_1 - BC \cdot BD_2 - CB \cdot CD_2 = BC(BD_1 + CD_1) - BC(BD_2 + CD_2)) = 0$, as desired.



4.3 Practice problems

Problem 4.1. (USAMO 2013/1) In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

Problem 4.2. (RMM SL 2017 G3) Let $ABCD$ be a convex quadrilateral and let P and Q be variable points inside this quadrilateral so that $\angle APB = \angle CPD = \angle AQB = \angle CQD$. Prove that the lines PQ obtained in this way all pass through a fixed point, or they are all parallel.

Problem 4.3. (Inspired by the above problem) In trapezoid $ABCD$ with bases AB and CD, points P and Q are chosen such that $\angle BPC = \angle BQC = 180^\circ - \angle DPA = 180^\circ - \angle DQA$. If $U = AC \cap BD$ and $V = BC \cap DA$, prove that PQ passes through the projection of U onto the line through V and parallel to AB.

Problem 4.4. (Ukraine TST 2013/6) Let A, B, C, D, E, F be six points, no three collinear and no four concyclic. Let P, Q, R be the intersection points of perpendicular bisectors of pairs of segments (AD, BE), (BE, CF), (CF, DA), and P', Q', R' be the intersection points of perpendicular bisectors of pairs of segments (AE, BD), (BF, CE), (CA, DF). Show that $P \neq P'$, $Q \neq Q'$, $R \neq R'$ and prove that PP', QQ', RR' are concurrent or all parallel.

Problem 4.5. (IMO 2019/6) Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

5 Hints

Hint 1.2. Construct phantom point using Iran lemma.

Hint 1.4. Use midpoint of altitude lemma.

Hint 1.5. Draw tangents to the circumcircle at A, B, C.

Hint 1.6. Draw lines parallel to BC, CA, AB through A, B, C.

Hint 1.8. How are X and Y related to $\triangle AST$?

Hint 1.9. Construct points on the nine-point circle of BIC using Iran lemma. Prove that S lies on the polar of T wrt this circle.

Hint 2.1. Use the "degenerate case" of the lemma.

Hint 2.4. Find a quadrilateral similar to $ABCD$.

Hint 3.1. Use isogonal lemma with A_1 as the vertex of angle.

Hint 3.4. In the isogonal lemma, some intersection points may be at the infinity.

Hint 3.6. Use isogonal lemma with X as the vertex of angle.

Hint 3.7. Use isogonal lemma with P as the vertex of angle.

Hint 4.3. What do we know about P and Q from before?

Hint 4.4. The lemma also works for 0-radius circles.

Hint 4.5. Sum of two linear functions is linear.