

Notes on the derived model theorem

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June 8, 2020

This is based on the seminar notes of the derived model theorem. In addition to a proof of the derived model theorem, we give some basic result of tree representations and AD^+ .

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1 Tree representations

In this section, we shall give basic results for tree representations.

1.1 Homogeneous, weakly homogeneous, universally Baire

We shall show the following.

Theorem 1.1. Suppose λ is a limit of Woodin cardinals. Then for any set of reals A , the following are equivalent:

1. A is $<\lambda$ -homogeneously Suslin,
2. A is weakly $<\lambda$ -homogeneously Suslin,
3. A is λ -universally Baire.

Definition 1.2 (homogeneous tree). A tree T on $X \times Y$ is κ -homogeneous if there is a sequence $\bar{\mu} = \langle \mu_s \mid s \in {}^{<\omega}X \rangle$ such that:

1. for each $s \in {}^{<\omega}X$, μ_s is a κ -complete ultrafilter over $T[s]$,
2. for each $s, t \in {}^{<\omega}X$ such that $s \subseteq t$, μ_t projects to μ_s ,
3. for any $x \in p[T]$ and $\langle Z_n \mid n \in \omega \rangle$ such that $Z_n \in \mu_{x \upharpoonright n}$ for all $n \in \omega$, there is a $f \in {}^\omega Y$ such that for all $n \in \omega$, $f \upharpoonright n \in Z_n$.

Definition 1.3. Let A be a set of reals.

- A is κ -homogeneously Suslin if there is a κ -homogeneous tree T such that $A = p[T]$.
- A is $<\lambda$ -homogeneously Suslin if A is κ -homogeneously Suslin for all $\kappa < \lambda$.

We give another characterization of homogeneity.¹

Proposition 1.4. For a set of reals A , the following are equivalent:

1. A is κ -homogeneously Suslin,
2. there is a function

$$\bar{\mu}: {}^{<\omega}\omega \rightarrow \text{meas}_\kappa(Y) = \{\mu \mid \mu \text{ is a } \kappa\text{-complete ultrafilter over } {}^{<\omega}Y\}$$

such that for any $s, t \in {}^{<\omega}\omega$,

- (a) $\dim(\mu_t) = \text{dom}(t)$,
- (b) if $s \subseteq t$, then μ_t projects to μ_s ,
- (c) If $S_{\bar{\mu}} = \{x \in \mathbb{R} \mid \langle \mu_{x \upharpoonright n} \mid n \in \omega \rangle \text{ is countably complete}\}$, then $S_{\bar{\mu}} = A$.

Proof. The forward direction is clear. Given such $\bar{\mu}$, for each $x \in \mathbb{R} \setminus A$, let $\langle Z_n^x \mid n \in \omega \rangle$ witness that $\langle \mu_{x \upharpoonright n} \mid n \in \omega \rangle$ is not countably complete. We set

$$(s, u) \in T \iff u \in Y^{\text{dom}(s)} \wedge \forall x \supseteq s (x \notin A \implies u \in Z_{\text{dom}(s)}^x).$$

Then this gives a κ -homogeneous tree projecting to A . \square

Hom_κ denotes the collection of all κ -homogeneously Suslin set of reals. We set

$$\text{Hom}_{<\kappa} = \bigcap_{\lambda < \kappa} \text{Hom}_\lambda.$$

Hom_κ is closed under Wadge reducibility and countable intersection. Enough homogeneity gives determinacy.

Theorem 1.5 (Martin). If a set of reals A is \aleph_1 -homogeneously Suslin, then A is determined.

¹We can define κ -homogeneously without using a tree. However Suslinness is central notion in descriptive set theory. So we use this definition.

Theorem 1.6 (Martin). If κ is a measurable cardinal, then every Π_1^1 set of reals is κ -homogeneously Suslin and determined.

Definition 1.7 (weakly homogeneous tree). A tree T on $X \times Y$ is weakly κ -homogeneous if there is a sequence $\bar{\mu} = \langle \mu_{s,t} \mid (s,t) \in {}^{<\omega}X \oplus {}^{<\omega}\omega \rangle$ such that:

1. for each $(s,t) \in {}^{<\omega}X \oplus {}^{<\omega}\omega$, $\mu_{s,t}$ is a κ -complete ultrafilter over $T[s]$,
2. for each $(p,r), (q,s) \in {}^{<\omega}X \oplus {}^{<\omega}\omega$ such that $(p,r) \subseteq (q,s)$, $\mu_{q,s}$ projects to $\mu_{p,r}$,
3. for any $x \in p[T]$ and $\langle Z_s \mid s \in {}^{<\omega}\omega \rangle$ such that for all $s \in {}^{<\omega}\omega$, $Z_s \in \mu_{x \upharpoonright \text{lh}(s), s}$, there is a $y \in {}^\omega X$ and a $f \in {}^\omega Y$ such that for all $n \in \omega$, $f \upharpoonright n \in Z_{x \upharpoonright n, y \upharpoonright n}$.

Definition 1.8. Let A be a set of reals.

- A is weakly κ -homogeneously Suslin if there is a weakly κ -homogeneous tree T such that $A = p[T]$.
- A is weakly $<\lambda$ -homogeneously Suslin if A is weakly κ -homogeneously Suslin for all $\kappa < \lambda$.

Proposition 1.9. For a set of reals A , the following are equivalent:

1. A is weakly κ -homogeneously Suslin,
2. A is a projection of some κ -homogeneously Suslin set of reals.

Clearly, κ -homogeneously Suslin set is weakly κ -homogeneously Suslin.

Definition 1.10. Let T on $X \times Y$ and U on $X \times Y$ be trees. T and U are κ -absolute complements if for any $<\kappa$ -generic G over V ²,

$$V[G] \models p[T] = {}^\omega X \setminus p[U].$$

We say T is κ -absolutely complemented if for some U , T and U are κ -absolutely complements.

Definition 1.11. A set of reals A is κ -universally Baire if there are κ -absolute complements T and U such that $A = p[T]$.

Proposition 1.12. Let (T, U) and (R, S) be pairs of κ -absolute complements such that $p[T] = p[R]$. Then for any $<\kappa$ -generic G over V , $p[T] = p[R]$ in $V[G]$.

Given κ -universally Baire set A , it is well-defined to write $A^{V[G]}$ if G is $<\kappa$ -generic. For any weakly homogeneous tree T , we can construct another tree which projects to the complement of $p[T]$. This construction is called Martin–Solovay construction. Let $\langle r_i \mid i \in \omega \rangle$ be a standard enumeration of ${}^{<\omega}\omega$. For weakly κ -homogeneous tree T on $X \times Y$ and a any $(p,r), (q,s) \in {}^{<\omega}X \oplus {}^{<\omega}\omega$ such that $(p,r) \subseteq (q,s)$, let

$$i_{(p,r),(q,s)}: \text{Ult}(V, \mu_{(p,r)}) \rightarrow \text{Ult}(V, \mu_{(q,s)})$$

be a natural elementary embedding.

²for some poset \mathbb{P} such that $|\mathbb{P}| < \kappa$, G is \mathbb{P} -generic.

Definition 1.13 (Martin–Solovay tree). Let T be a weakly κ -homogeneous tree on $X \times Y$ with κ -homogeneous system $\bar{\mu}$ and α be ordinal. A Martin–Solovay tree $\text{ms}(\bar{\mu}, \alpha)$ is a tree on $X \times \text{Ord}$ consisting of a pair (p, t) such that:

1. $p \in {}^{<\omega}X$,
2. $t \in {}^{\text{lh}(p)}\text{Ord}$,
3. $t(0) < \alpha$,
4. for any $i, j < \text{lh}(p)$, if $r_i \subsetneq r_j$, then

$$t(j) < i_{(p \restriction \text{lh}(r_i), r_i), (p \restriction \text{lh}(r_j), r_j)}(t(i)).$$

A infinite path of $\text{ms}(\bar{\mu}, \alpha)$ gives a witness for ill-foundedness of a direct limit model. Then Martin–Solovay tree projects to the complements of $p[T]$.

Lemma 1.14. Let T be a weakly κ -homogeneous tree on $X \times Y$ with κ -homogeneous system $\bar{\mu}$. If $\alpha \geq |Y|^+$, then

$$p[\text{ms}(\bar{\mu}, \alpha)] = \mathbb{R} \setminus p[T].$$

Theorem 1.15. Let T be a weakly κ -homogeneous tree on $X \times Y$ with κ -homogeneous system $\bar{\mu}$. For sufficiently large α , T and $\text{ms}(\bar{\mu}, \alpha)$ are κ -absolute complements.

Proof. Let G be a $<\kappa$ -generic over V . Then $\text{ms}(\bar{\mu}, \theta)^V = \text{ms}(\bar{\mu}^*, \theta)^{V[G]}$. By absoluteness, $p[T] = \mathbb{R} \setminus \text{ms}(\bar{\mu}, \theta)$. \square

Corollary 1.16. If a set of reals A is weakly κ -homogeneously Suslin, then A is κ -universally Baire.

Up to here we can show without any additional assumptions.

Theorem 1.17 (Martin–Steel). Let δ be a Woodin cardinal and T be a weakly δ^+ -homogeneous tree on $X \times Y$ with weakly δ^+ -homogeneous system $\bar{\mu}$ such that $X \in V_\delta$. Then for sufficiently large θ , $\text{ms}(\bar{\mu}, \theta)$ is $<\delta$ -homogeneous.

This proof is very hard, so we omit it.

Corollary 1.18. Let δ be a Woodin cardinal and $A \subseteq \mathbb{R}^2$ be a δ^+ -homogeneously Suslin. Then $\neg \exists A$ is $<\delta$ -homogeneously Suslin.

Corollary 1.19. Let λ be a limit of Woodin cardinals. Then $\text{Hom}_{<\lambda}$ is closed under $\exists^{\mathbb{R}}$, negation and Wadge reducibility.

Corollary 1.20. If there are ω -many Woodin cardinals, then PD holds.

Lemma 1.21. Let λ be a limit of Woodin cardinals. Then there is a ordinal $\gamma_0 < \lambda$ such that $\text{Hom}_{\gamma_0} = \text{Hom}_{<\lambda}$.

Proof. Suppose not. Then for any $\kappa < \lambda$, $\text{Hom}_\kappa \supsetneq \text{Hom}_{<\lambda}$. This gives an infinite descending $<_w$ -chain. However Martin's proof that $<_w$ is well-founded gives a contradiction. \square

Proposition 1.22. For a tree T on $\omega \times Y$, the following are equivalent:

1. T is weakly κ -homogeneous,
2. there is a countable collection Σ of κ -complete ultrafilter over ${}^{<\omega}Y$ such that, $x \in p[T]$ if and only if there is a countably complete tower $\langle \mu_n \mid n \in \omega \rangle \in {}^\omega \Sigma$ such that for all $n \in \omega$, $T[x \restriction n] \in \mu_n$.

Proof. The forward direction is clear. We may assume that Σ is closed under projections. Define $\mu_{s,t}$ for each $(s, t) \in {}^{<\omega}\omega \oplus {}^{<\omega}\omega$, by induction on $|s|$:

$$\begin{aligned} \{ \mu \cap \mathcal{P}(T[s^\frown \langle i \rangle]) \mid \mu \in \Sigma \wedge T[s^\frown \langle i \rangle] \wedge \mu_{s,t} \text{ is a projection of } \mu \} \\ \subseteq \{ \mu_{s^\frown \langle i \rangle, t^\frown \langle j \rangle} \mid j \in \omega \}. \end{aligned}$$

We allow principal ultrafilters if necessary. Then this gives a weakly homogeneous system for T . \square

Theorem 1.23 (Woodin). Let δ be a Woodin cardinal. If T and U are trees on $\omega \times Z$ and δ^+ -absolute complements, then T is weakly $<\delta$ -homogeneous.

Proof. Let T and U be trees on $\omega \times Z$ and δ^+ -absolute complements. Let η be a sufficiently large regular cardinal such that $T, U \in V_\eta$. Then let T' and U' be the subtrees of T and U consisting all nodes definable over V_η from T, U, δ . Then $|T'| = |U'| = \delta$. A $\mathbb{Q}_{<\delta}$ -name for a real can be taken as a $\mathbb{Q}_{<\kappa}$ -name for some inaccessible $\kappa < \delta$, then we can find it in V_δ . So

$$\mathbb{Q}_{<\delta} \Vdash p[\tilde{T}'] = \mathbb{R} \setminus p[\tilde{U}']$$

and $p[T] = p[T']$. By proposition 1.22, if T' is weakly homogeneous, then so is T . Then we may assume that T and U are trees on $\omega \times \delta$. Fix $\kappa < \delta$. We may assume that κ is $<\delta$ - T -strong. For each $\kappa < \lambda < \delta$, let $j_\lambda: V \rightarrow M_\lambda$ be a λ - T -strong embedding with a critical point κ . Put Σ_λ for each $(s, u) \in T \cap V_\lambda$ and $X \subseteq \kappa^{<\omega}$,

$$X \in \Sigma_\lambda(s, u) \iff u \in j_{\lambda(X)}.$$

Claim 1. For each $(s, u), (t, v) \in T \cap V_\lambda$,

1. $\Sigma_\lambda(s, u)$ is κ -complete ultrafilter,
2. $T[s] \cap V_\kappa \in \Sigma_\lambda(s, u)$,
3. if $(s, u) \subseteq (t, v)$, then $\Sigma_\lambda(t, v)$ projects to $\Sigma_\lambda(s, u)$,
4. for any $(x, f) \in [T \cap V_\lambda]$, $\langle \Sigma_\lambda(x \restriction n, f \restriction n) \mid n \in \omega \rangle$ is countably complete.

Proof of claim 1. Easy. \square

Let G be a $\mathbb{Q}_{<\delta}$ -generic over V and let $i: V \rightarrow N \subseteq V[G]$ be a canonical embedding.

Claim 2. $i(T)$ is weakly $i(\kappa)$ -homogeneous in N .

Proof of claim 2. We show that $\sigma = i'' \text{meas}_\kappa(\kappa)$ witnesses that $i(T)$ is weakly $i(\kappa)$ -homogeneous. By closure of N , $\sigma \in N$. For any $x \in p[i(T)] \cap N$, $x \in p[T]$, since we have that $p[T] \subseteq p[i(T)]$, $p[U] \subseteq p[i(U)]$ and $p[i(T)] \cap p[i(U)] = \emptyset$. Take a sufficiently large $\lambda < \delta = \omega_1^{V[G]}$ so that $x \in p[T \cap V_\lambda]$. Let f be such that $(x, f) \in [T \cap V_\lambda]$. By elementarity, $\langle i(\Sigma_\lambda)(x \restriction n, i(f) \restriction n) \mid n \in \omega \rangle$ is countably complete in N . Then $i(T)$ is weakly $i(\kappa)$ -homogeneous in N . \square

By elementarity, T is weakly κ -homogeneous in V . \square

We have shown the following.

Theorem 1.24. Suppose λ is a limit of Woodin cardinals. Then for any set of reals A , the following are equivalent:

1. A is $<\lambda$ -homogeneously Suslin,
2. A is weakly $<\lambda$ -homogeneously Suslin,
3. A is λ -universally Baire.

1.2 The tree production lemma

We shall show the tree production lemma. Roughly speaking, formulas with certain generic absoluteness define universally Baire sets. Let φ be a formula and a be a parameter. Suppose that $X \prec_n V$ is a sufficiently elementary submodel with $\kappa, a \in X$. Let $\pi: X \simeq N$ be a collapsing map. X is (φ, a, κ) -generically correct if and only if for any poset $\mathbb{P} \in H_{\pi(\kappa)}^N$, any \mathbb{P} -generic $g \in V$ over N and any $x \in N[g] \cap \mathbb{R}$,

$$N[g] \models \varphi[x, \pi(a)] \iff V \models \varphi[x, a].$$

Lemma 1.25. Let $\varphi(v_0, v_1)$ be a formula, let a be a parameter and let κ be an infinite cardinal. Suppose that M and σ are such that

- M is transitive,
- $H_\kappa \cup \{\kappa\} \subseteq M$,
- $\sigma: M \rightarrow V$ is sufficiently elementary so that $a \in \text{ran}(\sigma)$ and $\text{cp}(\sigma) > \kappa$.

Suppose that C is a collection of countable $X \prec M$ such that $\sigma''X$ is (φ, a, κ) -generically correct, and C contains a club of $\mathcal{P}_{\omega_1}(M)$. Then there are trees T and U such that for any $\mathbb{P} \in H_\kappa$ and any \mathbb{P} -generic G over V ,

$$V[G] \models p[T] = \{x \in \mathbb{R}^{V[G]} \mid \varphi[x, a]\} \wedge p[U] = \{x \in \mathbb{R}^{V[G]} \mid \neg \varphi[x, a]\}.$$

Proof. Let $F: M^{<\omega} \rightarrow M$ be such that for any $X \in \mathcal{P}_{\omega_1}(M)$, if $F''X^{<\omega} \subseteq X$, then $X \prec M$ and $\sigma''X$ is (φ, a, κ) -generically correct. Put $\sigma(a^*) = a$. Let T be a tree on $\omega \times M \times H_\kappa$ such that $(s, t, u) \in T$ if and only if

1. if $2m + 2 < \text{lh}(s)$, then $t(2m + 2) = F(t \circ r_m)$,
2. if $\text{lh}(s) > 0$, then $t(0) = \mathbb{P} \in H_\kappa$ is a poset,
3. if $\text{lh}(s) > 1$, then $t(1)$ is a \mathbb{P} -name and $u(0) \Vdash t(1): \omega \rightarrow \omega$,
4. if $2m + 3 < \text{lh}(s)$, then $t(2m + 3) = u(m)$,
5. $\{u(m) \mid u < \text{lh}(s)\} \subseteq \mathbb{P}$ has a common extension,
6. if $2m + 2 < \text{lh}(s)$ and $t(m)$ is a \mathbb{P} -dense subset, then $u(2m + 2) \in t(m)$,
7. if $2m + 3 < \text{lh}(s)$, then $u(2m + 3) \Vdash t(1)(m) = s(m)$,
8. if $\text{lh}(s) > 1$, then $u(1) \Vdash \varphi[t(1), \check{a}^*]$.

We also define a tree U on $\omega \times M \times H_\kappa$ replacing $\varphi[t(1), \check{a}^*]$ by $\neg\varphi[t(1), \check{a}^*]$.

Claim 1.

- $p[T] = \{x \in \mathbb{R} \mid \varphi[x, a]\}$.
- $p[U] = \{x \in \mathbb{R} \mid \neg\varphi[x, a]\}$.

Proof of claim 1. Fix $x \in p[T]$. Put $(x, f, g) \in [T]$ and $X = \text{ran}(f)$. By definition of T , $F''X^{<\omega} \subseteq X$. Hence $X \prec M$ and $\sigma''X$ is (φ, a, κ) -generically correct. Let $\mathbb{P} \in X \cap H_\kappa$ be a poset, let G be a \mathbb{P} -generic over X and $\tau \in X$ be a \mathbb{P} -name such that

1. there is a $p \in G$ such that $p \Vdash \dot{\tau}: \omega \rightarrow \omega$,
2. for all $n \in \omega$, there is a $p_n \in G$ such that $p_n \Vdash \dot{\tau}(n) = x(n)$,
3. there is a $q \in G$ such that $q \Vdash \varphi[\dot{\tau}, \check{a}^*]$.

Since σ is sufficiently elementary, $\sigma''X$ satisfies the same as X . As $\sigma''X$ is (φ, a, κ) -generically correct, $V \models \varphi[x, a]$. \square

Suppose that $\mathbb{P} \in H_\kappa$ is a poset and G is \mathbb{P} -generic over V .

Claim 2.

- for all $x \in \mathbb{R}^{V[G]}$, if $V[G] \models \varphi[x, a]$, then $x \in p[T]^{V[G]}$.
- for all $x \in \mathbb{R}^{V[G]}$, if $V[G] \models \neg\varphi[x, a]$, then $x \in p[U]^{V[G]}$.

Proof of claim 2. Let $x \in \mathbb{R}^{V[G]}$ be such that $V[G] \models \varphi[x, a]$ and let τ be such that $\tau_G = x$. Let $X \prec M$ be closed under F and such that $\mathbb{P} \in X$. Let $f: \omega \rightarrow X$ and $g: \omega \rightarrow X \cap G$ be surjection such that they obey the definition of T . Hence $x \in p[T]^{V[G]}$. \square

□

Theorem 1.26 (Tree production lemma, Woodin). Let $\varphi(v_0, v_1)$ be a formula and let a be a parameter. Let δ be a Woodin cardinal. Suppose that

- (generic absoluteness) if G is $<\delta$ -generic over V and H is $<\delta^+$ -generic over $V[G]$, then for any $x \in \mathbb{R} \cap V[G]$,

$$V[G] \models \varphi[x, a] \iff V[G][H] \models \varphi[x, a],$$

- (stationary tower correctness) if G is $\mathbb{Q}_{<\delta}$ -generic over V and $j: V \rightarrow M \subseteq V[G]$ is a canonical embedding, then for any $x \in \mathbb{R} \cap V[G]$,

$$V[G] \models \varphi[x, a] \iff M \models \varphi[x, j(a)].$$

Then there are trees T and U such that for any $<\delta$ -generic G over V ,

$$V[G] \models p[T] = \{x \in \mathbb{R}^{V[G]} \mid \varphi[x, a]\} \wedge p[U] = \{x \in \mathbb{R}^{V[G]} \mid \neg \varphi[x, a]\}.$$

Then $\{x \in \mathbb{R} \mid \varphi[x, a]\}$ is δ -universally Baire.

Proof. It is enough to construct T_κ and U_κ satisfying the claim for each $\kappa < \delta$. Fix $\kappa < \delta$. Let M be transitive, let σ and a^* be such that

- $H_{\kappa^+} \subseteq M$,
- $|M| < \delta$,
- $\sigma: M \rightarrow V$ is sufficiently elementary, $a = \sigma(a^*)$ and $\sigma \restriction \kappa^+ = \text{id}$.

Let $D \subseteq \mathcal{P}_{\omega_1}(M)$ be the collection of countable $X \prec M$ such that $\sigma''X$ is (φ, a, κ) -generically correct. It is enough to show that D contains a club. Otherwise. Suppose that $(\mathcal{P}_{\omega_1}(M) \setminus D) \in \mathbb{Q}_{<\delta}$. Let G be $\mathbb{Q}_{<\delta}$ -generic over V such that $(\mathcal{P}_{\omega_1}(M) \setminus D) \in G$, let $j: V \rightarrow N \subseteq V[G]$ be a canonical embedding. Since $j''M \in j(\mathcal{P}_{\omega_1}(M) \setminus D)$ and $j''M \prec j(M)$, $j(\sigma)''(j''M)$ is not $(\varphi, j(a), j(\kappa))$ -generically correct in N . Note that the transitive collapse of $j(\sigma)''(j''M)$ is M . Let $g \in N$ be $<\kappa$ -generic over M . Then for any $x \in \mathbb{R} \cap M[g]$,

$$\begin{aligned} M[g] \models \varphi[x, a^*] &\iff V[g] \models \varphi[x, a] \\ &\iff V[G] \models \varphi[x, a] \\ &\iff N \models \varphi[x, j(a)] \end{aligned}$$

Then $j(\sigma)''(j''M)$ is $(\varphi, j(a), j(\kappa))$ -generically correct in N . This is a contradiction. □

1.3 Scales on $\text{Hom}_{<\lambda}$ sets

We shall show the following.

Theorem 1.27 (Steel). Suppose λ is a limit of Woodin cardinals. Then $\text{Hom}_{<\lambda}$ has the scale property.

Suppose that δ is a Woodin cardinal. Let G be a $(V, \mathbb{Q}_{<\delta})$ -generic and $j: V \rightarrow M \subseteq V[G]$ be a canonical embedding.

Proposition 1.28. Suppose that δ is a Woodin cardinal. Let G be a $\mathbb{Q}_{<\delta}$ -generic over V and $j: V \rightarrow M \subseteq V[G]$ be a canonical embedding. Let μ be a δ^+ -complete ultrafilter over I with $\mu \in V$. Then for any $A \subseteq j(I)$ such that $A \in M$,

$$A \in j(\mu) \iff \exists B \in \mu (j(B) \subseteq A).$$

Proof. For the forward direction, fix $A \in j(\mu)$. Put $A = [f]_G$ where $f \in V$. We may assume that $f: \mathcal{P}_{\omega_1}(Z) \rightarrow \mu$ for some $Z \in V_\delta$. Put

$$B = \bigcup \{f(X) \mid X \in \mathcal{P}_{\omega_1}(Z)\} \in \mu.$$

Then we have that $j(B) \subseteq A$. The other direction is clear. \square

Proposition 1.29. Suppose that δ is a Woodin cardinal. Let G be a $\mathbb{Q}_{<\delta}$ -generic over V and $j: V \rightarrow M \subseteq V[G]$ be a canonical embedding. Let μ be a δ^+ -complete ultrafilter over I with $\mu \in V$. Then $\text{Ult}(M, j(\mu)) \simeq \text{Ult}^*(M, \mu^*)$ where $\text{Ult}^*(M, \mu^*)$ is a ultrapower of M using functions $f: I \rightarrow M$ in $V[G]$.

Proof.

Claim 1. Suppose that $f: I \rightarrow M$ is in $V[G]$. Then for some $h \in M$,

$$f(u) = h(j(u)) \text{ for } \mu\text{-a.e. } u.$$

Proof of claim 1. Working in $V[G]$. Let $\dot{f}_G = f$. For each $u \in I$, let $g \in V$, $Z_u \in V_\delta$ and $a_u \in G$ be such that $\text{dom}(g) = \mathcal{P}_{\omega_1}(Z_u)$ and $a_u \Vdash \dot{f}(\check{u}) = [\check{g}]_{\dot{G}}$. Since μ^* is δ^+ -complete, let a and Z be such that $a_u = a$ and $Z_u = Z$ for μ^* -a.e. u . Say on $B \in \mu^*$. Working in V . For each $u \in B$, take g_u with a domain $\mathcal{P}_{\omega_1}(Z)$ such that $a \Vdash \dot{f}(\check{u}) = [\check{g}_u]_{\dot{G}}$. Define for each $X \in \mathcal{P}_{\omega_1}(Z)$, $h_X(u) = g_u(X)$. Then for all $u \in B$,

$$[\lambda X. h_X]_G(j(u)) = [g_u]_G = f(u).$$

\square

Define $\pi: \text{Ult}^*(M, \mu^*) \rightarrow \text{Ult}(M, j(\mu))$ by

$$\pi([f]_{\mu^*}) = [h_f]_{j(\mu)}$$

where $h_f \in M$ is such that $h_f(j(u)) = f(u)$ for μ^* -a.e. u . It is easy to show that π is well-defined and an isomorphism between $\text{Ult}(M, j(\mu))$ and $\text{Ult}^*(M, \mu^*)$. \square

Proposition 1.30. Suppose that δ is a Woodin cardinal. Let G be a $\mathbb{Q}_{<\delta}$ -generic over V and $j: V \rightarrow M \subseteq V[G]$ be a canonical embedding. Let μ be a δ^+ -complete ultrafilter over I with $\mu \in V$. Let $\nu \leq_{RK} \mu$ via $p: I \rightarrow J$. Let

$$i: \text{Ult}(V, \nu) \rightarrow \text{Ult}(V, \mu)$$

be an induced embedding and

$$i^*: \text{Ult}(V[G], \nu^*) \rightarrow \text{Ult}(V[G], \mu^*)$$

be a liftup. Let

$$j(i): \text{Ult}(M, j(\nu)) \rightarrow \text{Ult}(M, j(\mu))$$

be an induced embedding. Then $j(i) = i^* \upharpoonright \text{Ult}(M, j(\nu))$.

Proof. It is enough to show that the following diagram commutes:

$$\begin{array}{ccc} \text{Ult}^*(M, \nu^*) & \xrightarrow{i^*} & \text{Ult}^*(M, \mu^*) \\ \sigma \uparrow & & \uparrow \tau \\ \text{Ult}(M, j(\nu)) & \xrightarrow{j(i)} & \text{Ult}(M, j(\mu)) \end{array}$$

where σ and τ are isomorphisms. The rest is easy to show. \square

Corollary 1.31. Suppose that δ is a Woodin cardinal. Let G be a $\mathbb{Q}_{<\delta}$ -generic over V and $j: V \rightarrow M \subseteq V[G]$ be a canonical embedding.

1. Let $\bar{\mu}^* = \langle \mu_s^* \mid s \in {}^{<\omega}\omega \rangle$ is a homogeneity system in $V[G]$, where each μ_s is a δ^+ complete ultrafilter in V . Then for any γ ,

$$\text{ms}(\bar{\mu}^*, \gamma)^{V[G]} = \text{ms}(\langle j(\mu_s) \mid s \in {}^{<\omega}\omega \rangle, \gamma)^M$$

$$\text{So } (S_{\bar{\mu}^*})^{V[G]} = (S_{\langle j(\mu_s) \mid s \in {}^{<\omega}\omega \rangle})^M.$$

2. Let $\lambda > \delta$ be a limit of Woodin cardinals. Then $\text{Hom}_{<\lambda}^{V[G]}$ is a Wadge initial segment of $j(\text{Hom}_{<\lambda})$.

Theorem 1.32 (Steel). Suppose λ is a limit of Woodin cardinals. Then $\text{Hom}_{<\lambda}$ has the scale property.

Proof. Let $\kappa_0 < \lambda$ be such that $\text{Hom}_{\kappa_0} = \text{Hom}_{<\lambda}$. Let $\delta_0, \delta_1, \delta_2$ be Woodin cardinals such that $\kappa_0 < \delta_0 < \delta_1 < \delta_2 < \lambda$. Fix $B \in \text{Hom}_{<\lambda}$. It is enough to show that there is a scale $\bar{\varphi} = \{\varphi_n\}_{n \in \omega}$ on B such that

$$A(n, x, y) \iff \varphi_n(x) \leq \varphi_n(y)$$

is δ_1^+ -universally Baire. Then A is κ_0 -homogeneous. Let $\bar{\mu} = \langle \mu_s \mid s \in {}^{<\omega}\omega \rangle$ be a δ_2^+ -homogeneity system such that

$$\mathbb{R} \setminus B = S_{\bar{\mu}}.$$

Put $T = \text{ms}(\bar{\mu}, \theta)$ for sufficiently large θ . Let $\bar{\varphi} = \{\varphi_n\}_{n \in \omega}$ be a left-most branch scale of T . Then $\bar{\varphi}$ is a scale on $B = p[T]$. For definability, we use the tree production lemma. Let $\varphi(v_0, T)$ be the definition of A using a parameter T . For generic absoluteness, let G be $< \delta_2$ -generic over V and let H be δ_2 -generic over $V[G]$. Then for any $x \in V[G]$,

$$x \in p[T]^{V[G]} \iff x \in p[T]^{V[G][H]}$$

and

$$l_x^{V[G]} = l_x^{V[G][H]}$$

where l_x is the left-most branch of T_x . Then

$$V[G] \models l_x \restriction n \leq_{\text{lex}} l_y \restriction n \iff V[G][H] \models l_x \restriction n \leq_{\text{lex}} l_y \restriction n.$$

For stationary tower corectness. Let G be $\mathbb{Q}_{<\delta_2}$ -generic over V and let $j: V \rightarrow M \subseteq V[G]$ be a canonical embedding. Since θ is sufficiently large, $j(\theta) = \theta$. Then

$$\begin{aligned} j(T) &= j(\text{ms}(\langle \mu_s \mid s \in {}^{<\omega}\omega, \theta \rangle)) \\ &= \text{ms}(\langle j(\mu_s) \mid s \in {}^{<\omega}\omega, \theta \rangle) \\ &= \text{ms}(\langle \mu_s^* \mid s \in {}^{<\omega}\omega, \theta \rangle) \\ &= T. \end{aligned}$$

Then

$$V[G] \models \varphi[(n, x, y), T] \iff M \models \varphi[(n, x, y), j(T)].$$

□

1.4 Projective absoluteness

Theorem 1.33 (Woodin). Let λ be a limit of Woodin cardinals, let $A \in \text{Hom}_{<\lambda}$ and let G be $<\lambda$ -generic over V . Then

$$(\text{HC}^V, \in, A) \equiv (\text{HC}^{V[G]}, \in, A^{V[G]}).$$

Proof. Fix $A \in \text{Hom}_{<\lambda}$. For each formula $\varphi(\vec{v}, A)$ and each $\kappa < \lambda$, we shall construct κ -homogeneous tree $T_{\varphi, \kappa}$ such that for any $<\kappa$ -generic G over V ,

$$V[G] \models p[T_{\varphi, \kappa}] = \{\vec{y} \in \mathbb{R}^{<\omega} \mid \varphi[\vec{y}, A^{V[G]}\}\}.$$

By induction on a complexity of φ . If $\varphi \equiv \exists x \psi(x, A)$ where ψ is Σ_0 , then $T_{\varphi, \kappa}$ is a κ -homogeneous tree of A . Suppose that $\varphi \equiv \neg \exists w \psi$. Let δ be a Woodin cardinal such that $\kappa < \delta < \lambda$ and let T_{ψ, δ^+} be a δ^+ -homogeneous tree on $\omega \times \omega \times Z$ which we have defined. Let S be a weakly δ^+ -homogeneous tree on $\omega \times \omega \times Z$ such that $p[S] = \exists^{\mathbb{R}} p[T_{\psi, \delta^+}]$ with a weak homogeneity system $\bar{\mu}$. This equality holds in any generic extension. Put $T_{\varphi, \kappa} = \text{ms}(\bar{\mu}, \theta)$ for sufficiently large θ . □

Theorem 1.34 ($(\Sigma_1^2)^{\text{Hom}_{<\lambda}}$ -absoluteness, Woodin). Let λ be a limit of Woodin cardinals, let $A \in \text{Hom}_{<\lambda}$ and let G be $<\lambda$ -generic over V . Let φ be a formula in $\mathcal{L}_{\in, \dot{A}, \dot{B}}$. Then

$$\begin{aligned} \exists B \in \text{Hom}_{<\lambda}^V(\text{HC}^V, \in, A, B) \models \varphi &\iff \\ \exists B \in \text{Hom}_{<\lambda}^{V[G]}(\text{HC}^{V[G]}, \in, A^{V[G]}, B) \models \varphi. \end{aligned}$$

Proof. Forward direction follows from the projective absoluteness. Let $\delta < \lambda$ be a sufficiently large Woodin cardinal and let H be $\mathbb{Q}_{<\delta}$ -generic over V such that $G \in V[H]$. Since

$$\exists B \in \text{Hom}_{<\lambda}^{V[G]}(\text{HC}^{V[G]}, \in, A^{V[G]}, B) \models \varphi,$$

then

$$\exists B \in \text{Hom}_{<\lambda}^{V[H]}(\text{HC}^{V[H]}, \in, A^{V[H]}, B) \models \varphi.$$

Let $j: V \rightarrow M \subseteq V[H]$ be a canonical embedding. Since $\text{Hom}_{<\lambda}^{V[H]}$ is a Wadge initial segment of $j(\text{Hom}_{<\lambda})$ and $j(A) = A^{V[H]}$,

$$M \models \exists B \in j(\text{Hom}_{<\lambda})(\text{HC}, \in, j(A), B) \models \varphi.$$

By elementarity of j ,

$$\exists B \in \text{Hom}_{<\lambda}^V(\text{HC}^V, \in, A, B) \models \varphi.$$

□

2 AD^+

AD^+ is a technical variant of AD and a generalization of $\text{AD}^{L(\mathbb{R})}$.

Definition 2.1. Θ is the supremum of λ such that there is a surjection $f: \mathbb{R} \rightarrow \lambda$.

Under $\text{AD} + \text{DC}_{\mathbb{R}}$, Θ is a limit cardinal by Moschovakis' coding lemma.

Definition 2.2 (Woodin). AD^+ is the conjunction of the following statements.

1. $\text{DC}_{\mathbb{R}}$,
2. Every set of reals is ∞ -Borel,
3. $<\Theta$ -ordinal determinacy.

It is an open question whether AD implies AD^+ . For $n \in \omega$,

$$A_n(x) \iff x(n_0) = n_1.$$

Definition 2.3 (∞ -Borel code). Let $\kappa \in \text{Ord}$. $S = \langle T, \varphi \rangle$ is a κ -Borel code if T is a well-founded tree on some $\lambda < \kappa$ and φ is a function from the collection of terminal nodes of T to $\{A_n \mid n \in \omega\}$. BC_κ denotes the collection of κ -Borel codes. $\text{BC}_\infty = \bigcup_{\kappa \in \text{Ord}} \text{BC}_\kappa$.

It is useful to regard an ∞ -Borel code as a tree on for some ordinal κ . Let $\langle T, \varphi \rangle$ be an ∞ -Borel code. φ can be extended to T :

$$\varphi(s) = \mathbb{R} \setminus \bigcup_{t \supseteq s, \text{lh}(t) = \text{lh}(s) + 1} \varphi(t).$$

Put $A_T = \varphi(T) = \varphi(\emptyset)$.

Definition 2.4. A set of reals A is ∞ -Borel if $A = \varphi(T)$ for some $\langle T, \varphi \rangle \in \text{BC}_\infty$.

Proposition 2.5. For a set of reals A , the following are equivalent:

1. A is ∞ -Borel,
2. there is a formula φ and $S \subseteq \kappa$ for some $\kappa \in \text{Ord}$, such that

$$A(x) \iff L[S, x] \models \varphi[S, x].$$

Corollary 2.6. Every Suslin set of reals is ∞ -Borel.

Being ∞ -Borel is local property. For a set of reals A , let $\Pi(A)$ be either $\Pi_1^1(A)$ or $\Pi_2^1(A)$ which has the prewellordering property, and let

$$\Delta(A) = \Pi(A) \cap \check{\Pi}(A).$$

Let $\delta(A)$ be a prewellordering of $\Pi(A)$.

Theorem 2.7. For any ∞ -Borel set A , A has an ∞ -Borel code in $\Delta(A)$ with respect to any $\Pi(A)$ -norm on a $\Pi(A)$ -complete set.

Proof. Let T be an ∞ -Borel code for A . For a ∞ -Borel code T , let A_T be an interpretation of T . Let \equiv be an equivalence relation on BC_∞ :

$$T \equiv T' \iff A_T = A_{T'}.$$

If $\text{BC}_{\delta(A)}^{L[A, \equiv]} = \text{BC}_\infty^{L[A, \equiv]}$, then A has an ∞ -Borel code in $\Delta(A)$. Otherwise. Then there is an $\text{BC}_{\delta(A)}$ -antichain $\langle S_\alpha \mid \alpha < \delta(A) \rangle$ of length $\delta(A)$. This gives a prewellordering \leq^* of length $\delta(A)$ with a code $S = \bigvee_{\alpha \leq \beta} S_\alpha \times S_\beta$ in $\text{BC}_{\delta(A)+}$. Then either $A \leq_w \leq^*$ or $A \leq_w \not\leq^*$. \square

Theorem 2.8 (Solovay's basis theorem). If

$$L(\mathbb{R}) \models \exists A \subseteq \mathbb{R} \varphi[A]$$

where φ is Σ_1^2 , then

$$L(\mathbb{R}) \models \exists A \in \Delta_1^2 \varphi[A].$$

Theorem 2.9 (Martin–Steel). Assume $\text{AD}^{L(\mathbb{R})}$. Then $(\Sigma_1^2)^{L(\mathbb{R})}$ has the scale property.

Corollary 2.10. Assume $\text{AD} + V = L(\mathbb{R})$. Then every set of reals is ∞ -Borel.

Theorem 2.11 (Moschovakis–Woodin). Assume $\text{AD} + \text{DC}_{\mathbb{R}}$. Let $\lambda < \Theta$. If $A \subseteq \lambda^\omega$ is Suslin-co-Suslin, then A is determined.

Theorem 2.12 (Kechris). Assume $\text{AD} + V = L(\mathbb{R})$. Then $\text{DC}_{\mathbb{R}}$ holds.

Corollary 2.13. Assume $V = L(\mathbb{R})$. Then AD implies AD^+ .

3 The derived model theorem

Let λ be a limit of Woodin cardinals and G be $\text{Coll}(\omega, < \lambda)$ -generic over V . Put

$$\mathbb{R}^* = \mathbb{R}_G^* = \bigcup_{\alpha < \lambda} \mathbb{R} \cap V[G \restriction \alpha]$$

and

$$\text{Hom}^* = \text{Hom}_G^* = \{p[T] \cap \mathbb{R}^* \mid \exists \alpha < \lambda (T \in V[G \restriction \alpha] \wedge V[G \restriction \alpha] \models T \text{ is } \lambda\text{-absolutely complemented})\}.$$

Or equivalently, for any $\alpha < \lambda$ and $A \in \text{Hom}_{<\lambda}^{V[G \restriction \alpha]}$, define

$$A^* = \bigcup_{\alpha < \beta < \lambda} A^{V[G \restriction \beta]}$$

and put

$$\text{Hom}^* = \{A^* \mid \exists \alpha < \lambda (A \in \text{Hom}_{<\lambda}^{V[G \restriction \alpha]})\}.$$

Then $L(\mathbb{R}^*, \text{Hom}^*)$ is called a derived model of V at λ . The model depends on G , but by homogeneity, first order theory of $L(\mathbb{R}^*, \text{Hom}^*)$ doesn't depend on G .

Theorem 3.1 (Derived model theorem, Woodin). Let λ be a limit of Woodin cardinals and $L(\mathbb{R}^*, \text{Hom}^*)$ be a derived model at λ . Then

1. $\mathbb{R}^* = \mathbb{R}^{V[G]} \cap L(\mathbb{R}^*, \text{Hom}^*)$,
2. $\text{Hom}^* = \{A \subseteq \mathbb{R}^* \mid A \text{ is Suslin-co-Suslin in } L(\mathbb{R}^*, \text{Hom}^*)\}$,
3. $L(\mathbb{R}^*, \text{Hom}^*) \models \text{AD}^+$.

Theorem 3.2. Let λ be a limit of Woodin cardinals, let G be $\text{Coll}(\omega, < \lambda)$ -generic over V , and let φ be a formula in $\mathcal{L}_{\in, A, B}$. Suppose that $A \in \text{Hom}_{<\lambda}^{V[G \restriction \alpha]}$ for some $\alpha < \lambda$. If

$$\exists B \subseteq \mathbb{R}_G^* (B \in L(\mathbb{R}_G^*, \text{Hom}_G^*) \wedge (HC_G^*, \in, A^*, B) \models \varphi),$$

then

$$\exists B (B \in \text{Hom}_{<\lambda}^{V[G \restriction \alpha]} \wedge (HC^{V[G \restriction \alpha]}, \in, A, B) \models \varphi).$$

Proposition 3.3. Let λ be a limit of Woodin cardinals, let G be $\text{Coll}(\omega, < \lambda)$ -generic over V . Then for any $\alpha < \lambda$ and $A \in \text{Hom}_{< \lambda}^{V[G \restriction \alpha]}$,

$$(HC^{V[G \restriction \alpha]}, \in, A) \prec (HC^*, \in, A^*).$$

Proof. This follows by the projective absoluteness and Tarski's elementary chain lemma. \square

Proof of theorem 3.1 modulo theorem 3.2. Fix $x \in \mathbb{R}^{V[G]} \cap L(\mathbb{R}^*, \text{Hom}^*)$. Let $A^* \in \text{Hom}^*$ and $y \in L(\mathbb{R}^*, \text{Hom}^*)$ be such that x is ordinal definable over $L(\mathbb{R}^*, \text{Hom}^*)$ from A^*, y . We may assume that $A, y \in V[G \restriction \alpha]$ for some $\alpha < \lambda$. Since the forcing is sufficiently homogeneous, then $x \in V[G \restriction \alpha]$. To show that Hom^* set is Suslin-co-Suslin in $L(\mathbb{R}^*, \text{Hom}^*)$, it is enough to show that Hom^* set is Suslin in $L(\mathbb{R}^*, \text{Hom}^*)$. This follows by theorem 1.27, since we can express using only real quantifications with a code of a scale. For the converse. Let A be Suslin-co-Suslin in $L(\mathbb{R}^*, \text{Hom}^*)$ witnessed by $T, U \in L(\mathbb{R}^*, \text{Hom}^*)$. By the same argument, $T, U \in V[G \restriction \alpha]$ for some $\alpha < \lambda$. Since $p[T] = \mathbb{R}^* \setminus p[U]$, for all $\alpha < \beta < \lambda$,

$$V[G \restriction \beta] \models p[T] = \mathbb{R} \setminus p[U].$$

This implies $p[T]$ is λ -universally Baire in $V[G \restriction \alpha]$. Then $A \in \text{Hom}^*$. First we shall show that $L(\mathbb{R}^*, \text{Hom}^*) \models \text{AD}$. Suppose not. Then

$$\exists B \subseteq \mathbb{R}^* (B \in L(\mathbb{R}^*, \text{Hom}^*) \wedge (HC^*, \in, B) \models B \text{ is not determined}).$$

By theorem 3.2, this contradicts theorem 1.5. That $L(\mathbb{R}^*, \text{Hom}^*) \models \text{DC}_{\mathbb{R}}$ is easy to show. The other axioms have the form $\forall A \subseteq \mathbb{R} \exists B \subseteq \text{Ord} \varphi$. By the locality of ∞ -Borel codes and Moschovakis' coding lemma, we can reduce " $\exists B \subseteq \text{Ord}$ " to the real quantifier. Then $L(\mathbb{R}^*, \text{Hom}^*) \models \text{AD}^+$ by theorem 2.11 and corollary 2.6. This completes the proof. \square

A General theory of sharps

In this section, we give a general theory of sharps introduced by Solovay in [7]. Some properties of A^\sharp which we shall prove is based on both [16] and [17], and is an easy generalization of that of 0^\sharp . Let A be a transitive set. Let $\varphi(x_0, \dots, x_n)$ be a formula in the language of set theory with constant symbols $\{\dot{a} \mid a \in A \cup \{A\}\}$. A Skolem term for φ is a function $\tau_\varphi: L(A)^n \rightarrow L(A)$ such that

$$\tau_\varphi(\bar{x}) = \begin{cases} y & \text{if } y \text{ is the unique } y \text{ such that } \varphi(y, \bar{x}) \text{ in } L(A), \\ \emptyset & \text{otherwise.} \end{cases}$$

For $A \subseteq B \subseteq L(A)$, $H^{L(A)}(B)$ denotes a Skolem hull of B . $\mathcal{L}_{\in, A}$ is the language of set theory with constant symbols $\{\dot{a} \mid a \in A \cup \{A\}\} \cup \{c_n \mid n \in \omega\}$ and closed under a combination of Skolem terms. $\{c_n \mid n \in \omega\}$ is to represent for indiscernibles. $\mathcal{L}_{\in, A}^*$ is the language without constant symbols $\{c_n \mid n \in \omega\}$.

Definition A.1. EM blueprint for A is the theory in $\mathcal{L}_{\in, A}$ of some structure $(L_\kappa(A), \in, \dot{a}(a \in A \cup \{A\}))$ where $Y \in H_\kappa$ or $\kappa = \text{Ord}$ and $\langle c_n \mid n \in \omega \rangle$ is the increasing enumeration of indiscernibles for $(L_\kappa(A), \in, \dot{a}(a \in A \cup \{A\}))$.

Definition A.2. Let Σ be a EM blueprint for A and α be a ordinal. $\Gamma(\Sigma, \alpha)$ is if it exists, the unique model M such that:

1. $M \models \Sigma \upharpoonright \mathcal{L}_{\in, A}^*$,
2. there is a $I_\alpha \subseteq \text{Ord}^M$ such that $(I_\alpha, \in^M) \simeq (\alpha, \in)$ and I_α is a set of indiscernibles for M ,
3. $H^M(I_\alpha \cup \{\dot{a}^M \mid a \in A \cup \{A\}\}) = M$.

Definition A.3. A EM blueprint Σ for A is a remarkable character for A if Σ satisfies the following:

1. for all $\alpha \in \text{Ord}$, $\Gamma(\Sigma, \alpha)$ is well-founded,
2. for any term $t(x_0, \dots, x_{n-1})$ in $\mathcal{L}_{\in, A}^*$,

$$t(c_0, \dots, c_{n-1}) \in \text{Ord} \implies t(c_0, \dots, c_{n-1}) < c_n$$

is in Σ ,

3. for any term $t(x_0, \dots, x_{m+n})$ in $\mathcal{L}_{\in, A}^*$,

$$t(c_0, \dots, c_{m+n}) < c_m \implies t(c_0, \dots, c_{m+n}) < t(c_0, \dots, c_{m-1}, c_{m+n-1}, \dots, c_{m+2n-1})$$

is in Σ ,

4. Σ satisfies the witness condition: if $\exists x \varphi(x) \in \Sigma$, then there is a term t all of whose constants for indiscernibles appear on $\varphi(x)$ and $\varphi(t) \in \Sigma$.

Almost the same thing for 0^\sharp holds. Let Σ be a remarkable character for A .

Proposition A.4. For a limit ordinal α , let I_α be a set of indiscernibles for $\Gamma(\Sigma, \alpha)$. Then I_α is unbounded in $\text{Ord}^{\Gamma(\Sigma, \alpha)}$.

Proof. Fix $y \in \text{Ord}^{\Gamma(\Sigma, \alpha)}$. Let $\tau, x_1 < \dots < x_n \in I_\alpha$ and $\bar{a} \in (A \cup \{A\})^{<\omega}$ such that τ is a Skolem term and $y = \tau[x_1, \dots, x_n, \bar{a}]$. Since Σ is a remarkable character for A , $\tau[x_1, \dots, x_n, \bar{a}] < x_{n+1}$ for any $x_n < x_{n+1} \in I_\alpha$. \square

Proposition A.5. Let $\gamma < \alpha$ be limit ordinals. Put $I_\alpha = \{i_\xi \mid \xi < \alpha\}$. Then for any $x \in \text{Ord}^{\Gamma(\Sigma, \alpha)}$ such that $x < i_\gamma$,

$$x \in H^{\Gamma(\Sigma, \alpha)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma, \alpha)} \mid a \in A \cup \{A\}\}).$$

Proof. Fix $x \in \text{Ord}^{\Gamma(\Sigma, \alpha)}$ such that $x < i_\gamma$. Let τ , $x_1 < \dots < x_m < y_1 < \dots < y_n \in I_\alpha$ and $\bar{a} \in A \cup \{A\}$ be such that τ is a Skolem term, $y_1 = i_\gamma$ and

$$x = \tau[x_1, \dots, x_m, y_1, \dots, y_n, \bar{a}].$$

Let $w_1, \dots, w_n, z_1, \dots, z_n \in I_\alpha$ be such that

$$x_1 < \dots < x_m < w_1 < \dots < w_n < y_1 < \dots < y_n < z_1 < \dots < z_n.$$

Since $x < y_1$ and Σ is a remarkable character for A ,

$$\tau[x_1, \dots, x_m, y_1, \dots, y_n, \bar{a}] = \tau[x_1, \dots, x_m, z_1, \dots, z_n, \bar{a}].$$

By indiscernibility,

$$\tau[x_1, \dots, x_m, w_1, \dots, w_n, \bar{a}] < \tau[x_1, \dots, x_m, z_1, \dots, z_n, \bar{a}].$$

Then $x = \tau[x_1, \dots, x_m, w_1, \dots, w_n, \bar{a}] \in H^{\Gamma(\Sigma, \alpha)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma, \alpha)} \mid a \in A \cup \{A\}\})$. \square

Proposition A.6. Let α be a limit ordinal. Then I_α is closed in $\text{Ord}^{\Gamma(\Sigma, \alpha)}$.

Proof. Fix a limit ordinal $\gamma < \alpha$ and $x \in \text{Ord}^{\Gamma(\Sigma, \alpha)}$ such that $x < i_\gamma$. Then $x \in H^{\Gamma(\Sigma, \alpha)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma, \alpha)} \mid a \in A \cup \{A\}\}) = \Gamma(\Sigma, \gamma)$. Then for some $\xi < \gamma$, $x < i_\xi$. \square

Proposition A.7. For any uncountable cardinal κ such that $A \in H_\kappa$,

$$\Gamma(\Sigma, \kappa) \simeq L_\kappa(A).$$

Proof. Suppose not. Let $\beta > \kappa$ be such that $\Gamma(\Sigma, \kappa) \simeq L_\beta(A)$. Then

$$\kappa \in H^{\Gamma(\Sigma, \kappa)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma, \alpha)} \mid a \in A \cup \{A\}\})$$

for some limit ordinal $\gamma < \kappa$. However

$$|H^{\Gamma(\Sigma, \kappa)}(\{i_\xi \mid \xi < \gamma\} \cup \{\dot{a}^{\Gamma(\Sigma, \alpha)} \mid a \in A \cup \{A\}\})| < \kappa.$$

This is a contradiction. \square

Proposition A.8. Let κ and λ be uncountable cardinals such that $\kappa < \lambda$ and $A \in H_\kappa$. Then $I_\lambda \cap \kappa = I_\kappa$ and

$$H^{L_\lambda(A)}(I_\kappa \cup \{\dot{a}^{\Gamma(\Sigma, \alpha)} \mid a \in A \cup \{A\}\}) = L_\kappa(A) \prec L(A).$$

Proof. Let J be the first κ members of I_λ . Put

$$M = H^{L_\lambda(A)}(J \cup \{\dot{a}^{\Gamma(\Sigma, \alpha)} \mid a \in A \cup \{A\}\}) \simeq \Gamma(\Sigma, \kappa) \simeq L_\kappa(A).$$

Since $\text{Ord}^M = \kappa$ and M is transitive, then $M = L_\kappa(A)$ and $J = I_\kappa$. \square

Definition A.9. Let A be a transitive set. A^\sharp exists if there is a remarkable character for A .

Proposition A.10. Let A be a transitive set. The following are equivalent:

1. A^\sharp exists,
2. there is a club proper class indiscernibles I for $L(A)$ such that
 - $H^{L(A)}(I \cup A \cup \{A\}) = L(A)$,
 - for any uncountable cardinal κ such that $A \in H_\kappa$, $H^{L(A)}((I \cap \kappa)A \cup \{A\}) = L_\kappa(A)$.

Proposition A.11. Let A be a transitive set. If there is a Ramsey cardinal κ such that $A \in H_\kappa$, then A^\sharp exists.

B Strong partition cardinals below Θ

In this chapter, our base theory is $\text{ZF} + \text{AD} + \text{DC}_\mathbb{R}$. We shall show strong partition cardinals are cofinal below Θ .

Definition B.1. κ is a strong partition cardinal if $\kappa \rightarrow (\kappa)_{<\kappa}^\kappa$.

Proposition B.2. If $\kappa \rightarrow (\kappa)_{<\kappa}^\kappa$, then $\kappa \rightarrow (\kappa)_\mu^\mu$ for all $\mu < \kappa$.

Proof. Easy. □

Proposition B.3. Let A be a set of reals and $\delta = (\delta_1^2(A))^{L(A, \mathbb{R})}$

1. $(\Sigma_1^2(A))^{L(A, \mathbb{R})} = \Sigma_1(L_\delta(A, \mathbb{R})) \cap \mathcal{P}(\mathbb{R})$,
2. $(\Delta_1^2(A))^{L(A, \mathbb{R})} = L_\delta(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$.

Theorem B.4 (Solovay's basis theorem). Let A be a set of reals and x, y be reals. For any formula φ in second order arithmetic, if

$$\exists B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R}) (\text{HC} \models \varphi[A, B, x, y]),$$

then

$$\exists B \in (\Delta_1^2(A))^{L(A, \mathbb{R})} (\text{HC} \models \varphi[A, B, x, y]).$$

For $X \in (\Sigma_1^2(A))^{L(A, \mathbb{R})}$, put

$$x \in X \iff L(A, \mathbb{R}) \models \exists B \subseteq \mathbb{R} \psi[x, y, B]$$

where $y \in \mathbb{R}$. Define a norm on X :

$$\varphi(x) = \text{the least ordinal } \alpha \text{ such that } L_\alpha(A, \mathbb{R}) \models \exists B \subseteq \mathbb{R} \psi[x, y, B].$$

It is easy to show that φ is a $\Sigma_1^2(A)^{L(A, \mathbb{R})}$ -norm on X . By Solovay's basis theorem and Σ_0 -collection, $\Sigma_1^2(A)^{L(A, \mathbb{R})}$ is closed under $\forall^\mathbb{R}$. Then $(\delta_1^2(A))^{L(A, \mathbb{R})}$ is a strong partition cardinal.³ We have the following. The following theorem appears in [6].

³See [6].

Theorem B.5 ($\text{AD} + \text{DC}_{\mathbb{R}}$). Strong partition cardinals are cofinal below Θ .

Using strong partition cardinals, we can prove determinacy of ordinal games.

Theorem B.6 ($\text{AD} + \text{DC}_{\mathbb{R}}$, Moschovakis–Woodin). Let $\lambda < \Theta$. If $A \subseteq \lambda^\omega$ is Suslin-co-Suslin, then A is determined.

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