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## ノーション

- $S = \{z \in \{0, 1\}^K \mid \sum_{k=1}^K z_k = 1\}$

### 1 変分下界全体

$$\ln \int \int \sum_{z^n \in S^n} p(x^n, z^n, \theta, \pi | \alpha_0, \beta_0) d\theta d\pi = \ln \int \int \sum_{z^n \in S^n} q(z^n, \theta, \pi) \frac{p(x^n, z^n, \theta, \pi | \alpha_0, \beta_0)}{q(z^n, \theta, \pi)} d\theta d\pi \quad (1)$$

$$\geq \int \int \sum_{z^n \in S^n} q(z^n, \theta, \pi) \ln \frac{p(x^n, z^n, \theta, \pi | \alpha_0, \beta_0)}{q(z^n, \theta, \pi)} d\theta d\pi \quad (2)$$

$$= \int \int \sum_{z^n \in S^n} q(z^n, \theta, \pi) \ln p(x^n, z^n, \theta, \pi | \alpha_0, \beta_0) d\theta d\pi \quad (3)$$

$$- \int \int \sum_{z^n \in S^n} q(z^n, \theta, \pi) \ln q(z^n, \theta, \pi) d\theta d\pi \quad (4)$$

## 2 前半 (式 (3))

$$(3) = \int \int \sum_{\mathbf{z}^n \in S^n} q(\mathbf{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) \ln p(\mathbf{x}^n, \mathbf{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi} | \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) d\boldsymbol{\theta} d\boldsymbol{\pi} \quad (5)$$

$$= \underbrace{\int \sum_{\mathbf{z}^n \in S^n} q(\mathbf{z}^n) q(\boldsymbol{\theta}) \ln p(\mathbf{x}^n | \boldsymbol{\theta}, \mathbf{z}^n) d\boldsymbol{\theta}}_{\text{第 1 項}} + \underbrace{\int q(\boldsymbol{\theta}) \ln p(\boldsymbol{\theta} | \boldsymbol{\beta}_0) d\boldsymbol{\theta}}_{\text{第 2 項}} \quad (6)$$

$$+ \underbrace{\int \sum_{\mathbf{z}^n \in S^n} q(\mathbf{z}^n) q(\boldsymbol{\pi}) \ln p(\mathbf{z} | \boldsymbol{\pi}) d\boldsymbol{\pi}}_{\text{第 3 項}} + \underbrace{\int q(\boldsymbol{\pi}) \ln p(\boldsymbol{\pi} | \boldsymbol{\alpha}_0) d\boldsymbol{\pi}}_{\text{第 4 項}} \quad (7)$$

$$\text{第 1 項} = \int \sum_{\mathbf{z}^n \in S^n} q(\mathbf{z}^n) q(\boldsymbol{\theta}) \ln p(\mathbf{x}^n | \boldsymbol{\theta}, \mathbf{z}^n) d\boldsymbol{\theta} \quad (8)$$

$$= \sum_{i=1}^n \sum_{\mathbf{z}_i \in S} q(\mathbf{z}_i) \int q(\boldsymbol{\theta}) \ln p(\mathbf{x}_i | \boldsymbol{\theta}, \mathbf{z}_i) d\boldsymbol{\theta} \quad (9)$$

$$= \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \int q(\boldsymbol{\theta}_k) \ln \frac{\Gamma(\sum_{l=1}^d x_{l,i} + 1)}{\prod_{l=1}^d \Gamma(x_{l,i} + 1)} \prod_{l=1}^d \theta_{k,l}^{x_{l,i}} d\boldsymbol{\theta}_k \quad (10)$$

$$= \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \sum_{l=1}^d x_{l,i} \int q(\boldsymbol{\theta}_k) \ln \theta_{k,l} d\boldsymbol{\theta}_k \\ + \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \ln \Gamma \left( \sum_{l=1}^d x_{l,i} + 1 \right) - \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \sum_{l=1}^d \ln \Gamma(x_{l,i} + 1) \quad (11)$$

$$= \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \sum_{l=1}^d x_{l,i} \left\{ \psi(\beta_{n,k,l}^{(t)}) - \psi \left( \sum_{l'=1}^d \beta_{n,k,l'}^{(t)} \right) \right\} \\ + \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \ln \Gamma \left( \sum_{l=1}^d x_{l,i} + 1 \right) - \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \sum_{l=1}^d \ln \Gamma(x_{l,i} + 1) \quad (12)$$

$$\text{第 2 項} = \int q(\boldsymbol{\theta}) \ln p(\boldsymbol{\theta}|\beta_0) d\boldsymbol{\theta} \quad (13)$$

$$= \int q(\boldsymbol{\theta}) \ln \prod_{k=1}^K \left\{ C(\beta_0) \prod_{l=1}^d \theta_{k,l}^{\beta_{0,l}-1} \right\} d\boldsymbol{\theta} \quad (14)$$

$$= \int q(\boldsymbol{\theta}) \ln \left\{ C(\beta_0)^K \prod_{k=1}^K \prod_{l=1}^d \theta_{k,l}^{\beta_{0,l}-1} \right\} d\boldsymbol{\theta} \quad (15)$$

$$= K \ln C(\beta_0) + \sum_{l=1}^d (\beta_{0,l} - 1) \sum_{k=1}^K \int q(\boldsymbol{\theta}_k) \ln \theta_{k,l} d\boldsymbol{\theta}_k \quad (16)$$

$$= K \ln C(\beta_0) + \sum_{l=1}^d (\beta_{0,l} - 1) \sum_{k=1}^K \left( \psi(\beta_{n,k,l}^{(t)}) - \psi \left( \sum_{l'=1}^d \beta_{n,k,l'}^{(t)} \right) \right) \quad (17)$$

$$\text{第 3 項} = \int \sum_{\mathbf{z}^n \in S^n} q(\mathbf{z}^n) q(\boldsymbol{\pi}) \ln p(\mathbf{z}|\boldsymbol{\pi}) d\boldsymbol{\pi} \quad (18)$$

$$= \sum_{i=1}^n \sum_{\mathbf{z}_i \in S} q(\mathbf{z}_i) \int q(\boldsymbol{\pi}) \ln p(\mathbf{z}_i|\boldsymbol{\pi}) d\boldsymbol{\pi} \quad (19)$$

$$= \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \int q(\boldsymbol{\pi}) \ln \pi_k d\boldsymbol{\pi} \quad (20)$$

$$= \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \left( \psi(\alpha_{n,k}^{(t)}) - \psi \left( \sum_{k'=1}^K \alpha_{n,k'}^{(t)} \right) \right) \quad (21)$$

$$\text{第 4 項} = \int q(\boldsymbol{\pi}) \ln p(\boldsymbol{\pi}|\boldsymbol{\alpha}_0) d\boldsymbol{\pi} \quad (22)$$

$$= \int q(\boldsymbol{\pi}) \ln \left\{ C(\boldsymbol{\alpha}_0) \prod_{k=1}^K \pi_k^{\alpha_{0,k}-1} \right\} d\boldsymbol{\pi} \quad (23)$$

$$= \ln C(\boldsymbol{\alpha}_0) + \sum_{k=1}^K (\alpha_{0,k} - 1) \int q(\boldsymbol{\pi}) \ln \pi_k d\boldsymbol{\pi} \quad (24)$$

$$= \ln C(\boldsymbol{\alpha}_0) + \sum_{k=1}^K (\alpha_{0,k} - 1) \left( \psi(\alpha_{n,k}^{(t)}) - \psi \left( \sum_{k'=1}^K \alpha_{n,k'}^{(t)} \right) \right) \quad (25)$$

### 3 後半 (式 (4))

$$(4) = \int \int \sum_{\mathbf{z}^n \in S^n} q(\mathbf{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) \ln q(\mathbf{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) d\boldsymbol{\theta} d\boldsymbol{\pi} \quad (26)$$

$$= \underbrace{\sum_{\mathbf{z}^n \in S^n} q(\mathbf{z}^n) \ln q(\mathbf{z}^n)}_{\text{第 1 項}} + \underbrace{\int q(\boldsymbol{\theta}) \ln q(\boldsymbol{\theta}) d\boldsymbol{\theta}}_{\text{第 2 項}} + \underbrace{\int q(\boldsymbol{\pi}) \ln q(\boldsymbol{\pi}) d\boldsymbol{\pi}}_{\text{第 3 項}} \quad (27)$$

$$\text{第 1 項} = \sum_{\mathbf{z}^n \in S^n} q(\mathbf{z}^n) \ln q(\mathbf{z}^n) \quad (28)$$

$$= \sum_{i=1}^n \sum_{\mathbf{z}_i \in S} q(\mathbf{z}_i) \ln q(\mathbf{z}_i) \quad (29)$$

$$= \sum_{i=1}^n \sum_{k=1}^K r_{i,k}^{(t)} \ln r_{i,k}^{(t)} \quad (30)$$

$$\text{第 2 項} = \int q(\boldsymbol{\theta}) \ln q(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (31)$$

$$= \sum_{k=1}^K \int q(\boldsymbol{\theta}_k) \ln q(\boldsymbol{\theta}_k) d\boldsymbol{\theta}_k \quad (32)$$

$$= \sum_{k=1}^K \int q(\boldsymbol{\theta}_k) \ln \left\{ C(\beta_{n,k}^{(t)}) \prod_{l=1}^d \theta_{k,l}^{\beta_{n,k,l}^{(t)} - 1} \right\} d\boldsymbol{\theta}_k \quad (33)$$

$$= \sum_{k=1}^K \left\{ \ln C(\beta_{n,k}^{(t)}) + \sum_{l=1}^d (\beta_{n,k,l}^{(t)} - 1) \int q(\boldsymbol{\theta}_k) \ln \theta_{k,l} d\boldsymbol{\theta}_k \right\} \quad (34)$$

$$= \sum_{k=1}^K \left\{ \ln C(\beta_{n,k}^{(t)}) + \sum_{l=1}^d (\beta_{n,k,l}^{(t)} - 1) \left( \psi(\beta_{n,k,l}^{(t)}) - \psi \left( \sum_{l'=1}^d \beta_{n,k,l'}^{(t)} \right) \right) \right\} \quad (35)$$

$$\text{第 3 項} = \int q(\boldsymbol{\pi}) \ln q(\boldsymbol{\pi}) d\boldsymbol{\pi} \quad (36)$$

$$= \int q(\boldsymbol{\pi}) \ln \left\{ C(\boldsymbol{\alpha}_n^{(t)}) \prod_{k=1}^K \pi_k^{\alpha_{n,k}^{(t)} - 1} \right\} d\boldsymbol{\pi} \quad (37)$$

$$= \ln C(\boldsymbol{\alpha}_n^{(t)}) + \sum_{k=1}^K (\alpha_{n,k}^{(t)} - 1) \int q(\boldsymbol{\pi}) \ln \pi_k d\boldsymbol{\pi} \quad (38)$$

$$= \ln C(\boldsymbol{\alpha}_n^{(t)}) + \sum_{k=1}^K (\alpha_{n,k}^{(t)} - 1) \left( \psi(\alpha_{n,k}^{(t)}) - \psi \left( \sum_{k'=1}^K \alpha_{n,k'}^{(t)} \right) \right) \quad (39)$$