Notes on the derived model theorem

Yasuda Yasutomo

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We shall prove the derived model theorem. This is based on [1].

Contents

1 Homogeneous, weakly homogeneous, universally Baire

In this section, we shall give basic results for tree representations. We shall show the following.

Theorem 1.1. Suppose λ is a limit of Woodin cardinals. Then for any set of reals A, the following are equivalent:

1. A is $<\lambda$-homogeneously Suslin,
2. A is weakly $<\lambda$-homogeneously Suslin,
3. A is λ-universally Baire.

Definition 1.2 (homogeneous tree)**.** A tree T on $X \times Y$ is κ-homogeneous if there is a sequence $\bar{\mu} = \langle \mu_s \mid s \in {}^{<\omega}X \rangle$ such that:

1. for each $s \in {}^{<\omega}X$, μ_s is a κ-complete ultrafilter over $T[s]$,
2. for each $s, t \in {}^{<\omega}X$ such that $s \subseteq t$, μ_t projects to μ_s,
3. for any $x \in p[T]$ and $\langle Z_n \mid n \in \omega \rangle$ such that $Z_n \in \mu_{x \restriction n}$ for all $n \in \omega$, there is a $f \in {}^{\omega}Y$ such that for all $n \in \omega$, $f \restriction n \in Z_n$.

Definition 1.3. Let A be a set of reals.

- A is κ-homogeneously Suslin if there is a κ-homogeneous tree T such that $A = p[T]$.
- A is $<\lambda$-homogeneously Suslin if A is κ-homogeneously Suslin for all $\kappa < \lambda$.

We give another characterization of homogeneity. [1]

Proposition 1.4. For a set of reals A, the following are equivalent:

1. A is κ-homogeneously Suslin,
2. there is a function
$$\bar{\mu}\colon {}^{<\omega}\omega \to \mathrm{meas}_\kappa(Y) = \{\mu \mid \mu \text{ is a } \kappa\text{-complete ultrafilter over } {}^{<\omega}Y\}$$
such that for any $s, t \in {}^{<\omega}\omega$,
   (a) $\dim(\mu_t) = \mathrm{dom}(t)$,
   (b) if $s \subseteq t$, then μ_t projects to μ_s,
   (c) If $S_{\bar{\mu}} = \{x \in \mathbb{R} \mid \langle \mu_{x\restriction n} \mid n \in \omega \rangle \text{ is countably complete}\}$, then $S_{\bar{\mu}} = A$.

Proof. The forward direction is clear. Given such $\bar{\mu}$, for each $x \in \mathbb{R} \setminus A$, let $\langle Z^x_n \mid n \in \omega \rangle$ witness that $\langle \mu_{x\restriction n} \mid n \in \omega \rangle$ is not countably complete. We set
$$(s, u) \in T \iff u \in Y^{\mathrm{dom}(s)} \wedge \forall x \supseteq s(x \notin A \implies u \in Z^x_{\mathrm{dom}(s)}).$$
Then this gives a κ-homogeneous tree projecting to A. □

Hom_κ denotes the collection of all κ-homogeneously Suslin set of reals. We set
$$\mathrm{Hom}_{<\kappa} = \bigcap_{\lambda<\kappa} \mathrm{Hom}_\lambda.$$
Hom_κ is closed under Wadge reducibility and contable intersection. Enough homogeneity gives determinacy.

Theorem 1.5 (Martin)**.** If a set of reals A is $\aleph_1$-homogeneously Suslin, then A is determined.

[1]We can define κ-homogeneously without using a tree. However Suslinness is central notion in descriptive set theory. So we use this definition.

Theorem 1.6 (Martin)**.** If κ is a measurable cardinal, then every $\boldsymbol{\Pi}^1_1$ set of reals is κ-homogeneously Suslin and determined.

Definition 1.7 (weakly homogeneous tree)**.** A tree T on $X \times Y$ is weakly κ-homogeneous if there is a sequence $\bar{\mu} = \langle \mu_{s,t} \mid (s,t) \in {}^{<\omega}X \oplus {}^{<\omega}\omega \rangle$ such that:

1. for each $(s,t) \in {}^{<\omega}X \oplus {}^{<\omega}\omega$, $\mu_{s,t}$ is a κ-complete ultrafilter over $T[s]$,
2. for each $(p,r),(q,s) \in {}^{<\omega}X \oplus {}^{<\omega}\omega$ such that $(p,r) \subseteq (q,s)$, $\mu_{q,s}$ projects to $\mu_{p,r}$,
3. for any $x \in p[T]$ and $\langle Z_s \mid s \in {}^{<\omega}\omega \rangle$ such that for all $s \in {}^{<\omega}\omega$, $Z_s \in \mu_{x\restriction \mathrm{lh}(s),s}$, there is a $y \in {}^{\omega}X$ and a $f \in {}^{\omega}Y$ such that for all $n \in \omega$, $f \restriction n \in Z_{x\restriction n, y\restriction n}$.

Definition 1.8. Let A be a set of reals.

- A is weakly κ-homogeneously Suslin if there is a weakly κ-homogeneous tree T such that $A = p[T]$.
- A is weakly $<\lambda$-homogeneously Suslin if A is weakly κ-homogeneously Suslin for all $\kappa < \lambda$.

Proposition 1.9. For a set of reals A, the following are equivalent:

1. A is weakly κ-homogeneously Suslin,
2. A is a projection of some κ-homogeneously Suslin set of reals.

Cleary, κ-homogeneously Suslin set is weakly κ-homogeneously Suslin.

Definition 1.10. Let T on $X \times Y$ and U on $X \times Y$ be trees. T and U are κ-absolute complements if for any $<\kappa$-generic G over V [2],

$$V[G] \models p[T] = {}^{\omega}X \setminus p[U].$$

We say T is κ-sbsolutely complemented if for some U, T and U are κ-absolutely complements.

Definition 1.11. A set of reals A is κ-universally Baire if there are κ-absolute complements T and U such that $A = p[T]$.

Proposition 1.12. Let (T,U) and (R,S) be pairs of κ-absolute complements such that $p[T] = p[R]$. Then for any $<\kappa$-generic G over V, $p[T] = p[R]$ in $V[G]$.

Given κ-universally Baire set A, it is well-defined to write $A^{V[G]}$ if G is $<\kappa$-generic. For any weakly homogeneous tree T, we can construct another tree which projects to the complement of $p[T]$. This construction is called Martin–Solovay construction. Let $\langle r_i \mid i \in \omega \rangle$ be a standard enumeration of ${}^{<\omega}\omega$. For weakly κ-homogeneous tree T on $X \times Y$ and a any $(p,r),(q,s) \in {}^{<\omega}X \oplus {}^{<\omega}\omega$ such that $(p,r) \subseteq (q,s)$, let

$$i_{(p,r),(q,s)} \colon \mathrm{Ult}\big(V, \mu_{(p,r)}\big) \to \mathrm{Ult}\big(V, \mu_{(q,s)}\big)$$

be a natural elementary embedding.

[2] for some poset $\mathbb{P}$ such that $|\mathbb{P}| < \kappa$, G is $\mathbb{P}$-generic.

Definition 1.13 (Martin–Solovay tree)**.** Let T be a weakly κ-homogeneous tree on $X \times Y$ with κ-homogeneous system $\bar{\mu}$ and α be ordinal. A Martin–Solovay tree $\mathrm{ms}(\bar{\mu}, \alpha)$ is a tree on $X \times \mathrm{Ord}$ consisting of a pair (p, t) such that:

1. $p \in {}^{<\omega}X$,
2. $t \in {}^{\mathrm{lh}(p)}\mathrm{Ord}$,
3. $t(0) < \alpha$,
4. for any $i, j < \mathrm{lh}\,(p)$, if $r_i \subsetneq r_j$, then

$$t(j) < i_{(p \upharpoonright \mathrm{lh}(r_i), r_i), (p \upharpoonright \mathrm{lh}(r_j), r_j)}(t(i)).$$

A infinite path of $\mathrm{ms}(\bar{\mu}, \alpha)$ gives a witness for ill-foundedness of a direct limit model. Then Martin–Solovay tree projects to the complements of $p[T]$.

Lemma 1.14. Let T be a weakly κ-homogeneous tree on $X \times Y$ with κ-homogeneous system $\bar{\mu}$. If $\alpha \geq |Y|^+$, then

$$p[\mathrm{ms}(\bar{\mu}, \alpha)] = \mathbb{R} \setminus p[T].$$

Theorem 1.15. Let T be a weakly κ-homogeneous tree on $X \times Y$ with κ-homogeneous system $\bar{\mu}$. For sufficiently large α, T and $\mathrm{ms}(\bar{\mu}, \alpha)$ are κ-absolute complements.

Proof. Let G be a $<\kappa$-generic over V. Then $\mathrm{ms}(\bar{\mu}, \theta)^V = \mathrm{ms}(\bar{\mu}^*, \theta)^{V[G]}$. By absoluteness, $p[T] = \mathbb{R} \setminus\ \mathrm{ms}(\bar{\mu}, \theta)$. □

Corollary 1.16. If a set of reals A is weakly κ-homogeneously Suslin, then A is κ-universally Baire.

Up to here we can show without any additional assumptions.

Theorem 1.17 (Martin–Steel)**.** Let δ be a Woodin cardinal and T be a weakly δ^+-homogeneous tree on $X \times Y$ with weakly δ^+-homogeneous system $\bar{\mu}$ such that $X \in V_\delta$. Then for sufficiently large θ, $\mathrm{ms}(\bar{\mu}, \theta)$ is $<\delta$-homogeneous.

This proof is very hard, so we omit it.

Corollary 1.18. Let δ be a Woodin cardinal and $A \subseteq \mathbb{R}^2$ be a δ^+-homogeneously Suslin. Then $\neg\exists A$ is $<\delta$-homogeneously Suslin.

Corollary 1.19. Let λ be a limit of Woodin cardinals. Then $\mathrm{Hom}_{<\lambda}$ is closed under $\exists^{\mathbb{R}}$, negation and Wadge reducibility.

Corollary 1.20. If there are ω-many Woodin cardinals, then PD holds.

Lemma 1.21. Let λ be a limit of Woodin cardinals. Then there is a ordinal $\gamma_0 < \lambda$ such that $\mathrm{Hom}_{\gamma_0} = \mathrm{Hom}_{<\lambda}$.

Proof. Suppose not. Then for any $\kappa < \lambda$, $\mathrm{Hom}_\kappa \supsetneq \mathrm{Hom}_{<\lambda}$. This gives an infinite descending $<_w$-chain. However Martin's proof that $<_w$ is well-founded gives a contradiction. $\square$

Proposition 1.22. For a tree T on $\omega \times Y$, the following are equivalent:

1. T is weakly κ-homogeneous,
2. there is a countable collection Σ of κ-complete ultrafilter over ${}^{<\omega}Y$ such that, $x \in p[T]$ if and only if there is a countably complete tower $\langle \mu_n \mid n \in \omega\rangle \in {}^{\omega}\Sigma$ such that for all $n \in \omega$, $T[x \restriction n] \in \mu_n$.

Proof. The forward direction is clear. We may assume that Σ is closed under projections. Define $\mu_{s,t}$ for each $(s,t) \in {}^{<\omega}\omega \oplus {}^{<\omega}\omega$, by induction on $|s|$:

$$\{\mu \cap \mathcal{P}(T[s^\frown\langle i\rangle]) \mid \mu \in \Sigma \wedge T[s^\frown\langle i\rangle] \wedge \mu_{s,t} \text{ is a projection of } \mu\} \\ \subseteq \{\mu_{s^\frown\langle i\rangle, t^\frown\langle j\rangle} \mid j \in \omega\}.$$

We allow principal ultrafilters if necessary. Then this gives a weaky homogeneous system for T. $\square$

Theorem 1.23. Suppose that δ is a Woodin cardinal and $A \subseteq \mathbb{R}$ is δ^+-universally Baire. Then A is determined.

Theorem 1.24 (Woodin)**.** Let δ be a Woodin cardinal. If T and U are trees on $\omega \times Z$ and δ^+-absolute complements, then T is weakly $<\delta$-homogeneous.

Proof. Let T and U be trees on $\omega \times Z$ and δ^+-absolute complements. Let η be a sufficiently large regular cardinal such that $T, U \in V_\eta$. Then let T' and U' be the subtrees of T and U consisting all nodes definable over V_η from T, U, δ. Then $|T'| = |U'| = \delta$. A $\mathbb{Q}_{<\delta}$-name for a real can be taken as a $\mathbb{Q}_{<\kappa}$-name for some inaccessible $\kappa < \delta$, then we can find it in V_δ. So

$$\mathbb{Q}_{<\delta} \Vdash p[\check{T}'] = \mathbb{R} \setminus p[\check{U}']$$

and $p[T] = p[T']$. By proposition 1.22, if T' is weakly homogeneous, then so is T. Then we may assume that T and U are trees on $\omega \times \delta$. Fix $\kappa < \delta$. We may assume that κ is $<\delta$-T-strong. For each $\kappa < \lambda < \delta$, let $j_\lambda \colon V \to M_\lambda$ be a λ-T-storng embedding with a critical point κ. Put Σ_λ for each $(s,u) \in T \cap V_\lambda$ and $X \subseteq \kappa^{<\omega}$,

$$X \in \Sigma_\lambda(s,u) \iff u \in j_{\lambda(X)}.$$

Claim 1. For each $(s,u), (t,v) \in T \cap V_\lambda$,

1. $\Sigma_\lambda(s,u)$ is κ-complete ultrafilter,
2. $T[s] \cap V_\kappa \in \Sigma_\lambda(s,u)$,
3. if $(s,u) \subseteq (t,v)$, then $\Sigma_\lambda(t,v)$ projects to $\Sigma_\lambda(s,u)$,
4. for any $(x,f) \in [T \cap V_\lambda]$, $\langle \Sigma_\lambda(x \restriction n, f \restriction n) \mid n \in \omega\rangle$ is countably complete.

Proof of claim 1. Easy. □

Let G be a $\mathbb{Q}_{<\delta}$-generic over V and let $i\colon V \to N \subseteq V[G]$ be a canonical embedding.

Claim 2. $i(T)$ is weakly $i(\kappa)$-homogeneous in N.

Proof of claim 2. We show that $\sigma = i\text{"}\mathrm{meas}_\kappa(\kappa)$ witnesses that $i(T)$ is weakly $i(\kappa)$-homogeneous. By closure of N, $\sigma \in N$. For any $x \in p[i(T)] \cap N$, $x \in p[T]$, since we have that $p[T] \subseteq p[i(T)]$, $p[U] \subseteq p[i(U)]$ and $p[i(T)] \cap p[i(U)] = \emptyset$. Take a sufficiently large $\lambda < \delta = \omega_1^{V[G]}$ so that $x \in p[T \cap V_\lambda]$. Let f be such that $(x, f) \in [T \cap V_\lambda]$. By elementarity, $\langle i(\Sigma_\lambda)(x \restriction n, i(f) \restriction n) \mid n \in \omega \rangle$ is countably complete in N. Then $i(T)$ is weakly $i(\kappa)$-homogeneous in N . □

By elementarity, T is weakly κ-homogeneous in V. □

We have shown the following.

Theorem 1.25. Suppose λ is a limit of Woodin cardinals. Then for any set of reals A, the following are equivalent:

1. A is $<\lambda$-homogeneously Suslin,
2. A is weakly $<\lambda$-homogeneously Suslin,
3. A is λ-universally Baire.

2 The tree production lemma

We shall show the tree production lemma. Roughly speaking, formulas with certain generic absoluteness define universally Baire sets. Let φ be a formula and a be a parameter. Suppose that $X \prec_n V$ is a sufficiently elementary submodel with $\kappa, a \in X$. Let $\pi\colon X \simeq N$ be a collapsing map. X is (φ, a, κ)-generically correct if and only if for any poset $\mathbb{P} \in H^N_{\pi(\kappa)}$, any $\mathbb{P}$-generic $g \in V$ over N and any $x \in N[g] \cap \mathbb{R}$,

$$N[g] \models \varphi[x, \pi(a)] \iff V \models \varphi[x, a].$$

Lemma 2.1. Let $\varphi(v_0, v_1)$ be a formula, let a be a parameter and let κ be an infinite cardinal. Suppose that M and σ are such that

- M is transitive,
- $H_\kappa \cup \{\kappa\} \subseteq M$,
- $\sigma\colon M \to V$ is sufficiently elementary so that $a \in \mathrm{ran}(\sigma)$ and $\mathrm{cp}\,(\sigma) > \kappa$.

Suppose that C is a collection of countable $X \prec M$ such that $\sigma\text{"}X$ is (φ, a, κ)-generically correct, and C contains a club of $\mathcal{P}_{\omega_1}(M)$. Then there are trees T and U such that for any $\mathbb{P} \in H_\kappa$ and any $\mathbb{P}$-generic G over V,

$$V[G] \models p[T] = \{x \in \mathbb{R}^{V[G]} \mid \varphi[x, a]\} \wedge p[U] = \{x \in \mathbb{R}^{V[G]} \mid \neg\varphi[x, a]\}.$$

Proof. Let $F\colon M^{<\omega} \to M$ be such that for any $X \in \mathcal{P}_{\omega_1}(M)$, if $F"X^{<\omega} \subseteq X$, then $X \prec M$ and $\sigma"X$ is (φ, a, κ)-generically correct. Put $\sigma(a^*) = a$. Let T be a tree on $\omega \times M \times H_\kappa$ such that $(s, t, u) \in T$ if and only if

1. if $2m + 2 < \mathrm{lh}\,(s)$, then $t(2m + 2) = F(t \circ r_m)$,
2. if $\mathrm{lh}\,(s) > 0$, then $t(0) = \mathbb{P} \in H_\kappa$ is a poset,
3. if $\mathrm{lh}\,(s) > 1$, then $t(1)$ is a $\mathbb{P}$-name and $u(0) \Vdash t(1)\colon \omega \to \omega$,
4. if $2m + 3 < \mathrm{lh}\,(s)$, then $t(2m + 3) = u(m)$,
5. $\{u(m) \mid u < \mathrm{lh}\,(s)\} \subseteq \mathbb{P}$ has a common extension,
6. if $2m + 2 < \mathrm{lh}\,(s)$ and $t(m)$ is a $\mathbb{P}$-dense subset, then $u(2m + 2) \in t(m)$,
7. if $2m + 3 < \mathrm{lh}\,(s)$, then $u(2m + 3) \Vdash t(1)(m) = s(m)$,
8. if $\mathrm{lh}\,(s) > 1$, then $u(1) \Vdash \varphi[t(1), \check{a}^*]$.

We also define a tree U on $\omega \times M \times H_\kappa$ replacing $\varphi[t(1), \check{a}^*]$ by $\neg\varphi[t(1), \check{a}^*]$.

Claim 1.

- $p[T] = \{x \in \mathbb{R} \mid \varphi[x, a]\}$.
- $p[U] = \{x \in \mathbb{R} \mid \neg\varphi[x, a]\}$.

Proof of claim 1. Fix $x \in p[T]$. Put $(x, f, g) \in [T]$ and $X = \mathrm{ran}(f)$. By definition of T, $F"X^{<\omega} \subseteq X$. Hence $X \prec M$ and $\sigma"X$ is (φ, a, κ)-generically correct. Let $\mathbb{P} \in X \cap H_\kappa$ be a poset, let G be a $\mathbb{P}$-generic over X and $\tau \in X$ be a $\mathbb{P}$-name such that

1. there is a $p \in G$ such that $p \Vdash \dot{\tau}\colon \omega \to \omega$,
2. for all $n \in \omega$, there is a $p_n \in G$ such that $p_n \Vdash \dot{\tau}(n) = x(n)$,
3. there is a $q \in G$ such that $q \Vdash \varphi[\dot{\tau}, \check{a}^*]$.

Since σ is sufficiently elementary, $\sigma"X$ satisfies the same as X. As $\sigma"X$ is (φ, a, κ)-generically correct, $V \models \varphi[x, a]$. □

Suppose that $\mathbb{P} \in H_\kappa$ is a poset and G is $\mathbb{P}$-generic over V.

Claim 2.

- for all $x \in \mathbb{R}^{V[G]}$, if $V[G] \models \varphi[x, a]$, then $x \in p[T]^{V[G]}$.
- for all $x \in \mathbb{R}^{V[G]}$, if $V[G] \models \neg\varphi[x, a]$, then $x \in p[U]^{V[G]}$.

Proof of claim 2. Let $x \in \mathbb{R}^{V[G]}$ be such that $V[G] \models \varphi[x, a]$ and let τ be such that $\tau_G = x$. Let $X' \prec M[G]$ be such that $X = X' \cap M$ is closed under F and such that $\mathbb{P} \in X$. Let $f\colon \omega \to X$ and $g\colon \omega \to X \cap G$ be surjection such that they obey the definition of T. Hence $x \in p[T]^{V[G]}$. □

□

Theorem 2.2 (Tree production lemma, Woodin)**.** Let $\varphi(v_0, v_1)$ be a formula and let a be a parameter. Let δ be a Woodin cardinal. Suppose that

- (generic absoluteness) if G is $<\delta$-generic over V and H is $<\delta^+$-generic over $V[G]$, then for any $x \in \mathbb{R} \cap V[G]$,
$$V[G] \models \varphi[x, a] \iff V[G][H] \models \varphi[x, a],$$
- (stationary tower correctness) if G is $\mathbb{Q}_{<\delta}$-generic over V and $j \colon V \to M \subseteq V[G]$ is a canonical embedding, then for any $x \in \mathbb{R} \cap V[G]$,
$$V[G] \models \varphi[x, a] \iff M \models \varphi[x, j(a)].$$

Then there are trees T and U such that for any $<\delta$-generic G over V,

$$V[G] \models p[T] = \{x \in \mathbb{R}^{V[G]} \mid \varphi[x, a]\} \wedge p[U] = \{x \in \mathbb{R}^{V[G]} \mid \neg\varphi[x, a]\}.$$

Then $\{x \in \mathbb{R} \mid \varphi[x, a]\}$ is δ-universally Baire.

Proof. It is enough to construct T_κ and U_κ satisfying the claim for each $\kappa < \delta$. Fix $\kappa < \delta$. Let M be transitive, let σ and a^* be such that

- $H_{\kappa^+} \subseteq M$,
- $|M| < \delta$,
- $\sigma \colon M \to V$ is sufficiently elementary, $a = \sigma(a^*)$ and $\sigma \restriction \kappa^+ = \mathrm{id}$.

Let $D \subseteq \mathcal{P}_{\omega_1}(M)$ be the collection of countbale $X \prec M$ such that σ"X is (φ, a, κ)-generically correct. It is enough to show that D contains a club. Otherwise. Suppose that $(\mathcal{P}_{\omega_1}(M) \setminus D) \in \mathbb{Q}_{<\delta}$. Let G be $\mathbb{Q}_{<\delta}$-generic over V such that $(\mathcal{P}_{\omega_1}(M) \setminus D) \in G$, let $j \colon V \to N \subseteq V[G]$ be a canonical embedding. Simce j"$M \in j(\mathcal{P}_{\omega_1}(M) \setminus D)$ and j"$M \prec j(M)$, $j(\sigma)$"$(j$"$M)$ is not $(\varphi, j(a), j(\kappa))$-generically correct in N. Note that the transitive collapse of $j(\sigma)$"$(j$"$M)$ is M. Let $g \in N$ be $<\kappa$-generic over M. Then for any $x \in \mathbb{R} \cap M[g]$,

$$\begin{aligned} M[g] \models \varphi[x, a^*] &\iff V[g] \models \varphi[x, a] \\ &\iff V[G] \models \varphi[x, a] \\ &\iff N \models \varphi[x, j(a)] \end{aligned}$$

Then $j(\sigma)$"$(j$"$M)$ is $(\varphi, j(a), j(\kappa))$-generically correct in N. This is a contradiction. □

3 Scales on $\mathrm{Hom}_{<\lambda}$ sets

We shall show the following.

Theorem 3.1 (Steel)**.** Suppose λ is a limit of Woodin cardinals. Then $\mathrm{Hom}_{<\lambda}$ has the scale property.

Suppose that δ is a Woodin cardinal. Let G be a $(V, \mathbb{Q}_{<\delta})$-generic and $j\colon V \to M \subseteq V[G]$ be a canonical embedding.

Proposition 3.2. Suppose that δ is a Woodin cardinal. Let G be a $\mathbb{Q}_{<\delta}$-generic over V and $j\colon V \to M \subseteq V[G]$ be a canonical embedding. Let μ be a δ^+-complete ultrafilter over I with $\mu \in V$. Then for any $A \subseteq j(I)$ such that $A \in M$,

$$A \in j(\mu) \iff \exists B \in \mu (j(B) \subseteq A).$$

Proof. For the forward direction, fix $A \in j(\mu)$. Put $A = [f]_G$ where $f \in V$. We may assume that $f\colon \mathcal{P}_{\omega_1}(Z) \to \mu$ for some $Z \in V_\delta$. Put

$$B = \bigcup \{ f(X) \mid X \in \mathcal{P}_{\omega_1}(Z) \} \in \mu.$$

Then we have that $j(B) \subseteq A$. The other direction is clear. $\square$

Proposition 3.3. Suppose that δ is a Woodin cardinal. Let G be a $\mathbb{Q}_{<\delta}$-generic over V and $j\colon V \to M \subseteq V[G]$ be a canonical embedding. Let μ be a δ^+-complete ultrafilter over I with $\mu \in V$. Then $\mathrm{Ult}(M, j(\mu)) \simeq \mathrm{Ult}^*(M, \mu^*)$ where $\mathrm{Ult}^*(M, \mu^*)$ is a ultrapower of M using functions $f\colon I \to M$ in $V[G]$.

Proof.

Claim 1. Suppose that $f\colon I \to M$ is in V[G]. Then for some $h \in M$,

$$f(u) = h(j(u)) \text{ for } \mu\text{-a.e. } u.$$

Proof of claim 1. Working in $V[G]$. Let $\dot{f}_G = f$. For each $u \in I$, let $g \in V$, $Z_u \in V_\delta$ and $a_u \in G$ be such that $\mathrm{dom}(g) = \mathcal{P}_{\omega_1}(Z_u)$ and $a_u \Vdash \dot{f}(\check{u}) = [\check{g}]_{\dot{G}}$. Since μ^* is δ^+-complete, let a and Z be such that $a_u = a$ and $Z_u = Z$ for μ^*-a.e. u. Say on $B \in \mu^*$. Working in V. For each $u \in B$, take g_u with a domain $\mathcal{P}_{\omega_1}(Z)$ such that $a \Vdash \dot{f}(\check{u}) = [\check{g_u}]_{\dot{G}}$. Define for each $X \in \mathcal{P}_{\omega_1}(Z)$, $h_X(u) = g_u(X)$. Then for all $u \in B$,

$$[\lambda X. h_X]_G(j(u)) = [g_u]_G = f(u).$$

$\square$

Define $\pi\colon \mathrm{Ult}^*(M, \mu^*) \to \mathrm{Ult}(M, j(\mu))$ by

$$\pi([f]_{\mu^*}) = [h_f]_{j(\mu)}$$

where $h_f \in M$ is such that $h_f(j(u)) = f(u)$ for μ^*-a.e. u. It is easy to show that π is well-defined and an isomorphism between $\mathrm{Ult}(M, j(\mu))$ and $\mathrm{Ult}^*(M, \mu^*)$. $\square$

Proposition 3.4. Suppose that δ is a Woodin cardinal. Let G be a $\mathbb{Q}_{<\delta}$-generic over V and $j\colon V \to M \subseteq V[G]$ be a canonical embedding. Let μ be a δ^+-complete ultrafilter over I with $\mu \in V$. Let $\nu \leq_{RK} \mu$ via $p\colon I \to J$. Let

$$i\colon \mathrm{Ult}(V,\nu) \to \mathrm{Ult}(V,\mu)$$

be an induced embedding and

$$i^*\colon \mathrm{Ult}(V[G],\nu^*) \to \mathrm{Ult}(V[G],\mu^*)$$

be a liftup. Let

$$j(i)\colon \mathrm{Ult}(M,j(\nu)) \to \mathrm{Ult}(M,j(\mu))$$

be an induced embedding. Then $j(i) = i^* \restriction \mathrm{Ult}(M,j(\nu))$.

Proof. It is enough to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ult}^*(M,\nu^*) \arrow[r, "i^*"] \arrow[d, leftrightarrow, "\sigma"'] & \mathrm{Ult}^*(M,\mu^*) \arrow[d, leftrightarrow, "\tau"] \\
\mathrm{Ult}(M,j(\nu)) \arrow[r, "j(i)"] & \mathrm{Ult}(M,j(\mu))
\end{tikzcd}$$

where σ and τ are isomorphisms. The rest is easy to show. $\square$

Corollary 3.5. Suppose that δ is a Woodin cardinal. Let G be a $\mathbb{Q}_{<\delta}$-generic over V and $j\colon V \to M \subseteq V[G]$ be a canonical embedding.

1. Let $\bar{\mu}^* = \langle \mu_s^* \mid s \in {}^{<\omega}\omega \rangle$ is a homogeneity system in $V[G]$, where each μ_s is a δ^+ complete ultrafilter in V. Then for any γ,

$$\mathrm{ms}(\bar{\mu}^*,\gamma)^{V[G]} = \mathrm{ms}(\langle j(\mu_s) \mid s \in {}^{<\omega}\omega \rangle, \gamma)^M$$

   So $(S_{\bar{\mu}^*})^{V[G]} = (S_{\langle j(\mu_s) \mid s \in {}^{<\omega}\omega \rangle})^M$.

2. Let $\lambda > \delta$ be a limit of Woodin cardinals. Then $\mathrm{Hom}_{<\lambda}^{V[G]}$ is a Wadge initial segment of $j(\mathrm{Hom}_{<\lambda})$.

Theorem 3.6 (Steel)**.** Suppose λ is a limit of Woodin cardinals. Then $\mathrm{Hom}_{<\lambda}$ has the scale property.

Proof. Let $\kappa_0 < \lambda$ be such that $\mathrm{Hom}_{\kappa_0} = \mathrm{Hom}_{<\lambda}$. Let δ_0, δ_1, δ_2 be Woodin cardinals such that $\kappa_0 < \delta_0 < \delta_1 < \delta_2 < \lambda$. Fix $B \in \mathrm{Hom}_{<\lambda}$. It is enough to show that there is a scale $\bar{\varphi} = \{\varphi_n\}_{n\in\omega}$ on B such that

$$A(n,x,y) \iff \varphi_n(x) \leq \varphi_n(y)$$

is δ_1^+-universally Baire. Then A is κ_0-homogeneous. Let $\bar{\mu} = \langle \mu_s \mid s \in {}^{<\omega}\omega \rangle$ be a δ_2^+-homogeneity system such that

$$\mathbb{R} \setminus B = S_{\bar{\mu}}.$$

Put $T = \mathrm{ms}(\bar{\mu}, \theta)$ for sufficiently large θ. Let $\bar{\varphi} = \{\varphi_n\}_{n\in\omega}$ be a left-most branch scale of T. Then $\bar{\varphi}$ is a scale on $B = p[T]$. For definability, we use the tree production lemma. Let $\varphi(v_0, T)$ be the definition of A using a parameter T. For generic absoluteness, let G be $< \delta_2$-generic over V and let H be δ_2-generic over $V[G]$. Then for any $x \in V[G]$,

$$x \in p[T]^{V[G]} \iff x \in p[T]^{V[G[H]}$$

and

$$l_x^{V[G]} = l_x^{V[G][H]}$$

where l_x is the left-most branch of T_x. Then

$$V[G] \models l_x \restriction n \leq_{\mathrm{lex}} l_y \restriction n \iff V[G][H] \models l_x \restriction n \leq_{\mathrm{lex}} l_y \restriction n.$$

For stationary tower corectness. Let G be $\mathbb{Q}_{<\delta_2}$-generic over V and let $j\colon V \to M \subseteq V[G]$ be a canonical embedding. Since θ is sufficiently large, $j(\theta) = \theta$. Then

$$\begin{aligned} j(T) &= j(\mathrm{ms}(\langle \mu_s \mid s \in {}^{<\omega}\omega, \theta)) \\ &= \mathrm{ms}(\langle j(\mu_s) \mid s \in {}^{<\omega}\omega\rangle, \theta) \\ &= \mathrm{ms}(\langle \mu_s^* \mid s \in {}^{<\omega}\omega\rangle, \theta) \\ &= T. \end{aligned}$$

Then

$$V[G] \models \varphi[(n, x, y), T] \iff M \models \varphi[(n, x, y), j(T)].$$

□

4 Projective absoluteness

Theorem 4.1 (Woodin)**.** Let λ be a limit of Woodin cardinals, let $A \in \mathrm{Hom}_{<\lambda}$ and let G be $<\lambda$-generic over V. Then

$$(\mathrm{HC}^V, \in, A) \equiv (\mathrm{HC}^{V[G]}, \in, A^{V[G]}).$$

Proof. Fix $A \in \mathrm{Hom}_{<\lambda}$. For each formula $\varphi(\vec{v}, A)$ and each $\kappa < \lambda$, we shall construct κ-homogeneous tree $T_{\varphi,\kappa}$ such that for any $<\kappa$-generic G over V,

$$V[G] \models p[T_{\varphi,\kappa}] = \{\vec{y} \in \mathbb{R}^{<\omega} \mid \varphi[\vec{y}, A^{V[G]}]\}.$$

By induction on a complexity of φ. If $\varphi \equiv \exists x \psi(x, A)$ where ψ is Σ_0, then $T_{\varphi,\kappa}$ is a κ-homogeneous tree of A. Suppose that $\varphi \equiv \neg\exists w \psi$. Let δ be a Woodin cardinal such that $\kappa < \delta < \lambda$ and let T_{ψ,δ^+} be a δ^+-homogeneous tree on $\omega \times \omega \times Z$ which we have defined. Let S be a weakly δ^+-homogeneous tree on $\omega \times \omega \times Z$ such that $p[S] = \exists^{\mathbb{R}} p[T_{\psi,\delta^+}]$ with a weak homogeneity system $\bar{\mu}$. This equality holds in any generic extension. Put $T_{\varphi,\kappa} = \mathrm{ms}(\bar{\mu}, \theta)$ for sufficiently large θ. □

Theorem 4.2 ($(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$-absoluteness, Woodin)**.** Let λ be a limit of Woodin cardinals, let $A \in \mathrm{Hom}_{<\lambda}$ and let G be $<\lambda$-generic over V. Let φ be a formula in $\mathcal{L}_{\in,\dot{A},\dot{B}}$. Then

$$\exists B \in \mathrm{Hom}^V_{<\lambda}(\mathrm{HC}^V, \in, A, B) \models \varphi \iff$$
$$\exists B \in \mathrm{Hom}^{V[G]}_{<\lambda}(\mathrm{HC}^{V[G]}, \in, A^{V[G]}, B) \models \varphi.$$

Proof. Forward direction follows from the projective absoluteness. Let $\delta < \lambda$ be a sufficiently large Woodin cardinal and let H be $\mathbb{Q}_{<\delta}$-generic over V such that $G \in V[H]$. Since

$$\exists B \in \mathrm{Hom}^{V[G]}_{<\lambda}(\mathrm{HC}^{V[G]}, \in, A^{V[G]}, B) \models \varphi,$$

then

$$\exists B \in \mathrm{Hom}^{V[H]}_{<\lambda}(HC^{V[H]}, \in, A^{V[H]}, B) \models \varphi.$$

Let $j\colon V \to M \subseteq V[H]$ be a canonical embedding. Since $\mathrm{Hom}^{V[H]}_{<\lambda}$ is a Wadge initial segment of $j(\mathrm{Hom}_{<\lambda})$ and $j(A) = A^{V[H]}$,

$$M \models \exists B \in j(\mathrm{Hom}_{<\lambda})(HC, \in, j(A), B) \models \varphi.$$

By elementarity of j,

$$\exists B \in \mathrm{Hom}^V_{<\lambda}(\mathrm{HC}^V, \in, A, B) \models \varphi.$$

□

5 AD$^+$

AD$^+$ is a technical variant of AD and a generalization of AD$^{L(\mathbb{R})}$.

Definition 5.1. Θ is the supremum of λ such that there is a surjection from $\mathbb{R}$ to λ.

Under $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$, Θ is a limit cardinal by Moschovakis' coding lemma.

Definition 5.2 (Woodin)**.** AD$^+$ is the conjunction of the following statements.

1. $\mathrm{DC}_{\mathbb{R}}$,
2. Every set of reals is ∞-Borel,
3. $<\Theta$-ordinal determinacy.

It it a open question whether AD implies AD$^+$. For $n \in \omega$,

$$A_n(x) \iff x(n_0) = n_1.$$

Definition 5.3 (∞-Borel code)**.** Let $\kappa \in \mathrm{Ord}$. $S = \langle T, \varphi \rangle$ is a κ-Borel code if T is a well-founded tree on some $\lambda < \kappa$ and φ is a function from the collection of terminal nodes of T to $\{A_n \mid n \in \omega\}$. BC_κ denotes the collection of κ-Borel codes. $\mathrm{BC}_\infty = \bigcup_{\kappa \in \mathrm{Ord}} \mathrm{BC}_\kappa$.

It is useful to regard an ∞-Borel code as a tree on for some ordinal κ. Let $\langle T, \varphi \rangle$ be an ∞-Borel code. φ can be extended to T:

$$\varphi(s) = \mathbb{R} \setminus \bigcup_{t \supseteq s, \mathrm{lh}(t) = \mathrm{lh}(s)+1} \varphi(t).$$

Put $A_T = \varphi(T) = \varphi(\emptyset)$.

Definition 5.4. A set of reals A is ∞-Borel if $A = \varphi(T)$ for some $\langle T, \varphi \rangle \in \mathrm{BC}_\infty$.

Proposition 5.5. For a set of reals A, the following are equivalent:

1. A is ∞-Borel,
2. there is a formula φ and $S \subseteq \kappa$ for some $\kappa \in \mathrm{Ord}$, such that

$$A(x) \iff L[S, x] \models \varphi[S, x].$$

Corollary 5.6. Every Suslin set of reals is ∞-Borel.

Being ∞-Borel is local property. For a set of reals A, let $\Pi(A)$ be either $\mathbf{\Pi}^1_1(A)$ or $\mathbf{\Pi}^1_2(A)$ which has the prewellodering property, and let

$$\Delta(A) = \Pi(A) \cap \check{\Pi}(A).$$

Let $\delta(A)$ be a prewellodering of $\Pi(A)$.

Theorem 5.7. For any ∞-Borel set A, A has an ∞-Borel code in $\Delta(A)$ with respect to any $\Pi(A)$-norm on a $\Pi(A)$-complete set.

Proof. Let T be an ∞-Borel code for A. For a ∞-Borel code T, let A_T be an interpretation of T. Let $\equiv$ be an equivalence relation on BC_∞:

$$T \equiv T' \iff A_T = A_{T'}.$$

If $\mathrm{BC}^{L[A,\equiv]}_{\delta(A)} = \mathrm{BC}^{L[A,\equiv]}_\infty$, then A has an ∞-Borel code in $\Delta(A)$. Otherwise. Then there is an $\mathrm{BC}_{\delta(A)}$-antichain $\langle S_\alpha \mid \alpha < \delta(A) \rangle$ of length $\delta(A)$. This gives a prewellodering $\leq^*$ of length $\delta(A)$ with a code $S = \bigvee_{\alpha \leq \beta} S_\alpha \times S_\beta$ in $\mathrm{BC}_{\delta(A)^+}$. Then either $A \leq_w \leq^*$ or $A \leq_w \not\leq^*$. $\square$

Theorem 5.8 (Solovay's basis theorem)**.** If

$$L(\mathbb{R}) \models \exists A \subseteq \mathbb{R} \varphi[A]$$

where φ is $\mathbf{\Sigma}^2_1$, then

$$L(\mathbb{R}) \models \exists A \in \mathbf{\Delta}^2_1 \varphi[A].$$

Theorem 5.9 (Martin–Steel)**.** Assume $\mathrm{AD}^{L(\mathbb{R})}$. Then $(\mathbf{\Sigma}^2_1)^{L(\mathbb{R})}$ has the scale property.

Corollary 5.10. Assume $\mathrm{AD} + V = L(\mathbb{R})$. Then every set of reals is ∞-Borel.

Theorem 5.11 (Moschovakis–Woodin)**.** Assume $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$. Let $\lambda < \Theta$. If $A \subseteq \lambda^\omega$ is Suslin-co-Suslin, then A is determined.

Theorem 5.12 (Kechris)**.** Assume $\mathrm{AD} + V = L(\mathbb{R})$. Then $\mathrm{DC}_{\mathbb{R}}$ holds.

Corollary 5.13. Assume $V = L(\mathbb{R})$. Then AD implies AD^+.

For more details, see [3] and [4].

6 The derived model theorem

We will show a part of the old derived model theorem. Let λ be a limit of Woodin cardinals and G be $\mathrm{Coll}\,(\omega, < \lambda)$-generic over V. Put

$$\mathbb{R}^* = \mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha]$$

and

$$\begin{aligned}\mathrm{Hom}^* = \mathrm{Hom}^*_G = \{p[T] \cap \mathbb{R}^* \mid \exists \alpha < \lambda (T \in V[G \restriction \alpha] \wedge \\ V[G \restriction \alpha] \models T \text{ is } \lambda\text{-absolutely complemented})\}.\end{aligned}$$

Or equivalently, for any $\alpha < \lambda$ and $A \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}$, define

$$A^* = \bigcup_{\alpha<\beta<\lambda} A^{V[G\restriction\beta]}$$

and put

$$\mathrm{Hom}^* = \{A^* \mid \exists \alpha < \lambda (A \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda})\}.$$

Then $L(\mathbb{R}^*, \mathrm{Hom}^*)$ is called a derived model of V at λ. The model depends on G, but by homogeneity, first order theory of $L(\mathbb{R}^*, \mathrm{Hom}^*)$ doesn't depend on G.

Theorem 6.1 (Derived model theorem, Woodin)**.** Let λ be a limit of Woodin cardinals and $L(\mathbb{R}^*, \mathrm{Hom}^*)$ be a derived model at λ. Then

1. $\mathbb{R}^* = \mathbb{R}^{V[G]} \cap L(\mathbb{R}^*, \mathrm{Hom}^*)$,
2. $\mathrm{Hom}^* = \{A \subseteq \mathbb{R}^* \mid A \text{ is Suslin-co-Suslin in } L(\mathbb{R}^*, \mathrm{Hom}^*)\}$,
3. $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathrm{AD}^+$.

we will show the reflection theorem in the next section.

Theorem 6.2. Let λ be a limit of Woodin cardinals, let G be $\mathrm{Coll}(\omega, <\lambda)$-generic over V, and let φ be a formula in $\mathcal{L}_{\in,\dot{A},\dot{B}}$. Suppose that $A \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}$ for some $\alpha < \lambda$. If

$$\exists B \subseteq \mathbb{R}^*_G(B \in L(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \wedge (HC^*_G, \in, A^*, B) \models \varphi),$$

then

$$\exists B(B \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda} \wedge (HC^{V[G\restriction\alpha]}, \in, A, B) \models \varphi).$$

Corollary 6.3. Let $L(\mathbb{R}^*, \mathrm{Hom}^*)$ be a derived model. Then for any $A \in \mathrm{Hom}^*$, Every $\Sigma^2_1(A)$-fact has a Suslin-co-Suslin-witness.

It is worth mentioning that AD^+ is characterized by Σ_1-reflection. See [18].

Theorem 6.4 (Woodin)**.** Assume $\mathrm{AD} + V = L(\mathcal{P}(\mathbb{R}))$. Then the following are equivalent:

1. AD^+,
2. Letting $\mathcal{S} = \{A \subseteq \mathbb{R} \mid A \text{ is Suslin-co-Suslin}\}$, $M_{\mathcal{S}} \prec_{\Sigma_1} V$. [3]

Proposition 6.5. Let λ be a limit of Woodin cardinals, let G be $\mathrm{Coll}(\omega, <\lambda)$-generic over V. Then for any $\alpha < \lambda$ and $A \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}$,

$$(HC^{V[G\restriction\alpha]}, \in, A) \prec (HC^*, \in, A^*).$$

Proof. This follows by the projective absoluteness and Tarski's elementary chain lemma. $\square$

Proof of theorem 3.1 modulo theorem 3.2. Fix $x \in \mathbb{R}^{V[G]} \cap L(\mathbb{R}^*, \mathrm{Hom}^*)$. Let $A^* \in \mathrm{Hom}^*$ and $y \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ be such that x is ordinal definable over $L(\mathbb{R}^*, \mathrm{Hom}^*)$ from A^*, y. We may assume that $A, y \in V[G \restriction \alpha]$ for some $\alpha < \lambda$. Since the forcing is sufficiently homogeneous, then $x \in V[G \restriction \alpha]$. To show that Hom^* set is Suslin-co-Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, it is enough to show that Hom^* set is Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$. This follows by theorem 3.1, since we can express using only real quantifications with a code of a scale. For the converse. Let A be Suslin-co-Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$ witnessed by $T, U \in L(\mathbb{R}^*, \mathrm{Hom}^*)$. By the same argument, $T, U \in V[G \restriction \alpha]$ for some $\alpha < \lambda$. Since $p[T] = \mathbb{R}^* \setminus p[U]$, for all $\alpha < \beta < \lambda$,

$$V[G \restriction \beta] \models p[T] = \mathbb{R} \setminus p[U].$$

This implies $p[T]$ is λ-universally Baire in $V[G \restriction \alpha]$. Then $A \in \mathrm{Hom}^*$. First we shall show that $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathrm{AD}$. Suppose not. Then

$$\exists B \subseteq \mathbb{R}^*(B \in L(\mathbb{R}^*, \mathrm{Hom}^*) \wedge (HC^*, \in, B) \models B \text{ is not determined}).$$

By theorem 6.2, this contradicts theorem 1.5. That $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathrm{DC}_{\mathbb{R}}$ is easy to show. The other axioms have the form $\forall A \subseteq \mathbb{R} \exists B \subseteq \mathrm{Ord}\, \varphi$. By the locality of ∞-Borel codes and Moschovakis' coding lemma, we can reduce "$\exists B \subseteq \mathrm{Ord}$" to the real quantifier. Then $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathrm{AD}^+$ by theorem 5.11 and corollary 5.6. This completes the proof. $\square$

[3]For any pointclass Γ, let M_Γ be the union of all trasitive structure $(M, \in)$ such that $(M, \in) \simeq (\mathbb{R}/E, F/E)$ for some $E, F \subseteq \mathbb{R} \times \mathbb{R}$ in Γ.

7 Proof of the reflection theorem

Theorem 7.1. Let λ be a limit of Woodin cardinals, let G be $\mathrm{Coll}(\omega, <\lambda)$-generic over V, and let φ be a formula in $\mathcal{L}_{\in,\dot{A},\dot{B}}$. Suppose that $A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}$ for some $\alpha < \lambda$. If

$$\exists B \subseteq \mathbb{R}^*_G (B \in L(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \wedge (HC^*_G, \in, A^*, B) \models \varphi),$$

then

$$\exists B (B \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]} \wedge (HC^{V[G\restriction\alpha]}, \in, A, B) \models \varphi).$$

Lemma 7.2. Let λ be a limit of Woodin cardinals and $b \in \mathbb{Q}_{<\lambda}$. Then there are stationarily many $X \in \mathcal{P}_{\omega_1}(V_\lambda)$ such that

- $X \cap (\cup b) \in b$,
- for any successor Woodin $\delta \in X$ such that $\cup b \in V_\delta$ and any maximal antichain in $\mathbb{Q}_{<\delta}$ $A \in X$, X captures A.

Proof. See [5]. □

Lemma 7.3 (Woodin)**.** Let λ be a limit of Woodin cardinals and H be $\mathrm{Coll}(\omega, <\lambda)$-generic over V. Let $\lambda < \alpha \in \mathrm{Ord}$. For any $b \in \mathbb{Q}_\lambda$, there is $G \subseteq \mathbb{Q}_\lambda$ such that

1. $b \in G$,
2. for any successor Woodin $\delta < \lambda$ such that $\cup b \in V_\delta$, $G \cap \mathbb{Q}_{<\delta}$ is $\mathbb{Q}_{<\delta}$-generic over V,
3. $\alpha \in \mathrm{wfp}(\mathrm{Ult}(V, G))$,
4. $\mathbb{R} \cap \mathrm{Ult}(V, G) = \mathbb{R}^*_H$,
5. Hom^* is a Wadge initial segment of $i_G(\mathrm{Hom}_{<\lambda})$, where i_G is the canonical embedding $i_G \colon V \to \mathrm{Ult}(V, G)$.

Proof. By the homogeneity of the collpase, it is enough to find some G and H. Fix b, α and λ. Put $\theta = \alpha + \omega$. Let $X \prec V_\theta$ be countable and in the stationary set given by lemma 7.2 such that $\alpha, b, \lambda \in X$. Let $\pi \colon N \simeq X$ be the uncollapsing. Put $\pi(\langle\bar{\alpha}, \bar{\lambda}\rangle), \bar{b}) = \langle\alpha, \lambda, b\rangle$. Let G

$$a \in G \iff \pi[\cup a] \in \pi(a).$$

It is clear that $\bar{b} \in G$ and for any successor Woodin $\delta < \bar{\lambda}$ in N such that $\cup\bar{b} \in V^N_\delta$, $G \cap \mathbb{Q}^N_{<\delta}$ is $\mathbb{Q}^N_{<\delta}$-generic over N. To see that $\bar{\alpha} \in \mathrm{wfp}(\mathrm{Ult}(N, G))$, define $\sigma \colon \mathrm{Ult}(N, G) \to V_\theta$ by

$$\sigma([f]_G) = \pi(f)(\pi[\cup\mathrm{dom}(f)])$$

where $f \in N$ is such that $\mathrm{dom}(f) = \mathcal{P}_{\omega_1}(V_\gamma)^N$ for some $\gamma < \bar{\lambda}$. It is easy to show that σ is well-defined and Σ_0-elementary. Thus $\bar{\alpha} \in \mathrm{wfp}(\mathrm{Ult}(N,G))$. To see that $\mathbb{R}^*_H = \mathbb{R} \cap \mathrm{Ult}(N,G)$ for some $\mathrm{Coll}\left(\omega, <\bar{\lambda}\right)$-generic H over N. Put

$$D = \mathbb{R} \cap \mathrm{Ult}(N,G) = \{y_n \mid n \in \omega\}.$$

We will construct $\mathrm{Coll}\left(\omega, <\bar{\lambda}\right)$-generic H over N and a sequence of ordinals $\{\alpha_n \mid n \in \omega\}$ such that

- $\lim \alpha_n = \bar{\lambda}$,
- $H \restriction \alpha_n$ is coded by a real in $\mathrm{Ult}(N,G)$,
- for each $n \in \omega$, $y_n \in N[H \restriction \alpha_{n+1}]$.

This is an easy induction. Let $i_G \colon N \to \mathrm{Ult}(N,G)$ be the canonical embdding. We will see that Hom^*_H is a Wadge initial segment of $i_G(\mathrm{Hom}^N_{<\bar{\lambda}})$. We will use this argument later. Fix $A \in \mathrm{Hom}^{N[H\restriction\eta]_{<\bar{\lambda}}}$ where $\eta < \bar{\lambda}$. Let γ be a sufficiently large successor Woodin cardinal in N such that $H \restriction \eta \in N[G \cap \mathbb{Q}^N_{<\gamma}]$. We may assume that $A = A^{N[G\cap\mathbb{Q}^N_{<\gamma}]}$. Let $i_\gamma \colon N \to \mathrm{Ult}\left(N, G \cap \mathbb{Q}^N_{<\gamma}\right)$ be the canonical embedding. It is enough to show that for any successor Woodin cardinal $\delta > \gamma$ in N,

$$i_{\gamma,\delta}(A) = A^{N[G\cap\mathbb{Q}^N_{<\delta}]}$$

where $i_{\gamma,\delta}$ is the factor embedding. Let $\bar{\mu} = \langle \mu^*_s \mid s \in {}^{<\omega}\omega\rangle$ be a δ^+-homogeneous system of A in $N[G \cap \mathbb{Q}^N_{<\gamma}]$ such that $A = S_{\bar{\mu}}^{N[G\cap\mathbb{Q}^N_{<\gamma}]}$ where each $\mu_s \in N$. Then

$$i_{\gamma,\delta}(A) = S^{\mathrm{Ult}\left(N, G\cap\mathbb{Q}^N_{<\delta}\right)}_{\langle i_\delta(\mu_s) \mid s \in {}^{<\omega}\omega\rangle} = S^{N[G\cap\mathbb{Q}^N_{<\delta}]}_{\langle \mu^{**}_s \mid s\in {}^{<\omega}\omega\rangle} = A^{N[G\cap\mathbb{Q}^N_{<\delta}]}.$$

□

Proof of theorem 7.1. Let G be $\mathrm{Coll}\,(\omega, <\lambda)$-generic and $A \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}$ where $\alpha < \lambda$. Suppose that

$$\exists B \subseteq \mathbb{R}^*(B \in L(\mathbb{R}^*, \mathrm{Hom}^*) \wedge (HC^*, \in, A^*, B) \models \varphi).$$

We call such B φ-witness for A^*. It is enough to find a φ-witness for A in $\mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}$. By the $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$-absoluteness, it is enough to find a φ-witness for $A^{V[G\restriction\beta]}$ in $\mathrm{Hom}^{V[G\restriction\beta]}_{<\lambda}$ for some $\alpha < \beta < \lambda$. First, assume there is $C^* \in \mathrm{Hom}^*$ such that there is a φ-witness for A^* in $L(C^*, \mathbb{R}^*)$. We may assume that

- $A, C \in \mathrm{Hom}^V_{<\lambda}$,
- $A \leq_w C$.

Let γ_0 be the least such that there is a φ-witness for A^* B and $A^*, B \in L_{\gamma_0}(C^*, \mathbb{R}^*)$. Since $B \in L_{\gamma_0}(C^*, \mathbb{R}^*)$, let $x_0 \in \mathbb{R}^*$ be such that B is ordinal definable over $L_{\gamma_0}(C^*, \mathbb{R}^*)$ from parameters x_0, A^*, C^*. We may assume that $x_0 \in V$. Put

$$y \in B \iff L_{\gamma_0}(C^*, \mathbb{R}^*) \models \psi[\langle x_0, A^*, C^* \rangle, y].$$

We will find an absolute definition of B. Put

$$\bar{\varphi}(v_0, v_1) = \text{" } v_0 \text{ is a } \varphi\text{-witness for } v_1\text{"}$$

and

$$\begin{aligned} \theta(v, u) = \text{" } & v \text{ is } \langle v_0, v_1, v_2 \rangle \text{ where } L(v_2, \mathbb{R}) \models \exists B \bar{\varphi}[B, v_1] \text{ and} \\ & \text{if } \gamma_0 \text{ is the least such that } L_{\gamma_0}(v_2, \mathbb{R}) \models \exists B \bar{\varphi}[B, v_1], \\ & \text{then } L_{\gamma_0}(v_2, \mathbb{R}) \models \psi[v, u]\text{"}. \end{aligned}$$

Claim 1. For any $g \in HC^*$ and for any $y \in \mathbb{R} \cap V[g]$,

$$V[g] \models \theta[\langle x_0, A^{V[g]}, C^{V[g]} \rangle, y] \iff y \in B.$$

Proof of claim 1. Let H be $\mathrm{Coll}\,(\omega, < \lambda)$-generic over $V[g]$ such that $V[g][H] = V[G]$. Let $K \subseteq \mathbb{Q}^{V[g]}_{<\lambda}$ be such that

- $\gamma_0 \in \mathrm{wfp}(\mathrm{Ult}(V[g], K))$,
- $\mathbb{R} \cap \mathrm{Ult}(V[g], K) = \mathbb{R}^*_H$,
- Hom^*_H is a Wadge initial segment of $i_K(\mathrm{Hom}^{V[g]}_{<\lambda})$ where $i_K \colon V[g] \to \mathrm{Ult}(V[g], K)$ is the canonical embedding.

Since for any successor Woodin $\delta < \lambda$ in $V[g]$,

$$i_\delta(A^{V[g]}) = A^{V[g][K \cap \mathbb{Q}^{V[g]}_{<\delta}]},$$

then

$$i_K(A^{V[g]}) = A^*_G = A^*_H,\ i_K(C^{V[g]}) = C^*_G = C^*_H.$$

This is the same argument as in lemma 7.3. Then

$$\mathrm{Ult}(V[g], K) \models L_{\gamma_0}(C^*, \mathbb{R}) \models \exists B \bar{\varphi}[B, A^*]$$

and

$$y \in B \iff \mathrm{Ult}(V[g], K) \models \theta[\langle x_0, A^*, C^* \rangle, y].$$

Thus

$$y \in B \iff V[g] \models \theta[\langle x_0, A^{V[g]}, C^{V[g]} \rangle, y].$$

□

In particular for any $y \in \mathbb{R}^V$,

$$V \models \theta[\langle x_0, A, C\rangle, y] \iff y \in B.$$

In V, $\mathbb{R}^V \cap B$ is a φ-witness for A. It is enough to show that $\mathbb{R}^V \cap B \in \mathrm{Hom}_{<\lambda}$. We use the tree production lemma. Working in V. Let (T, U) and (R, S) be λ-absolute complements such that $p[T] = A$ and $p[R] = C$. Let τ be such that

$$\tau[\langle x_0, T, R\rangle, y] \iff \theta[\langle x_0, p[T], p[R]\rangle, y].$$

Generic absoluteness follows from claim 1. To see that stationary tower correctness holds, let $\delta < \lambda$ be a Woodin cardinal and L be $\mathbb{Q}_{<\delta}$-generic over V. Then

$$V[L] \models p[T] = p[i(T)], p[R] = p[i(R)]$$

where $i \colon V \to M \subseteq V[L]$ is the canonical embedding. For any $y \in \mathbb{R} \cap V[L]$,

$$\begin{aligned} & M \models \tau[\langle x_0, i(T), i(R)\rangle y] \\ \iff & M \models \theta[\langle x_0, p[i(T)], p[i(R)]\rangle, y] \\ \iff & V[L] \models \theta[\langle x_0, p[T], p[R]\rangle, y] \\ \iff & V[L] \models \tau[\langle x_0, T, R\rangle, y]. \end{aligned}$$

Then by the tree production lemma, $\mathbb{R}^V \cap B \in \mathrm{Hom}_{<\lambda}$. This completes the first case. From now on, we may assume that for any $C^* \in \mathrm{Hom}^*$, there is no φ-witness for A^* in $L(C^*, \mathbb{R}^*)$. By the first case,

$$\forall C \in \mathrm{Hom}^* \, L(C, \mathbb{R}^*) \models \mathrm{AD}^+.$$

Claim 2. For any $C \in \mathrm{Hom}^*$, $C^\sharp$ exists and $C^\sharp \in \mathrm{Hom}^*$.

Proof of claim 2. Fix $C \in \mathrm{Hom}^*$. Let $D \in \mathrm{Hom}^*$ be such that $D \notin L(C, \mathbb{R}^*)$. Then $C \leq_w D$ by Wadge lemma. Since $L(D, \mathbb{R}^*) \models \mathrm{AD}^+$, $C^\sharp$ exists in $L(D, \mathbb{R}^*)$. So $C^\sharp$ exists. Let B_n be the type of the first n-indiscernibles. Since $C^\sharp = \bigoplus B_n$ and $B_n \leq_w D$ for each $n \in \omega$, $C^\sharp \leq_w D$. $\square$

Claim 3. For any $g \in HC^*$, $\mathrm{Hom}_{<\lambda}^{V[g]}$ is closed under sharps.

Proof of claim 3. Easy. Use $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$-absoluteness. $\square$

Let γ_0 be the least such that $L_{\gamma_0}(\mathbb{R}^*, \mathrm{Hom}^*) \models \bar{\varphi}[B, A^*]$ for some B, and for all $C \in \mathrm{Hom}^*$, $|C|_w < \gamma_0$.

Claim 4. Let $g \in HC^*$ and $H \subseteq \mathbb{Q}_{<\lambda}^{V[g]}$ be as in lemma 7.3. Then

$$i_H(\mathrm{Hom}_{<\lambda}^{V[g]}) = \mathrm{Hom}^*.$$

Proof of claim 4. It is easy to show that Hom^* is a Wadge initial segment of $i_H(\mathrm{Hom}_{<\lambda}^{V[g]})$. Suppose otherwise. Fix $C \in i_H(\mathrm{Hom}_{<\lambda}^{V[g]}) \setminus \mathrm{Hom}^*$ which is Wadge minimal such that

- $\mathrm{Ult}(V[g], H) \models \mathrm{Hom}^* = \{A \mid A <_w C\}$,
- $L_{\gamma_0}(\mathbb{R}^*, \mathrm{Hom}^*) \subseteq L(C, \mathbb{R})^{\mathrm{Ult}(V[g],H)}$.

By claim 3,

$$\mathrm{Ult}(V[g], H) \models \exists\, B \in i_H(\mathrm{Hom}^{V[g]}_{<\lambda}) B \text{ is a } \varphi\text{-witness for } A.$$

Then there is $B^* \in \mathrm{Hom}^*$ such that B^* is a φ-witness for A^*. This contradicts our assumption. $\square$

Recall that γ_0 is the least such that $L_{\gamma_0}(\mathbb{R}^*, \mathrm{Hom}^*) \models \bar{\varphi}[B, A^*]$ for some B, and for all $C \in \mathrm{Hom}^*$, $|C|_w < \gamma_0$. Let $x_0 \in \mathbb{R}^*$ and $C^* \in \mathrm{Hom}^*$ be such that such B is ordinal definable over $L_{\gamma_0}(\mathbb{R}^*, \mathrm{Hom}^*)$ from parameters x_0, A^*, C^*. We may assume that $x_0, C \in V$. Put

$$y \in B \iff L_{\gamma_0}(\mathbb{R}^*, \mathrm{Hom}^*) \models \psi[\langle x_0, A^*, C^*, \mathrm{Hom}^* \rangle, y]$$

and

$$\begin{aligned} \theta(v, u) = \text{''}\ v \text{ is } \langle v_0, v_1, v_2, v_3 \rangle & \text{ where } L(\mathbb{R}, v_3) \models \exists B \bar{\varphi}[B, v_1] \text{ and} \\ & \text{if } \gamma_0 \text{ is the least such that } L_{\gamma_0}(\mathbb{R}, v_3) \models \exists B \bar{\varphi}[B, v_1] \\ & \forall C \in v_3 (|C|_w < \gamma_0) \\ & \text{then } L_{\gamma_0}(\mathbb{R}, v) \models \psi[v, u]\text{''}. \end{aligned}$$

Claim 5. For any $g \in HC^*$ and any $y \in \mathbb{R} \cap V[g]$,

$$V[g] \models \theta[\langle x_0, A^{V[g]}, C^{V[g]}, \mathrm{Hom}^{V[g]}_{<\lambda}, y] \iff y \in B.$$

Proof of claim 5. This is the same as the previous one. $\square$

Then we can show than $\mathbb{R}^V \cap B \in \mathrm{Hom}_{<\lambda}$ is a φ-witness for A the same manner as the first case. This completes the proof of theorem 7.1. $\square$

A General theory of sharps

In this section, we give a general theory of sharps introduced by Solovay in [7]. Some properties of $A^\sharp$ which we shall prove is based on both [16] and [17], and is an easy generalization of that of $0^\sharp$. Let A be a transitive set. Let $\varphi(x_0, \ldots, x_n)$ be a formula in the language of set theory with constant symbols $\{\dot{a} \mid a \in A \cup \{A\}\}$. A Skolem term for φ is a function $\tau_\phi \colon L(A)^n \to L(A)$ such that

$$\tau_\varphi(\bar{x}) = \begin{cases} y & \text{if } y \text{ is the unique } y \text{ such that } \varphi(y, \bar{x}) \text{ in } L(A), \\ \emptyset & \text{otherwise.} \end{cases}$$

For $A \subseteq B \subseteq L(A)$, $H^{L(A)}(B)$ denotes a Skolem hull of B. $\mathcal{L}_{\in,A}$ is the language of set theory with constant symbols $\{\dot{a} \mid a \in A \cup \{A\}\} \cup \{c_n \mid n \in \omega\}$ and closed under a combination of Skolem terms. $\{c_n \mid n \in \omega\}$ is to represnet for indiscernibles. $\mathcal{L}^*_{\in,A}$ is the language without constant symbols $\{c_n \mid n \in \omega\}$.

Definition A.1. EM blueprint for A is the theory in $\mathcal{L}_{\in,A}$ of some structure $(L_\kappa(A), \in, \dot{a}(a \in A \cup \{A\}))$ where $Y \in H_\kappa$ or $\kappa = \mathrm{Ord}$ and $\langle c_n \mid n \in \omega \rangle$ is the increasing enumeration of indiscernibles for $(L_\kappa(A), \in, \dot{a}(a \in A \cup \{A\}))$.

Definition A.2. Let Σ be a EM blueprint for A and α be a ordinal. $\Gamma(\Sigma, \alpha)$ is if it exists, the unique model M such that:

1. $M \models \Sigma \restriction \mathcal{L}^*_{\in,A}$,
2. there is a $I_\alpha \subseteq \mathrm{Ord}^M$ such that $(I_\alpha, \in^M) \simeq (\alpha, \in)$ and I_α is a set of indiscernibles for M,
3. $H^M(I_\alpha \cup \{\dot{a}^M \mid a \in A \cup \{A\}) = M$.

Definition A.3. A EM blueprint Σ for A is a remarkable character for A if Σ satisfies the following:

1. for all $\alpha \in \mathrm{Ord}$, $\Gamma(\Sigma, \alpha)$ is well-founded,
2. for any term $t(x_0, \dots, x_{n-1})$ in $\mathcal{L}^*_{\in,A}$,
$$t(c_0, \dots, c_{n-1}) \in \mathrm{Ord} \implies t(c_0, \dots, c_{n-1}) < c_n$$
is in Σ,
3. for any term $t(x_0, \dots, x_{m+n})$ in $\mathcal{L}^*_{\in,A}$,
$$t(c_0, \dots, c_{m+n}) < c_m \implies t(c_0, \dots, c_{m+n}) < t(c_0, \dots, c_{m-1}, c_{m+n-1}, \dots, c_{m+2n-1})$$
is in Σ,
4. Σ satisfies the witness condition: if $\exists x \varphi(x) \in \Sigma$, then there is a term t all of whose constants for indiscernibles appear on $\varphi(x)$ and $\varphi(t) \in \Sigma$.

Almost the same thing for $0^\sharp$ holds. Let Σ be a remarkable character for A.

Proposition A.4. For a limit ordinal α, let I_α be a set of indiscernibles for $\Gamma(\Sigma, \alpha)$. Then I_α is unbounded in $\mathrm{Ord}^{\Gamma(\Sigma,\alpha)}$.

Proof. Fix $y \in \mathrm{Ord}^{\Gamma(\Sigma,\alpha)}$. Let τ, $x_1 < \cdots < x_n \in I_\alpha$ and $\bar{a} \in (A \cup \{A\})^{<\omega}$ such that τ is a Skolem term and $y = \tau[x_1, \dots, x_n, \bar{a}]$. Since Σ is a remarkable character for A, $\tau[x_1, \dots, x_n, \bar{a}] < x_{n+1}$ for any $x_n < x_{n+1} \in I_\alpha$. □

Proposition A.5. Let $\gamma < \alpha$ be limit ordinals. Put $I_\alpha = \{i_\xi \mid \xi < \alpha\}$. Then for any $x \in \mathrm{Ord}^{\Gamma(\Sigma,\alpha)}$ such that $x < i_\gamma$,
$$x \in H^{\Gamma(\Sigma,\alpha)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma,\alpha)} \mid a \in A \cup \{A\}\}).$$

Proof. Fix $x \in \mathrm{Ord}^{\Gamma(\Sigma,\alpha)}$ such that $x < i_\gamma$. Let τ, $x_1 < \cdots < x_m < y_1 < \cdots < y_n \in I_\alpha$ and $\bar{a} \in A \cup \{A\}$ be such that τ is a Skolem term, $y_1 = i_\gamma$ and

$$x = \tau[x_1, \ldots, x_m, y_1, \ldots, y_n, \bar{a}].$$

Let $w_1, \ldots, w_n, z_1, \ldots, z_n \in I_\alpha$ be such that

$$x_1 < \cdots < x_m < w_1 < \cdots < w_n < y_1 < \cdots < y_n < z_1 < \cdots < z_n.$$

Since $x < y_1$ and Σ is a remarkable character for A,

$$\tau[x_1, \ldots, x_m, y_1, \ldots, y_n, \bar{a}] = \tau[x_1, \ldots, x_m, z_1, \ldots, z_n, \bar{a}].$$

By indiscernibility,

$$\tau[x_1, \ldots, x_m, w_1, \ldots, w_n, \bar{a}] < \tau[x_1, \ldots, x_m, z_1, \ldots, z_n, \bar{a}].$$

Then $x = \tau[x_1, \ldots, x_m, w_1, \ldots, w_n, \bar{a}] \in H^{\Gamma(\Sigma,\alpha)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma,\alpha)} \mid a \in A \cup \{A\}\})$. □

Proposition A.6. Let α be a limit ordinal. Then I_α is closed in $\mathrm{Ord}^{\Gamma(\Sigma,\alpha)}$.

Proof. Fix a limit ordinal $\gamma < \alpha$ and $x \in \mathrm{Ord}^{\Gamma(\Sigma,\alpha)}$ such that $x < i_\gamma$. Then $x \in H^{\Gamma(\Sigma,\alpha)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma,\alpha)} \mid a \in A \cup \{A\}) = \Gamma(\Sigma, \gamma)$. Then for some $\xi < \gamma$, $x < i_\xi$. □

Proposition A.7. For any uncountable cardinal κ such that $A \in H_\kappa$,

$$\Gamma(\Sigma, \kappa) \simeq L_\kappa(A).$$

Proof. Suppose not. Let $\beta > \kappa$ be such that $\Gamma(\Sigma, \kappa) \simeq L_\beta(A)$. Then

$$\kappa \in H^{\Gamma(\Sigma,\kappa)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma,\alpha)} \mid a \in A \cup \{A\}\})$$

for some limit ordinal $\gamma < \kappa$. However

$$|H^{\Gamma(\Sigma,\kappa)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma,\alpha)} \mid a \in A \cup \{A\}\})| < \kappa.$$

This is a contradiction. □

Proposition A.8. Let κ and λ be uncountable cardinals such that $\kappa < \lambda$ and $A \in H_\kappa$. Then $I_\lambda \cap \kappa = I_\kappa$ and

$$H^{L_\lambda(A)}(I_\kappa \cup \{\dot{a}^{\Gamma(\Sigma,\alpha)} \mid a \in A \cup \{A\}\}) = L_\kappa(A) \prec L(A).$$

Proof. Let J be the first κ members of I_λ. Put

$$M = H^{L_\lambda(A)}(J \cup \{\dot{a}^{\Gamma(\Sigma,\alpha)} \mid a \in A \cup \{A\}\}) \simeq \Gamma(\Sigma, \kappa) \simeq L_\kappa(A).$$

Since $\mathrm{Ord}^M = \kappa$ and M is transitive, then $M = L_\kappa(A)$ and $J = I_\kappa$. □

Definition A.9. Let A be a trasitive set. $A^\sharp$ exists if there is a remarkable character for A.

Proposition A.10. Let A be a trasitive set. The following are equivalent:

1. $A^\sharp$ exists,
2. there is a club proper class indiscernibles I for $L(A)$ such that
   - $H^{L(A)}(I \cup A \cup \{A\}) = L(A)$,
   - for any uncountable cardinal κ such that $A \in H_\kappa$, $H^{L(A)}((I \cap \kappa)A \cup \{A\}) = L_\kappa(A)$.

Proposition A.11. Let A be a trasitive set. If there is a Ramsey cardinal κ such that $A \in H_\kappa$, then $A^\sharp$ exists.

B Strong partition cardinals below Θ

In this chapter, our base theory is ZF + AD + $\mathrm{DC}_{\mathbb{R}}$. We shall show strong partition cardinals are cofinal below Θ.

Definition B.1. κ is a strong partition cardinal if $\kappa \to (\kappa)^\kappa_{<\kappa}$.

Proposition B.2. If $\kappa \to (\kappa)^\kappa_{<\kappa}$, then $\kappa \to (\kappa)^\mu_\mu$ for all $\mu < \kappa$.

Proof. Easy. $\square$

Proposition B.3. Let A be a set of reals and $\delta = (\boldsymbol{\delta}^2_1(A))^{L(A,\mathbb{R})}$

1. $(\boldsymbol{\Sigma}^2_1(A))^{L(A,\mathbb{R})} = \boldsymbol{\Sigma}_1(L_\delta(A,\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$,
2. $(\boldsymbol{\Delta}^2_1(A))^{L(A,\mathbb{R})} = L_\delta(A,\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$.

Theorem B.4 (Solovay's basis theorem)**.** Let A be a set of reals and x, y be reals. For any formula φ in second order arithmetic, if

$$\exists B \in \mathcal{P}(\mathbb{R}) \cap L(A,\mathbb{R})(\mathrm{HC} \models \varphi[A,B,x,y]),$$

then

$$\exists B \in (\boldsymbol{\Delta}^2_1(A))^{L(A,\mathbb{R})}(\mathrm{HC} \models \varphi[A,B,x,y]).$$

For $X \in (\boldsymbol{\Sigma}^2_1(A))^{L(A,\mathbb{R})}$, put

$$x \in X \iff L(A,\mathbb{R}) \models \exists B \subseteq \mathbb{R}\psi[x,y,B]$$

where $y \in \mathbb{R}$. Define a norm on X:

$$\varphi(x) = \text{ the laest ordinal } \alpha \text{ such that } L_\alpha(A,\mathbb{R}) \models \exists B \subseteq \mathbb{R}\psi[x,y,B].$$

It it easy to show that φ is a $\boldsymbol{\Sigma}^2_1(A)^{L(A,\mathbb{R})}$-norm on X. By Solovay's basis theorem and Σ_0-collection, $\boldsymbol{\Sigma}^2_1(A)^{L(A,\mathbb{R})}$ is closed under $\forall^{\mathbb{R}}$. Then $(\boldsymbol{\delta}^2_1(A))^{L(A,\mathbb{R})}$ is a strong partition cardinal. [4] We have the following. The following theorem appears in [6].

[4] See [6].

Theorem B.5 ($\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$)**.** Strong partition cardinals are cofinal below Θ.

Using strong partition cardinals, we can prove determinacy of ordinal games.

Theorem B.6 ($\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$, Moschovakis–Woodin)**.** Let $\lambda < \Theta$. If $A \subseteq \lambda^\omega$ is Suslin-co-Suslin, then A is determined.

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