Compressive Sensing, A Brief Tutorial

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Camera: But only a Single Pixel!

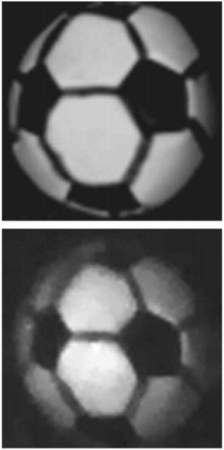


Figure: (Top) Image from Normal Camera (Bottom) Same image obtained from 1600 measurements of Single-Pixel Camera

Sparse Signals

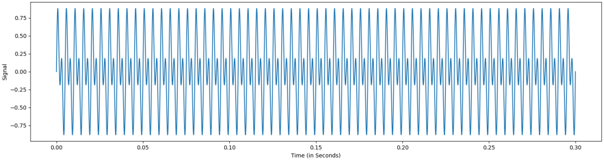


Figure: 1D Signal, sparse in Fourier domain

Compressive Sensing in MRI

TECHNOLOGY | MAGNETIC RESONANCE IMAGING (MRI) | JANUARY 04, 2017

Siemens Introduces Compressed Sensing Acceleration Technology for Faster MRI Scans

Enables MR imaging up to 10 times faster without compromising image quality

At RSNA 2016, Siemens Healthineers unveiled its groundbreaking Compressed Sensing technology, which overcomes a major limitation of magnetic resonance imaging (MRI): long acquisition times. With Compressed Sensing, MRI scans can be shortened dramatically. For example, cardiac cine imaging with Compressed Sensing can be performed in 16 seconds rather than the traditional four minutes, thanks to an innovative algorithm that reduces the amount of data required.2

Introduction

$y = Ax$ where $y \in \mathbb{R}^{m \times 1}$, $A \in \mathbb{R}^{m \times N}$, $x \in \mathbb{R}^{N \times 1}$

1. Case 1: $m = N$, **Linear system of equations**
2. Case 2: $m > N$, **Overdetermined system of equations**
    1. **Overdetermined system of equations** Least squared error solution: $x = (A^T A)^{-1} A^T y$
3. Case 3: $m \ll N$, **Underdetermined system of equations**

Why is the least squared error solution not applicable in Case 3?

Introduction

1. Need of extra information to solve underdetermined system
2. **Sparsity!**

Assume the position of non zero components of x, say set S are known. How do you obtain the solution in this case?

Compressive Sensing: Important Questions

1. Reconstruction of x from $y = Ax$ or from $y = Ax + \eta$
2. Design of measurement matrix A

Measurement Matrix A

Consider the matrix A as given below and explain what is wrong with it (in terms of Compressive Sensing)?

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sparsity and Compressibility

1. Firstly we define the ℓ_0 norm of vector x

$$||x||_{\ell_0} := \lim_{p \to 0} \{\sum_{i=1}^{N} |x_i|^p\}^{1/p} = \text{Number of non-zero entries in } x \tag{1}$$

2. Vector x is an s-sparse vector $\Longleftrightarrow ||x||_{\ell_0} \leq s$
3. Compressibility of vector x, $\sigma_s(x)$, is defined as:

$$\sigma_s(x) = ||x - x_s^*||_{\ell_2} \tag{2}$$

where x_s^* represents the best s term approximation to x

ℓ_0 Norm Minimization

The following statements are equivalent

1. x is unique s-sparse solution of $Az = y$
2. x can be reconstructed as unique solution of

$$x = \underset{z}{min.} \; ||z||_0 \text{ subject to } Az = y \tag{3}$$

Recovery of all s - sparse vectors

All the following statements are equivalent

1. For every s-sparse vector x, if $Az = Ax$, and both x and z are s sparse $\implies x = z$
2. $x \notin \mathcal{N}(A)$ if x is $2s$-sparse (except for 0)
3. Every set of $2s$ columns are independent

Is the last condition equivalent to saying $rank(A) >= 2s$?

Reconstruction Algorithms

1. Search for sparse vector x consistent with data

$$x = \underset{z}{min.} \; ||z||_{\ell_0} \text{ subject to } Az = y \tag{4}$$

2. Reconstruction for noisy measurements

$$x = \underset{z}{min.} \; ||z||_{\ell_0} \text{ subject to } ||Az - y||_2 \leq \epsilon \tag{5}$$

3. Computationally Intractable (How?)

Optimization Methods

- As seen before, the problem can be solved as

$$x = \underset{z}{min.} \; ||z||_{\ell_0} \text{ subject to } Az = y \tag{6}$$

- Approximation of problem: $0 < q < 1$, (P_q)

$$x = \underset{z}{min.} \; ||z||_{\ell_q} \text{ subject to } Az = y \tag{7}$$

- Convex Relaxation $q = 1$, (P_1)

$$x = \underset{z}{min.} \; ||z||_{\ell_1} \text{ subject to } Az = y \tag{8}$$

- Convex Relaxation for Noisy Measurements:

$$x = \underset{z}{min.} \; ||z||_{\ell_1} \text{ subject to } ||Az - y||_{\ell_2} \leq \epsilon \tag{9}$$

Optimization Methods

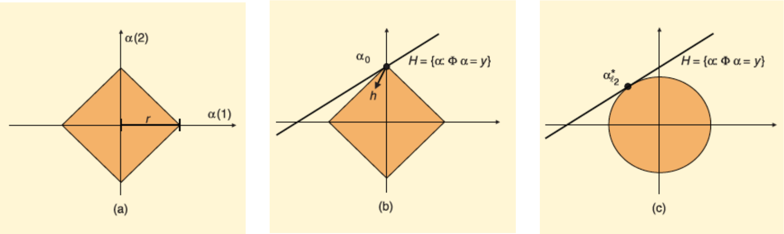


Figure: ℓ_2 minimization Vs. ℓ_1 minimization

Why ℓ_1 minimization works? The Math

1. Restricted Isometry Constant δ_s: smallest $\delta > 0$ such that the following property is satisfied for all s sparse vectors x:

$$(1-\delta)||x||_{\ell_2}^2 \leq ||Ax||_{\ell_2}^2 \leq (1+\delta)||x||_{\ell_2}^2 \tag{10}$$

2. Assume $\delta_{2s} < \sqrt{2} - 1$, and $||\eta||_{\ell_2} \leq \epsilon$. Recovering x from $y = Ax + \eta$ through ℓ_1 minimization provides the solution x^* and obeys the following error bound:

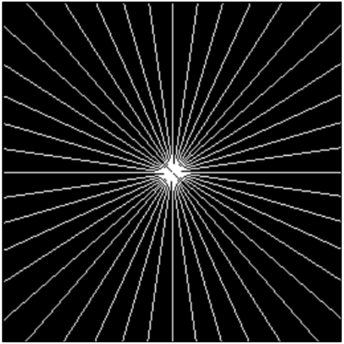
$$||x^* - x||_{\ell_2} \leq C_0 s^{-1/2}||x - x_s||_{\ell_1} + C_1\epsilon \tag{11}$$

Proof here! Not for the weak of heart

Candes, Emmanuel J. "The restricted isometry property and its implications for compressed sensing."

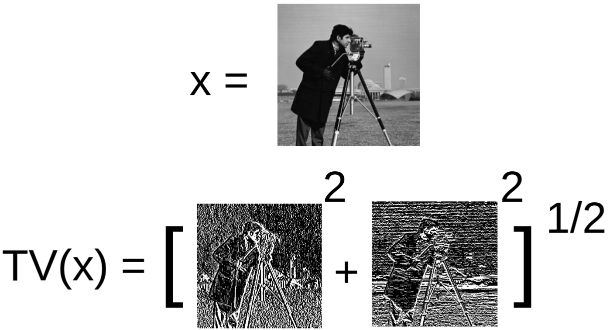
How it All Started: A Puzzling Numerical Experiment

(a) Logan Shepp Phantom



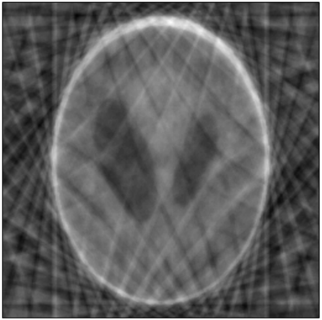
(b) Frequency measurements

How it All Started: A Puzzling Numerical Experiment



(a) Total Variation of Image

How it All Started: A Puzzling Numerical Experiment



(a) Reconstruction using frequency measurements (No regularization)

(b) Reconstruction using $x^* = \arg\min_x ||TV(x)||_{\ell_1}$ s. t. $A\mathscr{F}(x) = y$

Single Pixel - How it Works

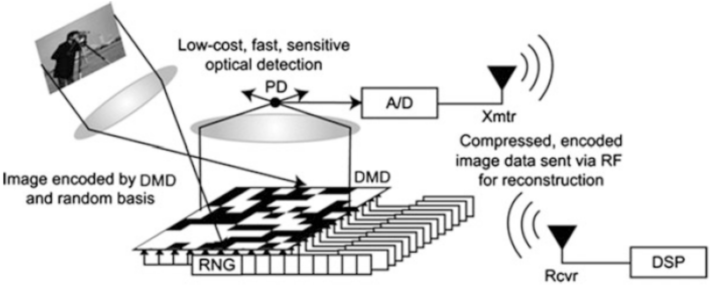


Figure: Schematic: Single Pixel Camera

Single Pixel - How it Works

$$y_{1,a} = \sum_{i \in \text{ switched on pixels}} x_i \tag{12}$$

$$y_{1,b} = \sum_{i \in \text{ switched off pixels}} x_i \tag{13}$$

$$y = y_{1,a} - y_{1,b} = \sum_i a_i x_i \tag{14}$$

where $a_i = +/-1$ with probability $1/2$ each. Writing number of measurements in matrix form as

$$y = Ax \tag{15}$$

We recover the true image as

$$x^* = \arg\min_x ||\Psi x||_{\ell_1} \text{ s.t. } ||Ax - y||_{\ell_2} \leq \eta \tag{16}$$

Greedy Algorithms: Orthogonal Matching Pursuit

1. Greedy procedure, expand support by one coefficient at a time.
2. Define the residue as $r_n = y - Ax_n$

$$j_{n+1} = \underset{j \in [N]}{\text{arg max.}} \ |(A^T r_n)_j| \tag{17}$$

3. Add the coefficient to "hypothesis" support

$$S_{n+1} = S_n \cup j_{n+1} \tag{18}$$

4. Find best solution lying on "hypothesis" support

$$x_{n+1} = \underset{supp(z) \subset S_{n+1}}{\text{arg min.}} \ ||y - Az||_2 \tag{19}$$

How can you explain the choice of index in step 1 of the algorithm?

Thank you