目次

ノーテーション

- $S = \{\boldsymbol{z} \in \{0,1\}^K \mid \sum_{k=1}^K z_k = 1\}$

1 変分下界全体

$$
\begin{aligned}
\ln \int\int \sum_{\boldsymbol{z}^n \in S^n} p(\boldsymbol{x}^n, \boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi} | \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) d\boldsymbol{\theta} d\boldsymbol{\pi} &= \ln \int\int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) \frac{p(\boldsymbol{x}^n, \boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi} | \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)}{q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi})} d\boldsymbol{\theta} d\boldsymbol{\pi} \quad (1) \\
&\geq \int\int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) \ln \frac{p(\boldsymbol{x}^n, \boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi} | \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)}{q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi})} d\boldsymbol{\theta} d\boldsymbol{\pi} \quad (2) \\
&= \int\int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) \ln p(\boldsymbol{x}^n, \boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi} | \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) d\boldsymbol{\theta} d\boldsymbol{\pi} \quad (3) \\
&\quad - \int\int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) \ln q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) d\boldsymbol{\theta} d\boldsymbol{\pi} \quad (4)
\end{aligned}
$$

2 前半 (式 (3))

$$(3) = \int \int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) \ln p(\boldsymbol{x}^n, \boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi} | \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) d\boldsymbol{\theta} d\boldsymbol{\pi} \tag{5}$$

$$= \underbrace{\int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n) q(\boldsymbol{\theta}) \ln p(\boldsymbol{x}^n | \boldsymbol{\theta}, \boldsymbol{z}^n) d\boldsymbol{\theta}}_{\text{第 1 項}} + \underbrace{\int q(\boldsymbol{\theta}) \ln p(\boldsymbol{\theta} | \boldsymbol{\beta}_0) d\boldsymbol{\theta}}_{\text{第 2 項}} \tag{6}$$

$$+ \underbrace{\int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n) q(\boldsymbol{\pi}) \ln p(\boldsymbol{z} | \boldsymbol{\pi}) d\boldsymbol{\pi}}_{\text{第 3 項}} + \underbrace{\int q(\boldsymbol{\pi}) \ln p(\boldsymbol{\pi} | \boldsymbol{\alpha}_0) d\boldsymbol{\pi}}_{\text{第 4 項}} \tag{7}$$

$$\text{第 1 項} = \int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n) q(\boldsymbol{\theta}) \ln p(\boldsymbol{x}^n | \boldsymbol{\theta}, \boldsymbol{z}^n) d\boldsymbol{\theta} \tag{8}$$

$$= \sum_{i=1}^{n} \sum_{\boldsymbol{z}_i \in S} q(\boldsymbol{z}_i) \int q(\boldsymbol{\theta}) \ln p(\boldsymbol{x}_i | \boldsymbol{\theta}, \boldsymbol{z}_i) d\boldsymbol{\theta} \tag{9}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \int q(\boldsymbol{\theta}_k) \ln \frac{\Gamma(\sum_{l=1}^{d} x_{l,i} + 1)}{\prod_{l=1}^{d} \Gamma(x_{l,i} + 1)} \prod_{l=1}^{d} \theta_{k,l}^{x_{l,i}} d\boldsymbol{\theta}_k \tag{10}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \sum_{l=1}^{d} x_{l,i} \int q(\boldsymbol{\theta}_k) \ln \theta_{k,l} d\boldsymbol{\theta}_k$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \ln \Gamma \left(\sum_{l=1}^{d} x_{l,i} + 1 \right) - \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \sum_{l=1}^{d} \ln \Gamma \left(x_{l,i} + 1 \right) \tag{11}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \sum_{l=1}^{d} x_{l,i} \left\{ \psi(\beta_{n,k,l}^{(t)}) - \psi \left(\sum_{l'=1}^{d} \beta_{n,k,l'}^{(t)} \right) \right\}$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \ln \Gamma \left(\sum_{l=1}^{d} x_{l,i} + 1 \right) - \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \sum_{l=1}^{d} \ln \Gamma \left(x_{l,i} + 1 \right) \tag{12}$$

$$
\begin{aligned}
\text{第 2 項} &= \int q(\boldsymbol{\theta}) \ln p(\boldsymbol{\theta}|\boldsymbol{\beta}_0) d\boldsymbol{\theta} && (13)\\
&= \int q(\boldsymbol{\theta}) \ln \prod_{k=1}^{K} \left\{ C(\boldsymbol{\beta}_0) \prod_{l=1}^{d} \theta_{k,l}^{\beta_{0,l}-1} \right\} d\boldsymbol{\theta} && (14)\\
&= \int q(\boldsymbol{\theta}) \ln \left\{ C(\boldsymbol{\beta}_0)^K \prod_{k=1}^{K} \prod_{l=1}^{d} \theta_{k,l}^{\beta_{0,l}-1} \right\} d\boldsymbol{\theta} && (15)\\
&= K \ln C(\boldsymbol{\beta}_0) + \sum_{l=1}^{d} (\beta_{0,l} - 1) \sum_{k=1}^{K} \int q(\boldsymbol{\theta}_k) \ln \theta_{k,l} d\boldsymbol{\theta}_k && (16)\\
&= K \ln C(\boldsymbol{\beta}_0) + \sum_{l=1}^{d} (\beta_{0,l} - 1) \sum_{k=1}^{K} \left(\psi(\beta_{n,k,l}^{(t)}) - \psi\left(\sum_{l'=1}^{d} \beta_{n,k,l'}^{(t)} \right) \right) && (17)
\end{aligned}
$$

$$
\begin{aligned}
\text{第 3 項} &= \int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n) q(\boldsymbol{\pi}) \ln p(\boldsymbol{z}|\boldsymbol{\pi}) d\boldsymbol{\pi} && (18)\\
&= \sum_{i=1}^{n} \sum_{\boldsymbol{z}_i \in S} q(\boldsymbol{z}_i) \int q(\boldsymbol{\pi}) \ln p(\boldsymbol{z}_i|\boldsymbol{\pi}) d\boldsymbol{\pi} && (19)\\
&= \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \int q(\boldsymbol{\pi}) \ln \pi_k d\boldsymbol{\pi} && (20)\\
&= \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \left(\psi(\alpha_{n,k}^{(t)}) - \psi\left(\sum_{k'=1}^{K} \alpha_{n,k'}^{(t)} \right) \right) && (21)
\end{aligned}
$$

$$
\begin{aligned}
\text{第 4 項} &= \int q(\boldsymbol{\pi}) \ln p(\boldsymbol{\pi}|\boldsymbol{\alpha}_0) d\boldsymbol{\pi} && (22)\\
&= \int q(\boldsymbol{\pi}) \ln \left\{ C(\boldsymbol{\alpha}_0) \prod_{k=1}^{K} \pi_k^{\alpha_{0,k}-1} \right\} d\boldsymbol{\pi} && (23)\\
&= \ln C(\boldsymbol{\alpha}_0) + \sum_{k=1}^{K} (\alpha_{0,k} - 1) \int q(\boldsymbol{\pi}) \ln \pi_k d\boldsymbol{\pi} && (24)\\
&= \ln C(\boldsymbol{\alpha}_0) + \sum_{k=1}^{K} (\alpha_{0,k} - 1) \left(\psi(\alpha_{n,k}^{(t)}) - \psi\left(\sum_{k'=1}^{K} \alpha_{n,k'}^{(t)} \right) \right) && (25)
\end{aligned}
$$

3 後半 (式 (4))

$$
\begin{aligned}
(4) &= \int\int \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) \ln q(\boldsymbol{z}^n, \boldsymbol{\theta}, \boldsymbol{\pi}) d\boldsymbol{\theta} d\boldsymbol{\pi} && (26) \\
&= \underbrace{\sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n) \ln q(\boldsymbol{z}^n)}_{\text{第 1 項}} + \underbrace{\int q(\boldsymbol{\theta}) \ln q(\boldsymbol{\theta}) d\boldsymbol{\theta}}_{\text{第 2 項}} + \underbrace{\int q(\boldsymbol{\pi}) \ln q(\boldsymbol{\pi}) d\boldsymbol{\pi}}_{\text{第 3 項}} && (27)
\end{aligned}
$$

$$
\begin{aligned}
\text{第 1 項} &= \sum_{\boldsymbol{z}^n \in S^n} q(\boldsymbol{z}^n) \ln q(\boldsymbol{z}^n) && (28) \\
&= \sum_{i=1}^{n} \sum_{\boldsymbol{z}_i \in S} q(\boldsymbol{z}_i) \ln q(\boldsymbol{z}_i) && (29) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{K} r_{i,k}^{(t)} \ln r_{i,k}^{(t)} && (30)
\end{aligned}
$$

$$
\begin{aligned}
\text{第 2 項} &= \int q(\boldsymbol{\theta}) \ln q(\boldsymbol{\theta}) d\boldsymbol{\theta} && (31) \\
&= \sum_{k=1}^{K} \int q(\boldsymbol{\theta}_k) \ln q(\boldsymbol{\theta}_k) d\boldsymbol{\theta}_k && (32) \\
&= \sum_{k=1}^{K} \int q(\boldsymbol{\theta}_k) \ln \left\{ C(\boldsymbol{\beta}_{n,k}^{(t)}) \prod_{l=1}^{d} \theta_{k,l}^{\beta_{n,k,l}^{(t)} - 1} \right\} d\boldsymbol{\theta}_k && (33) \\
&= \sum_{k=1}^{K} \left\{ \ln C(\boldsymbol{\beta}_{n,k}^{(t)}) + \sum_{l=1}^{d} (\beta_{n,k,l}^{(t)} - 1) \int q(\boldsymbol{\theta}_k) \ln \theta_{k,l} d\boldsymbol{\theta}_k \right\} && (34) \\
&= \sum_{k=1}^{K} \left\{ \ln C(\boldsymbol{\beta}_{n,k}^{(t)}) + \sum_{l=1}^{d} (\beta_{n,k,l}^{(t)} - 1) \left(\psi(\beta_{n,k,l}^{(t)}) - \psi\left(\sum_{l'=1}^{d} \beta_{n,k,l'}^{(t)} \right) \right) \right\} && (35)
\end{aligned}
$$

$$
\begin{aligned}
\text{第 3 項} &= \int q(\boldsymbol{\pi}) \ln q(\boldsymbol{\pi}) d\boldsymbol{\pi} && (36) \\
&= \int q(\boldsymbol{\pi}) \ln \left\{ C(\boldsymbol{\alpha}_n^{(t)}) \prod_{k=1}^{K} \pi_k^{\alpha_{n,k}^{(t)} - 1} \right\} d\boldsymbol{\pi} && (37) \\
&= \ln C(\boldsymbol{\alpha}_n^{(t)}) + \sum_{k=1}^{K} (\alpha_{n,k}^{(t)} - 1) \int q(\boldsymbol{\pi}) \ln \pi_k d\boldsymbol{\pi} && (38) \\
&= \ln C(\boldsymbol{\alpha}_n^{(t)}) + \sum_{k=1}^{K} (\alpha_{n,k}^{(t)} - 1) \left(\psi(\alpha_{n,k}^{(t)}) - \psi\left(\sum_{k'=1}^{K} \alpha_{n,k'}^{(t)} \right) \right) && (39)
\end{aligned}
$$