Quantum Error Correction Codes

3-Qubit and 9-Qubit Shor Codes

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October 11, 2025

Outline

3-Qubit Bit-Flip Code: Overview

Code Definition

The 3-qubit repetition code (Shor, 1995) encodes a single logical qubit into three physical qubits with protection against single bit-flip (X) errors.

Logical basis states:

$$|0\rangle_L = |000\rangle\,, \quad |1\rangle_L = |111\rangle \tag{1}$$

Arbitrary logical state:

$$|\psi\rangle_L = \alpha\,|000\rangle + \beta\,|111\rangle \tag{2}$$

where $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is the input state.

Code Distance and Error Correction Capability

Code Distance

The minimum Hamming distance between codewords:

$$d = \min_{i \neq j} d_H(|i\rangle_L, |j\rangle_L) = 3 \tag{3}$$

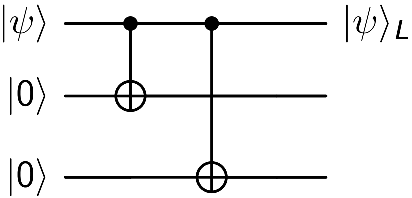
Error Correction Capability

Number of correctable errors:

$$t = \left\lfloor \frac{d-1}{2} \right\rfloor = 1 \tag{4}$$

This enables correction of **1 bit-flip error**.

Encoding Circuit



Encoding Unitary

$$U_{\text{enc}} = \text{CNOT}_{1,3} \cdot \text{CNOT}_{1,2} \quad (5)$$

The circuit creates the redundant encoding by copying the state to two additional qubits.

Error Detection and Syndrome Measurement

Syndrome Extraction

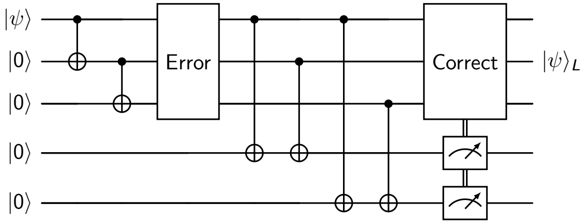
Error correction proceeds by measuring two parity-check operators without collapsing the encoded state. Two ancilla qubits extract syndrome information:

- **Syndrome 1:** Parity of qubits 1 and 2
- **Syndrome 2:** Parity of qubits 1 and 3

Syndrome (s_1, s_2)	**Error Location**	**Correction**
(0, 0)	No error	None
(1, 1)	Qubit 1	X_1
(1, 0)	Qubit 2	X_2
(0, 1)	Qubit 3	X_3

Table: Syndrome Table for 3-Qubit Bit-Flip Code

Complete Error Correction Circuit



Mathematical Analysis of Error Suppression

Consider a coherent rotation error $U_E = e^{i\epsilon X}$ with $\epsilon \ll 1$ acting on all qubits:

$$E = U_E^{\otimes 3} = (c_0 I + c_1 X)^{\otimes 3} \tag{6}$$

where $c_0 = \cos(\epsilon)$ and $c_1 = i\sin(\epsilon)$.

Mathematical Analysis of Error Suppression

The state after error becomes a superposition of 0, 1, 2, or 3 flips:

$$|\psi\rangle_E = (c_0 I^{\otimes 3} + c_1 E_1 + c_2 E_2 + c_3 E_3)|\psi\rangle_L \tag{7}$$

- E_1 represents the sum of single-flip terms
- E_2 represents the double-flip terms
- E_3 represents the triple-flip term

Expanding the error operation:

$$\begin{aligned} E\,|\psi\rangle_L = {} & c_0^3\,|\psi\rangle_L + c_1^3 X_1 X_2 X_3\,|\psi\rangle_L \\ & + c_0^2 c_1 (X_1 + X_2 + X_3)\,|\psi\rangle_L \\ & + c_0 c_1^2 (X_1 X_2 + X_2 X_3 + X_1 X_3)\,|\psi\rangle_L \end{aligned} \tag{8}$$

Error Amplitudes

Amplitude Coefficients

The amplitudes for different numbers of bit-flips are:

$$c_0 = \cos^3(\epsilon) \quad \text{(no flips)} \tag{9}$$

$$c_1 = i\cos^2(\epsilon)\sin(\epsilon) \quad \text{(single flip)} \tag{10}$$

$$c_2 = -\cos(\epsilon)\sin^2(\epsilon) \quad \text{(double flip)} \tag{11}$$

$$c_3 = -i\sin^3(\epsilon) \quad \text{(triple flip)} \tag{12}$$

After Syndrome Measurement and Correction

After syndrome measurement and correction (U_{QEC} is the corresponding operator):

$$
\begin{aligned}
U_{\mathrm{QEC}}\big(E|\psi\rangle_L|00\rangle\big) = {} & c_0^3|\psi\rangle_L|00\rangle \\
& + c_0^2 c_1 \sigma_x\sigma_I\sigma_I|\psi\rangle_L|11\rangle \\
& + c_0^2 c_1 \sigma_I\sigma_x\sigma_I|\psi\rangle_L|10\rangle \\
& + c_0^2 c_1 \sigma_I\sigma_I\sigma_x|\psi\rangle_L|01\rangle \\
& + c_0 c_1^2 \sigma_x\sigma_x\sigma_I|\psi\rangle_L|01\rangle \\
& + c_0 c_1^2 \sigma_I\sigma_x\sigma_x|\psi\rangle_L|11\rangle \\
& + c_0 c_1^2 \sigma_x\sigma_I\sigma_x|\psi\rangle_L|10\rangle \\
& + c_1^3 \sigma_x\sigma_x\sigma_x|\psi\rangle_L|00\rangle
\end{aligned} \tag{13}
$$

Collapsed States After Measurement

Ancilla Measurement	Collapsed State (after normalisation)
00	$c_0\lvert\psi\rangle_L + c_3\sigma_x\sigma_x\sigma_x\lvert\psi\rangle_L$
01	$c_1\lvert\psi\rangle_L + c_2\sigma_x\sigma_x\sigma_x\lvert\psi\rangle_L$
10	$c_1\lvert\psi\rangle_L + c_2\sigma_x\sigma_x\sigma_x\lvert\psi\rangle_L$
11	$c_1\lvert\psi\rangle_L + c_2\sigma_x\sigma_x\sigma_x\lvert\psi\rangle_L$

Table: State after ancilla measurement for different syndrome outcomes

Fidelity of an Unprotected Qubit

State After Error

For a single unprotected qubit, the error U rotates the state:

$$U|\psi\rangle = \cos(\epsilon)|\psi\rangle + i\sin(\epsilon)\sigma_x|\psi\rangle \tag{14}$$

The state is a superposition of:

- Original state (amplitude $\cos(\epsilon)$)
- Bit-flipped state (amplitude $i\sin(\epsilon)$)

Fidelity Calculation

The fidelity measures similarity to the original state:

$$F_{\text{unencoded}} = |\langle\psi|U|\psi\rangle|^2 = \cos^2(\epsilon) \tag{15}$$

Unprotected Qubit: Small Angle Approximation

Taylor Expansion

Using $\cos(\epsilon) \approx 1 - \frac{\epsilon^2}{2}$ for small ϵ:

$$F_{\text{unencoded}} = \cos^2(\epsilon) \approx \left(1 - \frac{\epsilon^2}{2}\right)^2 = 1 - \epsilon^2 + \frac{\epsilon^4}{4} \tag{16}$$

Key Result: Baseline Performance

To first order:

$$\boxed{F_{\text{unencoded}} \approx 1 - \epsilon^2} \tag{17}$$

The infidelity (error) scales as $\epsilon^{\mathbf{2}}$.

This is our benchmark for comparison.

$F_{\text{no error detected}}$

Syndrome Measurement Outcome

The "no error detected" syndrome ($s_1 = 0, s_2 = 0$) projects the state into a subspace containing only:

- Zero-error components ($c_0 I^{\otimes 3}$)
- Three-error components ($c_3 \sigma_x^{\otimes 3}$)

State After Measurement

The unnormalized state after syndrome measurement yields (0,0):

$$|\psi\rangle_{\text{post-no-error}} = (c_0 I^{\otimes 3} + c_3 \sigma_x^{\otimes 3})|\psi\rangle_L \tag{18}$$

Key insight: Three-qubit flips are undetectable because they transform $|000\rangle \leftrightarrow |111\rangle$ (both valid codewords).

Fidelity Calculation: No Error Detected

Ideal State

The ideal, perfectly clean state we want:

$$|\psi\rangle_{\text{ideal}} = I^{\otimes 3}|\psi\rangle_L \tag{19}$$

Fidelity Formula

Fidelity is the squared overlap with normalization:

$$F = \frac{|\langle\psi_{\text{ideal}}|\psi_{\text{post-no-error}}\rangle|^2}{\langle\psi_{\text{post-no-error}}|\psi_{\text{post-no-error}}\rangle} \tag{20}$$

The cross-terms $\langle\psi|_L \sigma_x^{\otimes 3}|\psi\rangle_L = 0$ because logical states are orthogonal.

Computing the Overlap

Numerator

Only the c_0 term overlaps with the ideal state:

$$|\langle\psi|_L c_0 I^{\otimes 3}|\psi\rangle_L|^2 = |c_0|^2 \tag{21}$$

Denominator (Normalization)

$$\langle\psi|_L(|c_0|^2 I^{\otimes 3} + |c_3|^2(\sigma_x^{\otimes 3})^\dagger \sigma_x^{\otimes 3})|\psi\rangle_L = |c_0|^2 + |c_3|^2 \tag{22}$$

Result

$$F_{\text{no error detected}} = \frac{|c_0|^2}{|c_0|^2 + |c_3|^2} \tag{23}$$

Substituting Amplitudes

Computing $|c_0|^2$ and $|c_3|^2$

$$|c_0|^2 = |\cos^3(\epsilon)|^2 = \cos^6(\epsilon) \tag{24}$$

$$|c_3|^2 = |-i\sin^3(\epsilon)|^2 = \sin^6(\epsilon) \tag{25}$$

Fidelity Expression

$$F_{\text{no error detected}} = \frac{\cos^6(\epsilon)}{\cos^6(\epsilon) + \sin^6(\epsilon)} \tag{26}$$

Dividing numerator and denominator by $\cos^6(\epsilon)$:

$$F = \frac{1}{1 + \tan^6(\epsilon)} \tag{27}$$

Small Angle Approximation

Approximation

For small ϵ, $\tan(\epsilon) \approx \epsilon$, so:

$$F_{\text{no error detected}} = \frac{1}{1+\epsilon^6} \approx 1 - \epsilon^6 \tag{28}$$

using the approximation $\frac{1}{1+x} \approx 1 - x$ for small x.

Key Result: No Error Detected

$$\boxed{F_{\text{no error detected}} \approx 1 - \epsilon^6} \tag{29}$$

The infidelity scales as $\boldsymbol{\epsilon^6}$ — a massive improvement!

Case 2 : $F_{\text{error detected}}$

Syndrome Measurement Outcome

When a single-bit-flip error is detected, the syndrome measurement projects the state into:

- Single-flip errors ($c_1 E_1$)
- Double-flip errors ($c_2 E_2$)

These terms anticommute with the stabilizers.

State After Measurement

The unnormalized state after detecting a single error:

$$|\psi\rangle_{\text{post-error}} = (c_1 E_1 + c_2 E_2)|\psi\rangle_L \tag{30}$$

State After Correction

Applying Correction

Apply correction operator X_1 (corresponding to measured syndrome). This:

- Flips the single-error term back to zero-error state
- Turns the double-error term into a three-error state

Corrected State

$$|\psi\rangle_{\text{corrected}} = X_1|\psi\rangle_{\text{post-error}} = (\text{terms} \propto c_1 I^{\otimes 3} + \text{terms} \propto c_2\sigma_x^{\otimes 3})|\psi\rangle_L \tag{31}$$

The ideal state after correction is:

$$|\psi\rangle_{\text{ideal}} = I^{\otimes 3}|\psi\rangle_L \tag{32}$$

Fidelity Calculation: Error Detected

Fidelity Formula

The fidelity is the normalized squared overlap:

$$F_{\text{error detected}} = \frac{|c_1|^2}{|c_1|^2 + |c_2|^2} \tag{33}$$

Only the c_1 component overlaps with the ideal state.

Computing $|c_1|^2$ and $|c_2|^2$

$$|c_1|^2 = |i\cos^2(\epsilon)\sin(\epsilon)|^2 = \cos^4(\epsilon)\sin^2(\epsilon) \tag{34}$$

$$|c_2|^2 = |-\cos(\epsilon)\sin^2(\epsilon)|^2 = \cos^2(\epsilon)\sin^4(\epsilon) \tag{35}$$

Simplifying the Expression

Fidelity Expression

$$F = \frac{\cos^4(\epsilon)\sin^2(\epsilon)}{\cos^4(\epsilon)\sin^2(\epsilon) + \cos^2(\epsilon)\sin^4(\epsilon)} \tag{36}$$

Dividing by $\cos^2(\epsilon)\sin^2(\epsilon)$

$$F = \frac{\cos^2(\epsilon)}{\cos^2(\epsilon) + \sin^2(\epsilon)} = \frac{\cos^2(\epsilon)}{1} = \cos^2(\epsilon) \tag{37}$$

Key Result: Error Detected

$$\boxed{F_{\text{error detected}} = \cos^2(\epsilon) \approx 1 - \epsilon^2} \tag{38}$$

Error Detected Case

Same as Unprotected Qubit

When an error is detected and corrected, the fidelity is:

$$F_{\text{error detected}} \approx 1 - \epsilon^2 \tag{39}$$

This is the **same** as the unprotected qubit!

Why?

- Detection happens when 1 or 2 qubits flip
- Correction fixes single flips, but residual double-flip errors remain
- These residual errors give $O(\epsilon^2)$ infidelity
- However, these events are **rare** (probability $\sim \epsilon^2$)

The key advantage comes from the **no error detected** case, which occurs most often!

Summary of Results

Scenario	Fidelity	Infidelity
Unprotected Qubit	$1 - \epsilon^2$	ϵ^2
Protected (No Error Detected)	$1 - \epsilon^6$	ϵ^6
Protected (Error Detected)	$1 - \epsilon^2$	ϵ^2

Table: Fidelity comparison for different scenarios

The Massive Improvement

Comparing Infidelities

- **Unprotected Qubit Infidelity:** $\approx \epsilon^2$
- **Protected Qubit Infidelity:** $\approx \epsilon^6$

Numerical Example

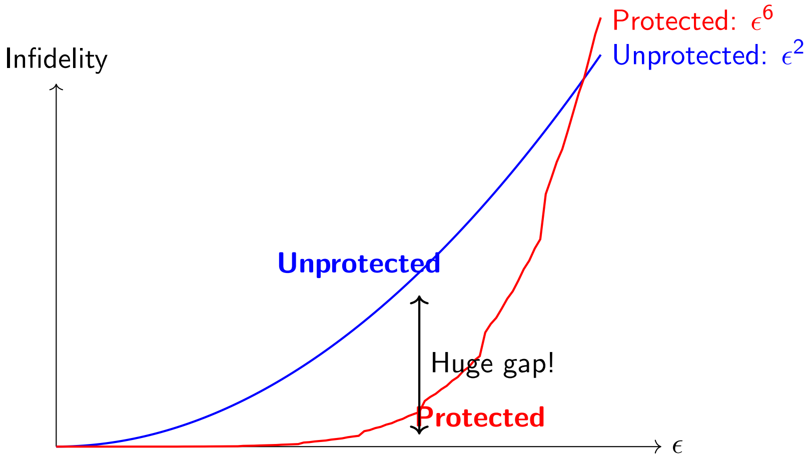
If $\epsilon = 0.01$ (1% rotation):

- Unprotected error: $(0.01)^2 = 0.0001$ (1 in 10,000)
- Protected error: $(0.01)^6 = 10^{-12}$ (1 in a trillion!)

Power of Quantum Error Correction

The error is suppressed from a **second-order effect** (ϵ^2) to a much smaller **sixth-order effect** (ϵ^6), making the logical qubit vastly more reliable than any physical qubit.

Visual Comparison



The gap between protected and unprotected grows dramatically as ϵ increases!

Key Takeaways

1. **Redundancy is powerful:** By encoding one logical qubit into three physical qubits, we suppress errors dramatically.

2. **Error scaling matters:** The transition from ϵ^2 to ϵ^6 scaling makes a trillion-fold difference in error rates.

3. **Undetectable errors dominate:** The three-qubit flip ($\sigma_x^{\otimes 3}$) is undetectable but extremely rare ($\sim \epsilon^6$).

4. **Foundation of fault tolerance:** This principle extends to larger codes, enabling practical quantum computation despite noisy hardware.

Quantum Error Correction Is Possible

3-Qubit Phase-Flip Code: Overview

Implementation Strategy

The phase flip error correction circuit uses a six-phase protocol:

1. Register initialization (3 data qubits + 2 ancilla qubits)
2. State encoding with Hadamard gates
3. Controlled error injection (random σ_z errors)
4. Syndrome measurement via parity checks
5. Error correction based on syndrome
6. State decoding

Key Insight: Hadamard gates transform phase errors into detectable bit-flip errors in the conjugate basis.

Phase-Flip Syndrome Measurement

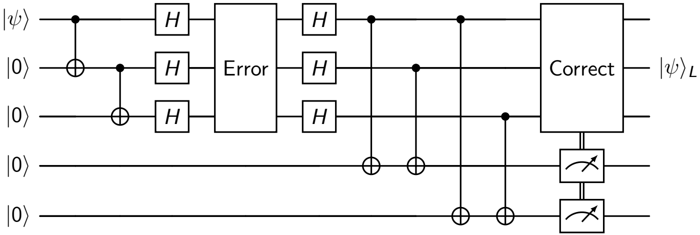
Syndrome Extraction

- **Syndrome 1:** Parity between qubits 0 and 1 (via ancilla[0])
- **Syndrome 2:** Parity between qubits 0 and 2 (via ancilla[1])

Syndrome (s_1, s_2)	**Error Location**	**Correction**
(0,0)	No error	None
(0,1)	Qubit 0	Z_2
(1,0)	Qubit 1	Z_1
(1,1)	Qubit 2	Z_0

Table: Syndrome mapping for phase flip error correction

Phase-Flip Error Correction Circuit



Limitations of 3-Qubit Codes

Problem with Single-Type Protection

The 3-qubit bit-flip code only protects against X errors. Phase errors Z cannot be detected:

$$Z_i(\alpha |000\rangle + \beta |111\rangle) = \alpha |000\rangle \pm \beta |111\rangle \tag{40}$$

Phase errors anticommute with the logical $\overline{X}$ operator and cannot be detected by X-parity checks.

Solution

Shor's 9-qubit code provides simultaneous protection against single X and/or Z errors through concatenated encoding.

Shor Code Construction

Logical Basis States

$$
\begin{aligned}
|0\rangle_L &= \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3} \\
|1\rangle_L &= \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}
\end{aligned} \tag{41}
$$

Code Parameters: $[[9, 1, 3]]$

- $n = 9$ physical qubits
- $k = 1$ logical qubit encoded
- $d = 3$ minimum distance
- Corrects any single Pauli error (X, Y, or Z)

Error Correction Structure

Two-Level Correction

The 9 qubits are divided into three blocks of three qubits each.

1. **Bit-flip correction (within each block):**
   - Apply 3-qubit repetition code correction circuit
   - Detect and correct X errors within blocks
2. **Phase-flip correction (between blocks):**
   - Measure relative phase between blocks using 6 CNOT gates
   - Compare blocks 1 and 2
   - Compare blocks 2 and 3

Syndrome Measurement

X-Error Syndromes

Three sets of two-bit syndromes, one for each block (similar to 3-qubit bit-flip code).

Z-Error Syndromes

$$s_1 = \text{parity}(\text{block}_1, \text{block}_2) \; ; \; s_2 = \text{parity}(\text{block}_2, \text{block}_3) \tag{42}$$

Syndrome (s_1, s_2)	**Error Block**	**Correction**
(0, 0)	No phase error	None
(1, 1)	Block 1	Z on any qubit in block 1
(1, 0)	Block 2	Z on any qubit in block 2
(0, 1)	Block 3	Z on any qubit in block 3

Degeneracy and Correction

Code Degeneracy

Different physical errors map to the same logical error.

- Any single Z error within a block produces the same syndrome
- Correction can apply Z to **any qubit in that block**

Degeneracy and Correction

Handling Y Errors

For a simultaneous bit and phase error on the same qubit:

$$Y_i = iX_iZ_i \tag{43}$$

- The X correction circuit handles the bit-flip component
- The Z correction circuit handles the phase-flip
- Together, they effectively correct the Y error

Shor Code Circuit

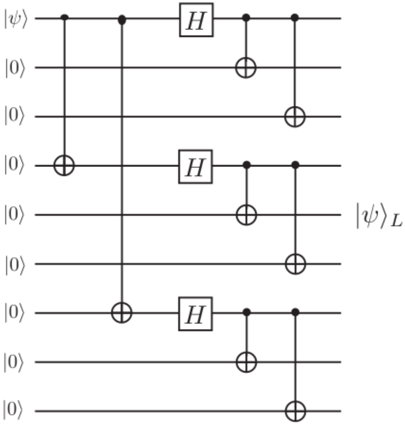


Figure: Encoding Circuit

Shor Code Circuit

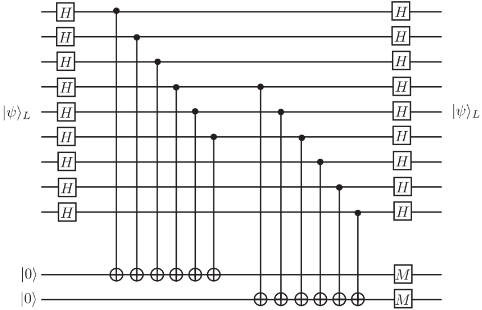


Figure: Circuit to perform Z-error correction

Summary

3-Qubit Codes

- **Bit-flip code:** Protects against X errors
- **Phase-flip code:** Protects against Z errors using Hadamard basis change
- Simple structure but limited to single error type

9-Qubit Shor Code

- **Universal protection:** Simultaneous correction of X, Y, and Z errors
- **Concatenated design:** Combines both 3-qubit codes
- First practical quantum error correction code

Summary

But are these codes effiecient? Can we do anything better?