LQ programming

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LQ DYNAMIC PROGRAMMING PROBLEMS

Q: ***What is a linear quadratic programming problem?***
A: ***Mathematical optimisation problem where:***

- **Linear**: law of motion for the state
- **Quadratic**: preferences

Tremendously flexible framework!

Law of motion

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1} \qquad t = 0, 1, 2\ldots$$

state vector
$n \times 1$

$n \times n$

$n \times k$

control vector
$k \times 1$

$n \times j$

zero mean shock, $Ew_t w_t' = I$
$j \times 1$

Example 1

Household budget constraint:

$$a_{t+1} + c_t = (1+r)a_t + y_t$$

assets

consumption

fixed interest rate

income $\{y_t\} \sim N(0, \sigma^2)$

map into linear regulator:

$$a_{t+1} = (1+r)a_t - c_t + \sigma w_{t+1}$$

state $\underset{1\times 1}{}$

$A = I_{1\times 1} + r$

$B = -I_{1\times 1}$

control $\underset{1\times 1}{}$

$C = \sigma_{1\times 1}$

$\{w_t\}$ standard normal

EXAMPLE 2

Problem: non-financial income has a zero mean and is often negative.

$\Longrightarrow$ Take $\mathbf{y_t} = \boldsymbol{\sigma} \mathbf{w_{t+1}} + \boldsymbol{\mu}$ for $\boldsymbol{\mu} > 0$.

Addition:
deviation of consumption from some "ideal" quantity $\bar{c}$

$\Longrightarrow$ Take control to be $\mathbf{u_t} := \mathbf{c_t} - \bar{\mathbf{c}}$.

$$a_{t+1} = (1+r)a_t - u_t - \bar{c} + \sigma w_{t+1} + \mu$$

EXAMPLE 2

$$a_{t+1} = (1+r)a_t - u_t - \bar{c} + \sigma w_{t+1} + \mu$$

Problem: Not linear! Affine

$\implies$ map into the linear regulator by adding a state variable:

$$\begin{pmatrix} a_{t+1} \\ 1 \end{pmatrix} = \begin{pmatrix} (1+r) & -\bar{c}+\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_t \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u_t + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} w_{t+1}$$

GO TO WORKBOOK: EXERCISE 1

Preferences

minimise a flow of losses:

$$\underbrace{x_t' R x_t}_{R:\ \text{symmetric positive semi-definite},\ n\times n} + \underbrace{u_t' Q u_t}_{Q:\ \text{symmetric positive definite},\ k\times k}$$

EXAMPLE 3

In the model of Example 2:

$$R = 0_{2x2} \quad Q = 1$$

Q: Why is R a 2×2 matrix?

GO TO WORKBOOK: EXERCISE 2

EXAMPLE 3

In the model of Example 2:

$$R = 0_{2x2} \quad Q = 1 \implies u_t^2 = (c_t - \bar{c})^2$$

Q: Why is R a 2×2 matrix?

GO TO WORKBOOK: EXERCISE 2

THE OBJECTIVE

FINITE HORIZON: $T \in \mathbb{N}$

Choose a sequence of controls $\{u_0, u_1, ...u_{T-1}\}$ to minimise:

$$E\left\{\sum_{t=0}^{T-1} \beta^t (x_t' R x_t + u_t' Q u_t) + \beta^T x_T' R_f x_T\right\}$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t + C w_{t+1}$$

$$x_0 \text{ given}$$

GO TO WORKBOOK: EXERCISE 3

Information

Decision for time-t control u_t can be made with knowledge of present and past shocks only.

i.e. $u_t = g_t(x_0, w_1, w_2, ..., w_t)$

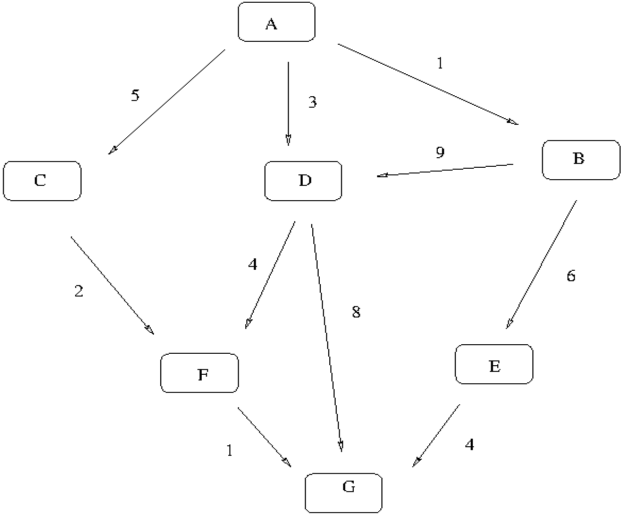
$\implies u_t = g_t(x_t)$

LQ: $g_t(\cdot)$ *linear*

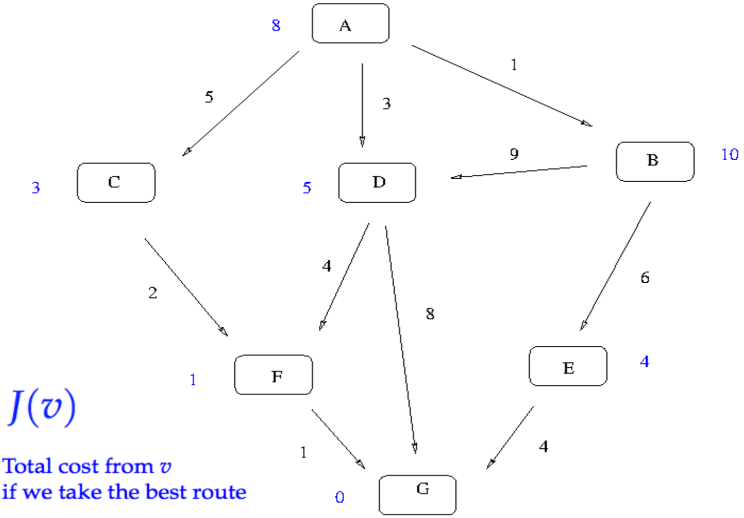
SOLUTION STRATEGY

Skip to Backward Induction

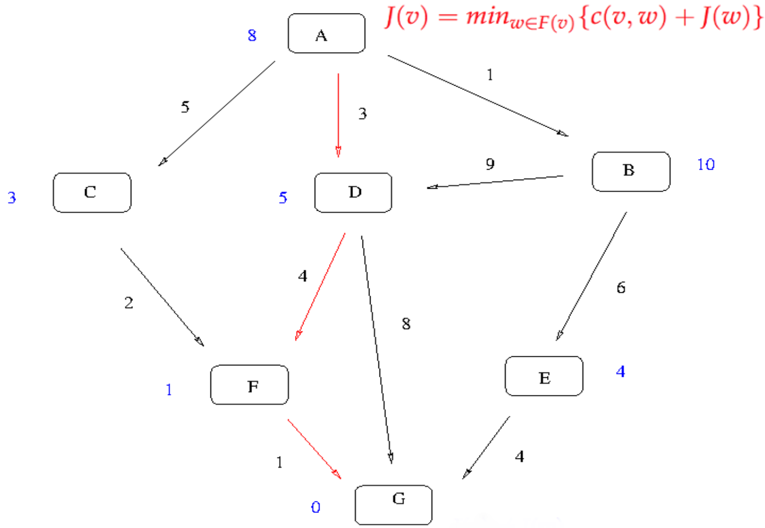
SHORTEST PATHS



Shortest Paths



SHORTEST PATHS



(Rough) sketch of the solution method:

In our problem, we do the same thing only backwards. This gives us a sequence of value functions $\{J_0, J_1, J_2 ... J_T\}$. But we don't know the functional form $J_t(x)$.

1. It turns out that every J_t has the form $J_t(x) = xP_t x + d_t$ where P_t is a $n \times n$ matrix and d_t is a constant

2. The optimal policy rule can be shown to be $u_t = F_t x_t$ where $F_t := (Q + \beta B P_{t+1} B)^{-1} \beta B P_{t+1} A$

3. This means that the state evolves as $x_{t+1} = (A - BF_t)x_t + Cw_{t+1}$

HALL'S PERMANENT INCOME MODEL

Hall's (1978) linear-quadratic permanent income model:

$$E_0 \sum_{t=0}^{\infty} \beta^t (c_t - \bar{c})^2, \beta \in (0,1) \tag{1}$$

$$a_{t+1} + c_t = (1+r)ra_t + y_t \tag{2}$$

$$y_t = \mu_y(1-\rho) + \rho y_{t-1} + \sigma(w_{t+1}) \tag{3}$$

c_t consumption
a_t assets
$\bar{c}$ 'bliss point' constant
$(1+r)$ time-invariant gross rate of return on savings
d_t exogenous endowment
ρ persistence of the endowment
σ scalar on zero mean shock
w_{t+1} zero mean shock

Computation

- Map this into a linear regulator problem by taking
  $u_t = c_t - \bar{c}$ as the control
  $x_t = [1 \; a_{t+1} \; y_t]'$ as the state.
- Parameters taken from Hansen, Sargent and Tallarini (1999). Maintain Hall's assumption that $\beta(1+r) = 1$.