CSC 860 / MATH 795: Stochastic Optimization & Simulation Methodology
Fall 2022 Assignment 4

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Exercises 5.1, 5.3, 6.3, 6.6

EXERCISE 5.1. Consider the following queueing problem. Consecutive inter-arrival times are continuous random variables, iid, with finite moments and unit mean. The mean service time is $\theta > 0$. Let $f_\theta(\cdot)$ be the well defined density of the corresponding service time $S_n(\theta)$ and assume that $\mathsf{Var}(S_n(\theta)) < \infty$ for all finite values of θ. We wish to minimise the cost of operation $C(\theta) = 1/\theta^2$ while satisfying $L(\theta) \stackrel{\text{def}}{=} \mathbb{P}(W(\theta) > w) \le \alpha$, for a constant w, where $W(\theta)$ is a random variable with the stationary waiting time distribution and $\alpha \in (0,1)$.

(a) Argue by using the results of Chapter 1, that at the optimal value θ^* the constraint must be active, that is, $L(\theta^*) = \alpha$.

(b) Let $\{\xi_n\}$ be the sequence of consecutive waiting times. Lindley's equation gives the dynamics of the waiting time process:

$$\xi_n = (\xi_{n-1} + S_n(\theta_n) - A_n)_+, \tag{5.16}$$

where A_n is the inter-arrival time between customers n and $n+1$, and $S_n(\theta)$ is the service time of customer n, given θ_n. Discuss the validity of the stochastic approximation procedure:

$$\theta_{n+1} = \theta_n - \epsilon_n Y_n,$$

for Y_n an estimator of $L(\theta) - \alpha$ obtained observing the process $\{\xi_n\}$. Specify your model (what will you use for Y_n) and verify the assumptions of Theorem 5.1, assuming that $\theta_n < 1$ infinitely often for this procedure.

(c) Instead of using one observation of the process ξ_n to produce Y_n, consider using an estimation interval with K observations. That is, use $\xi_{nK}, \ldots, \xi_{(n+1)K-1}$ to produce the estimate Y_n of the n-th interval. Write a program to simulate the queue under the service time control and experiment with various values of K. Plot the results and discuss them.

A) Since we are trying to minimize cost of operation C(θ) = 1/ θ ^2, C is a decreasing function of θ, so we want θ to be as large as possible. When θ is bigger, W(θ) is bigger. The bigger W(θ), the higher the probability W(θ) > w, so the bigger L(θ), and the constraint is L(θ) cannot be bigger than alpha, L(θ) - alpha = 0. So optimal value θ* is as big as possible to minimize C(θ) until the constraint is reached which is L(θ*) = alpha.

b) Can you use stochastic optimization to converge to θ* where L(θ*) = alpha? If θ is too big, then L(θ) - alpha = Y is positive, then θ n+1 is smaller than θ n, since were multiplying –e*+Y. If θ is too small, L(θ) is less than alpha so L(θ) - alpha = Y is negative and -e*-Y is positive, so we are increasing θ n+1. So, it is valid! Increase or decrease alpha to increase or decrease W(θ) to increase or decrease L(θ) until it equals alpha.

Our model for Yn: Yn is our estimator; we will use Yn to tell us if L(θ) equals alpha or not. The model we will use is below:

L(θ) = is the probability that W(θ) > w, either W(θ) is bigger than w, or it is not. If W(θ) is bigger than w, L(θ) = 1, otherwise, L(θ) = 0.

Yn = L(θ) - alpha, for alpha between (0,1) so it is either 1-alpha = Yn is positive, or it is 0 - alpha = Yn is negative. If Yn is positive, we decrease θ, if Yn is negative, we increase θ.

P(ξ_{n+1} = 0 | ξ_n = 1) = P01 --> decrease because 0 means W(θ) is not bigger than w

P(ξ_{n+1} = 1 | ξ_n = 1) = P11 --> increase, because 1 means W(θ) is bigger than w

Assumptions of theorem 5.1:

A1) transition probabilities are continuous because when θ_n is slightly changed, L(θ_n) is slightly changed, that slightly changes Yn, which slightly changes θ_{n+1}. P probability of θ increasing or decreasing changes slightly based off this. Additionally, the stationary measure for the fixed θ process of observed service times is tight and unique since θ, which is the mean service time, is known.

A2) We do not know W(θ), so we do not know L(θ), we only know θ so we can only take G() which is the long run average not g(). What we are doing in this problem is taking the long run average G(θ), such that each little g() comes together to form big G(), little g() is all the individual observations Yn and when adding them up, we get to the right place G(), and if each time you move more in the right direction you get there eventually.

A3) Yn = 1-alpha or 0-alpha, and alpha is between 0 and 1, so variance of Yn is between 0 and 1. Sum e^2*vi is less than e^2 because vi is less than 1, for some series step size e where sum of e is infinite and e^2 is finite, then if e^2*vi is less than something finite, it is finite.

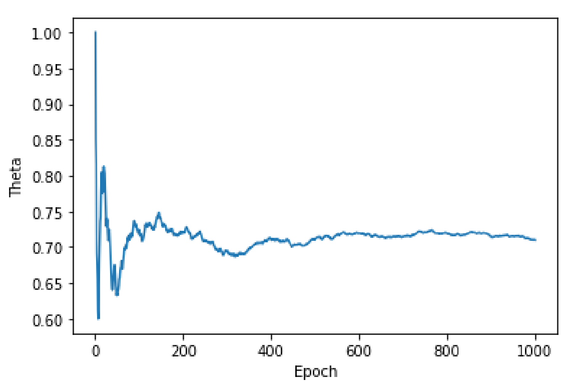
A4) As mentioned in A3, there's some series step size e where sum of e is infinite and sum of e^2 is finite

A5) there are no error terms here, θ is known.

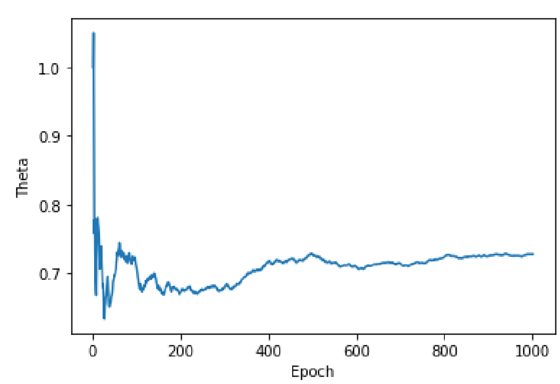
A6) as mentioned, θ is bounded, so the trajectories of the ODE are bounded and for any given θ, the ODE converges to the correct L(θ) - alpha = 0.

c)

`alpha:0.5, w:1.1, Initial θ: 0.5`

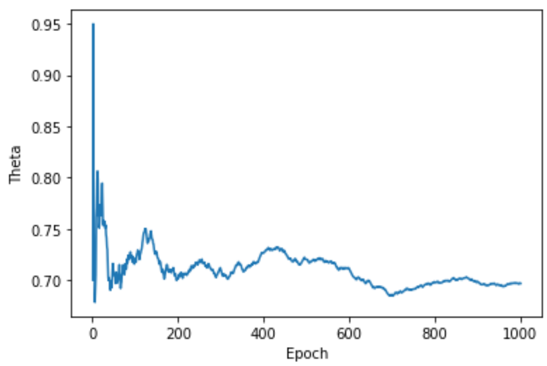


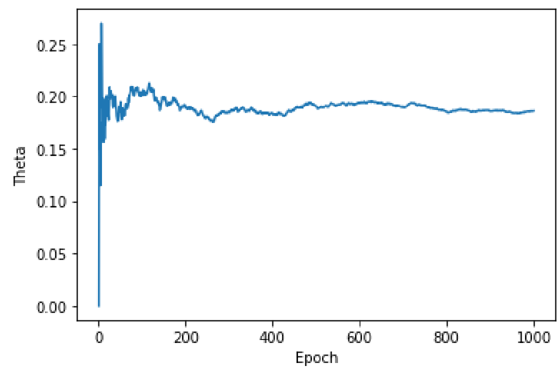
`alpha:0.5 w:1.1 Initial θ: 0.9`



`alpha:0.5 w:1.1 Initial θ: 0.2`

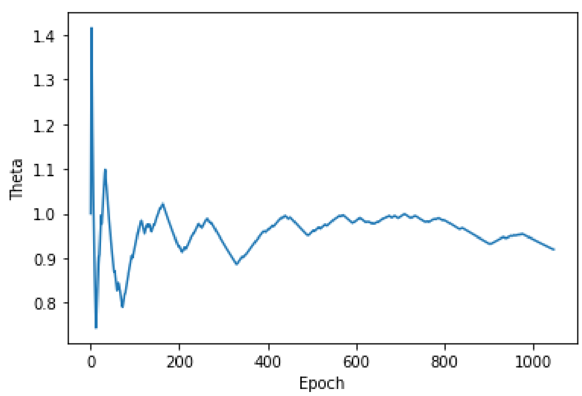
`alpha:0.5 w:0.2 Initial θ: 0.5`

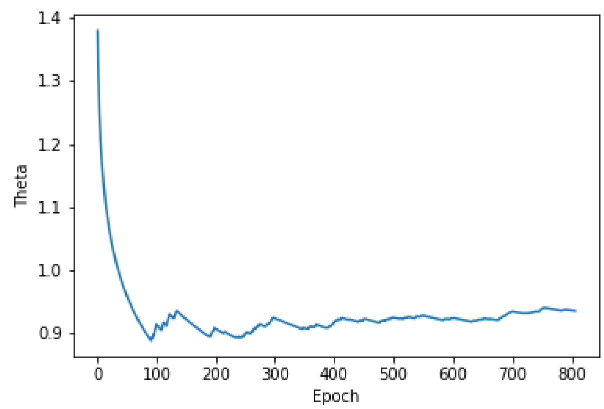




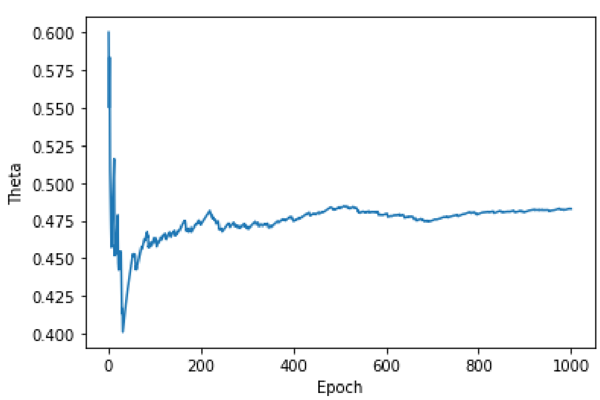
alpha:0.5 w:4.3 Initial θ: 0.5

alpha:0.88 w:1.1 Initial θ: 0.5





alpha:0.1 w:1.1 Initial θ: 0.5



Initial θ was set to .5 to put it somewhere in the middle and see where it converges to. When the initial θ is .9 it is obvious that it is too large, and it comes down right away, and when initial θ is .2 θ jumps around a bit in trying to find its correct path towards convergence.

W was set to 1.1, keeping it in the middle of where W(θ) might be, making w significantly smaller caused θ to converge to a smaller number, and making it significantly larger caused θ to converge to a larger number. This is because when w is larger, P(W(θ) > w) is smaller so L(θ) is smaller, making Yn smaller for constant alpha, causing θ to increase, and the opposite when w is smaller.

Initial alpha was set to .5, keeping it in the middle of [0,1] without it equaling x/5, for x = [0,5]. Increasing alpha caused θ to get bigger since Yn is smaller if we subtract a bigger number from L(θ), causing θ to increase, and the opposite happens when decreasing alpha

EXERCISE 5.3. Consider the model of the "automatic learning" exercise machine that adjusts the resistance to each user so as to enable them to reach their desired target heart rate (refer to Example 5.1, Example 5.2 and Example 5.4.

(a) Suppose that your algorithm considers batches of K intervals of .3 sec in order to get a better estimate of the person's heart rate, so that only every K measurements , θ_n changes:

$$\theta^{\epsilon}_{n+1} = \theta^{\epsilon}_{n} - \epsilon \left(200 \sum_{k=nK}^{(n+1)K-1} \xi_k - K\,\alpha. \right),$$

for $\alpha = 120$. Define the interpolation process $\vartheta^{\epsilon}(\cdot)$ as usual, that is:

$$\vartheta^{\epsilon}(t) = \theta_{m(t)}, \quad m(t) = \left\lfloor \frac{t}{\epsilon} \right\rfloor$$

Using Theorem 5.3, find the limiting ODE for this process. What is the dependency of the behaviour of the limiting ODE on K?

(b) Program the procedure for $K = 1, 10, 20$ and show the plots with $\epsilon = 0.001$. Discuss your results.

(c) Program the procedure $\theta_{n+1} = \theta_n + \epsilon_n Y_n$ with decreasing step sizes $\epsilon_n = \mathcal{O}(1/n)$, plot and discuss the results. If you were to patent the algorithm for the fitness industry, what scheme would you choose and why?

To find the limiting ODE we applied theorem 5.3. But first we check weather all the conditions of the theorem works.

The model is such that:

$$P\left(\xi_{n+1} = 1 \middle| \xi_n = 0, \theta\right) = p_{1,0}(\theta) = \theta$$
$$P\left(\xi_{n+1} = 1 \middle| \xi_n = 1, \theta\right) = p_{1,1}(\theta) = \theta^2$$

Probabilities are continuous and differentiable function of θ, also they are increase when θ increase.

So $p_{1,i} > 0$, it is ergodic MC. There is exist a unique stationary distribution $\pi_{\theta}(\square)$, and the set of the stationary measures is tight, because the sequence of the stationary measures is mass preserving, i.e. bounded in probabilities:

$$\mathrm{P}\left(\|\xi_n\| \geq L_n\right) \leq \alpha$$

Where $L_n > 0$ and $n > 0$.

Let say $\{\xi_n\}$ satisfy $\mathrm{E}\left(\|\xi_n\|^p\right) \leq r < \infty$, for some r>0 and p>0.

Then, using Markov inequality we have:

$P\left(\|\xi_n\| \geq L\right) \leq \frac{E\left\lfloor \|\xi_n\|^p \right\rfloor}{L^p} \leq \frac{r}{L^p} \to 0$ then $\{\xi_n\}$ uniformly tight, and stationary measure is also uniformly tight.

Y_n has no bias term, and measurements bounded by 200, so the variance of the process is uniformly bounded, conditions a4 and a5 of Theorem are satisfied.

So now we can find out $G(\theta)$.

Stochastic algorithm:

$$\theta^{\epsilon}_{n+1} = \theta^{\epsilon}_n - \epsilon\left(200 \sum_{k=nK}^{(n+1)K-1} \xi_k - K\alpha.\right),$$

Gives us feedback function:

$$Y_n = K\alpha - 200 \sum_{k=nK}^{(n+1)K-1} \xi_k$$

Then by definition:

$$E\left\lfloor Y_n | \Im_{n-1} \right\rfloor = g\left(\xi_{n-1}, \theta\right)$$

So

$$E\left[Y_n | \Im_{n-1}\right] = E\left[K\alpha\right] - 200 \cdot E\left|\sum_{k=nK}^{(n+1)K-1} \xi_k\right| = K\alpha - 200 \cdot E\left|\sum_{k=nK}^{(n+1)K-1} \xi_k\right|$$

$$E\left|\sum_{k=nK}^{(n+1)K-1} \xi_k\right| = E\left[\xi_{nK}\right] + E\left[\xi_{nK+1}\right] + \ldots + E\left[\xi_{(n+1)K-1}\right] =$$
$$= \left[(\xi_{nK} = 0) p_{1,0} + (\xi_{nK} = 1) p_{1,1}\right] + \left[(\xi_{nK} = 0) p_{1,0} + (\xi_{nK} = 1) p_{1,1}\right] + \ldots + \left[(\xi_{nK} = 0) p_{1,0} + (\xi_{nK} = 1) p_{1,1}\right] =$$
$$= \left[0 \cdot \theta + 1 \cdot \theta^2\right] + \left[0 \cdot \theta + 1 \cdot \theta^2\right] + \ldots + \left[0 \cdot \theta + 1 \cdot \theta^2\right] = K \cdot 1 \cdot \theta^2$$

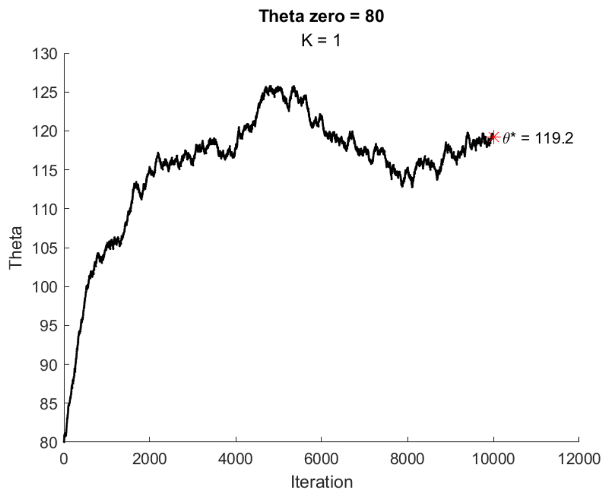
$$g\left(\xi_{n-1}, \theta_n\right) = K\left(\alpha - 200 \cdot \theta_n^2\right)$$
$$G(\theta) = \lim_{N \to \infty} \frac{1}{N} \sum_{n}^{N} K\left(\alpha - 200 \cdot \theta_n^2\right)$$

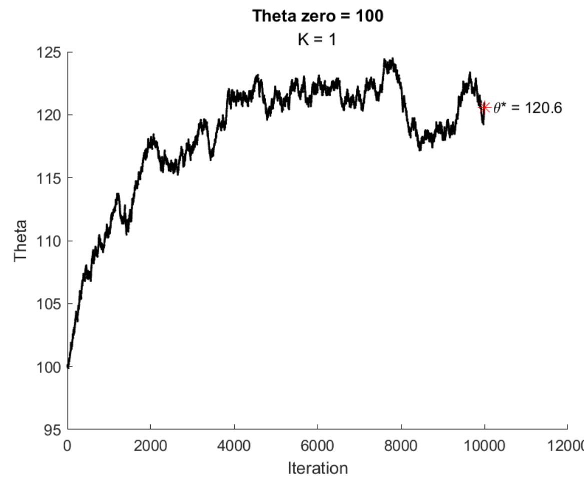
We have linear dependance of the limiting ODE on K.

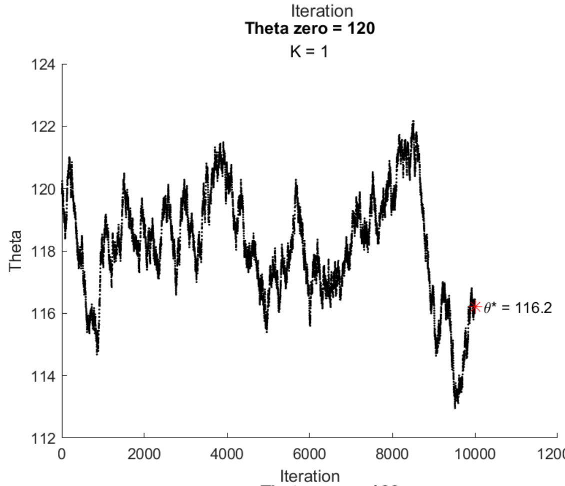
b) The θ is bounded and realistically heart rate could be between 80 and 200 beats per minute.

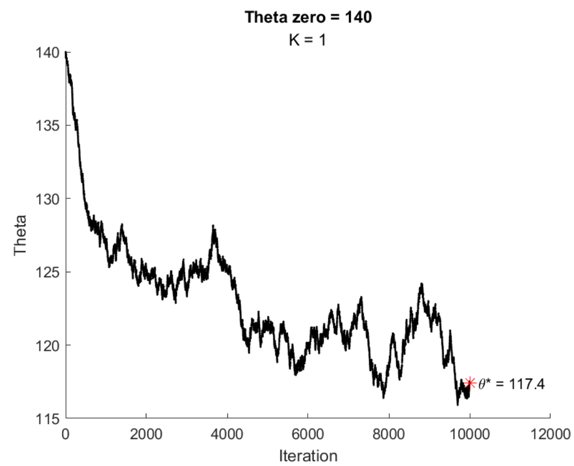
The initial θ will be between 80 and 3. Number of iterations were chosen 10000, which is not necessary because most of the times 6000 iteration are enough to reach the stable point.

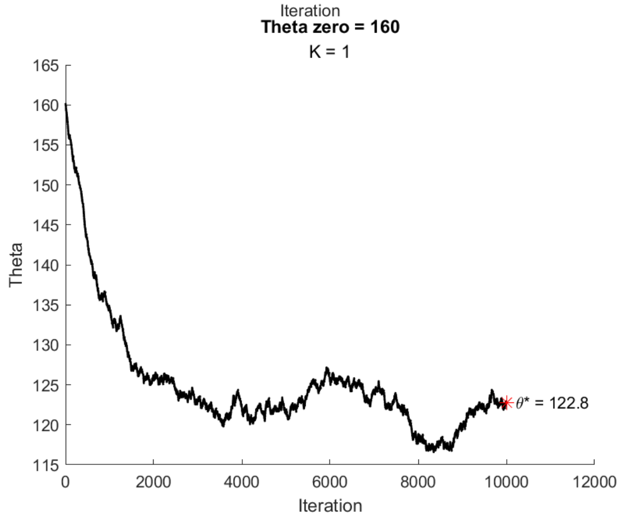
Here are results for the batch size K = 1

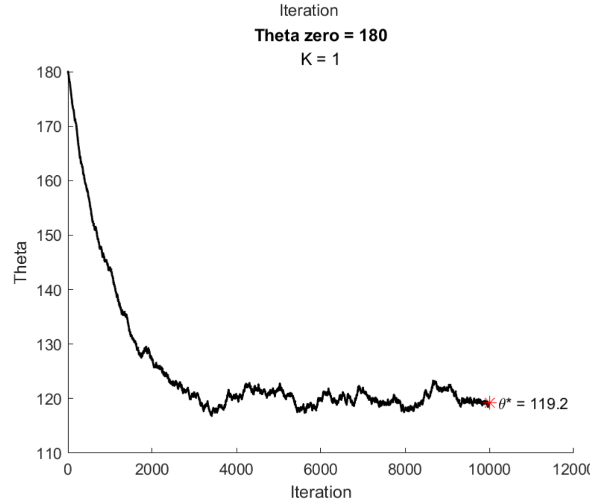


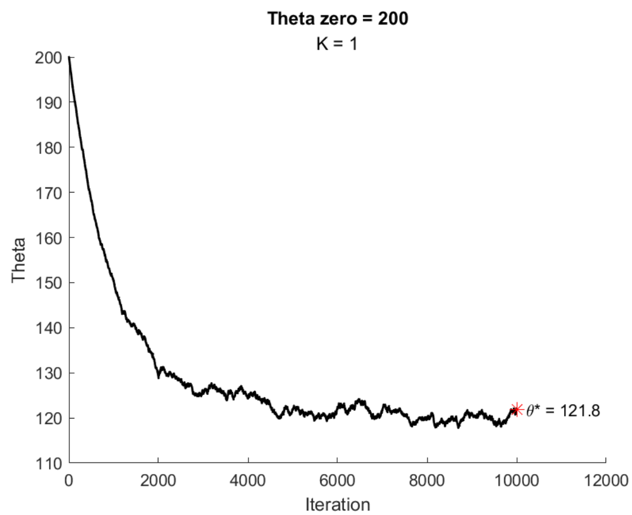






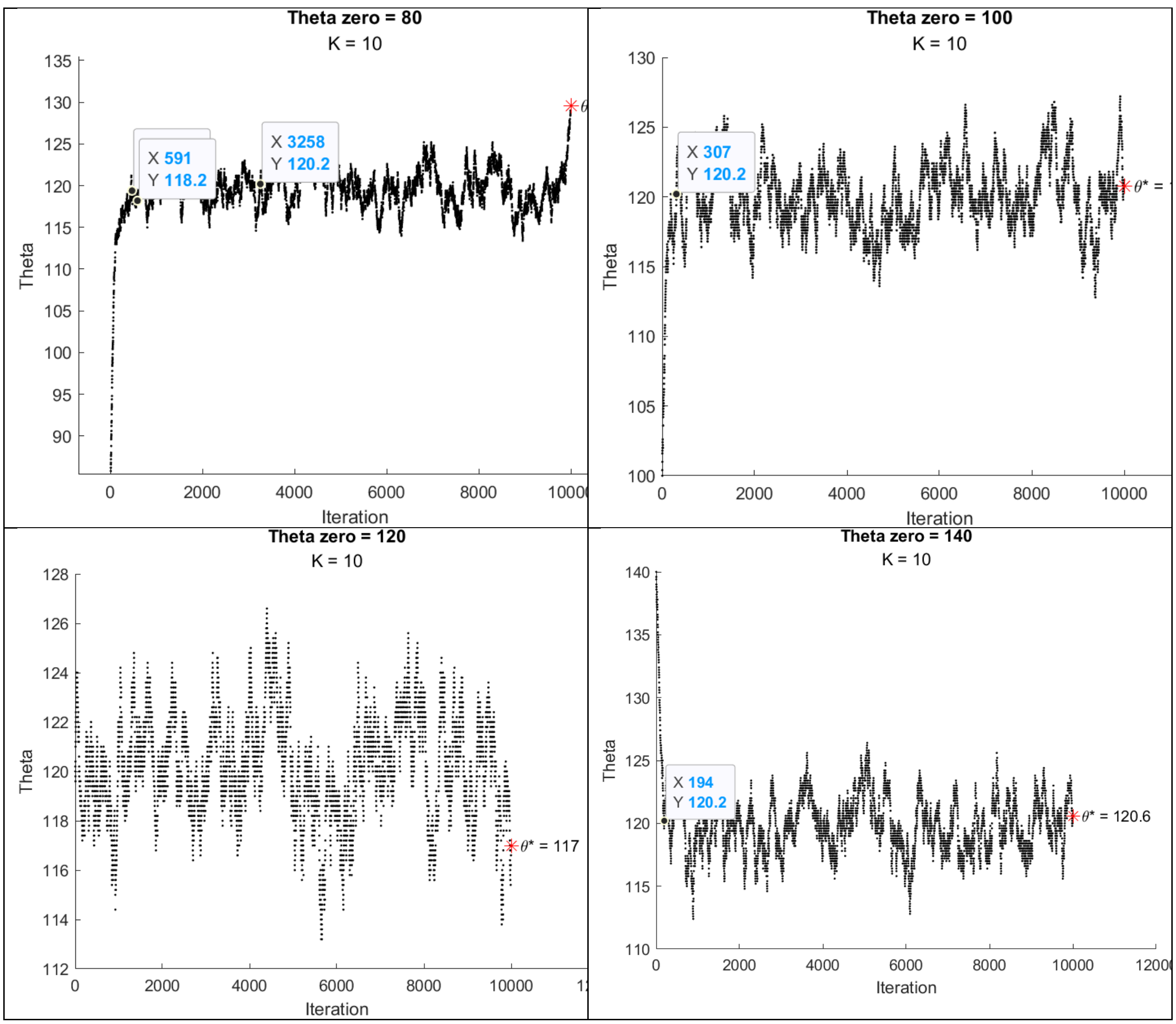


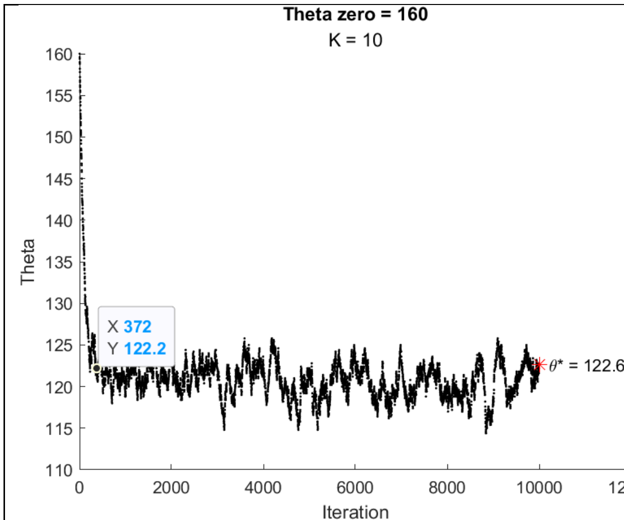


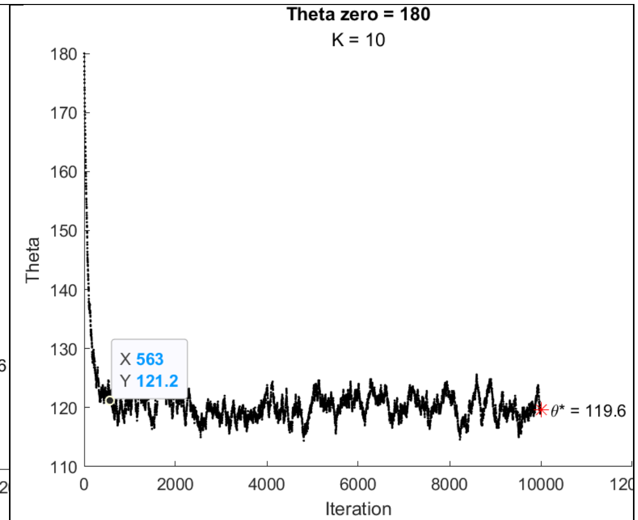


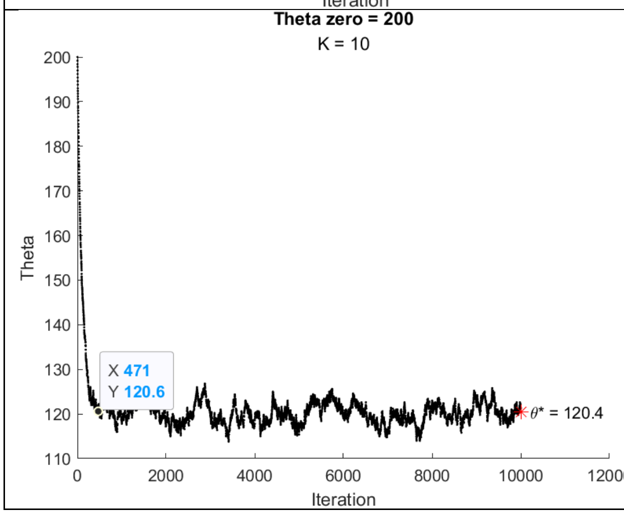
For the batch size K=1, we don't sum over the batches, just calculate the feedback function directly takin gone stochastic measurement. For the batch size K = 1 is takes longer of the algorithm to converge, for example for the θ zero 200 beats per minute we have to do around 4 thousand iterations to reach stable point. In other cases where the beginning heart rate was close enough to 120 (i.e θ = 80, 100, 140) it still takes around 2k iterations to reach stable point.

Here are results for the batch size K = 10



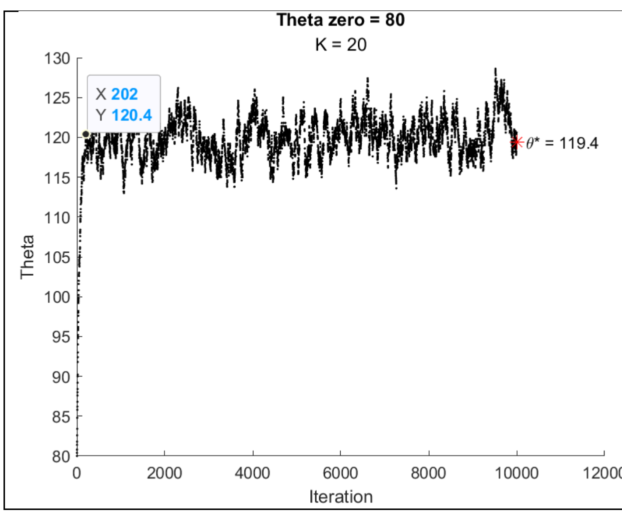


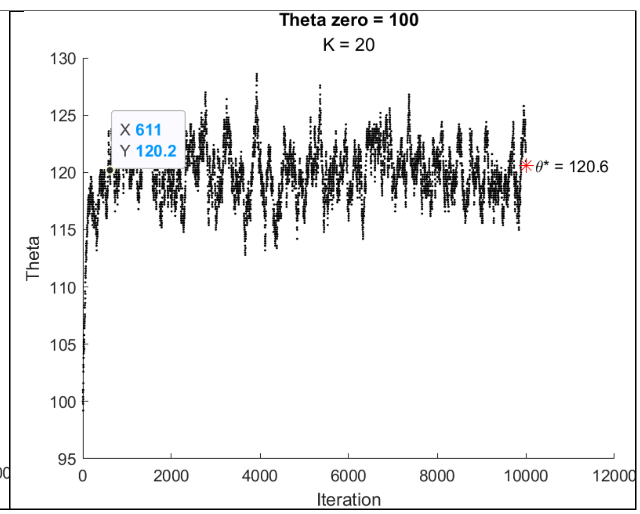


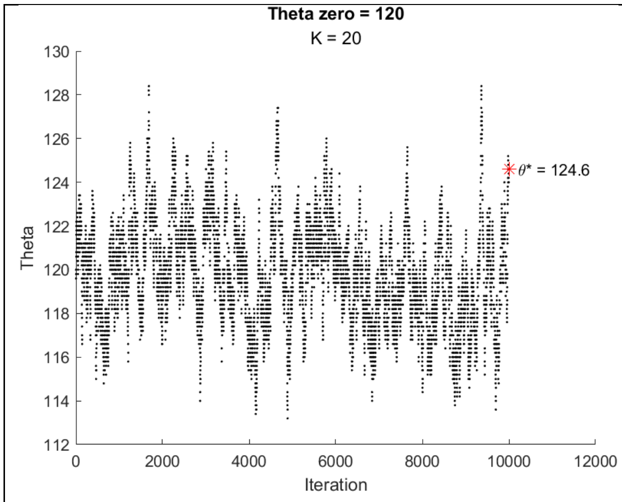


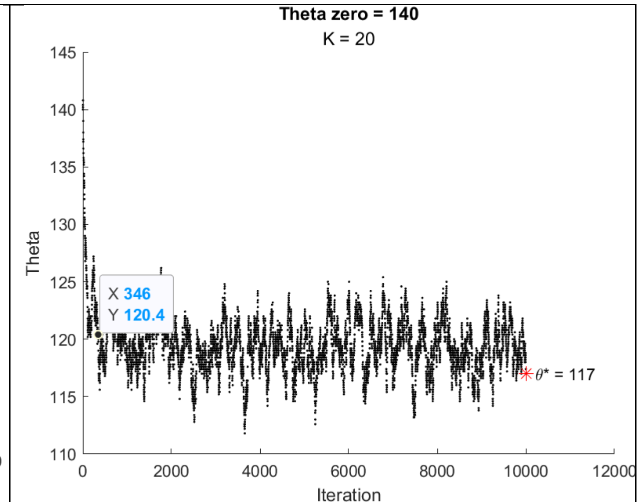
With the batch size K=10, algorithm converges faster. Now, for the θ zero =200 it took the algorithm around 500 iterations to get to stable point. Which is almost in 10 times less compared to algorithm with no batches.

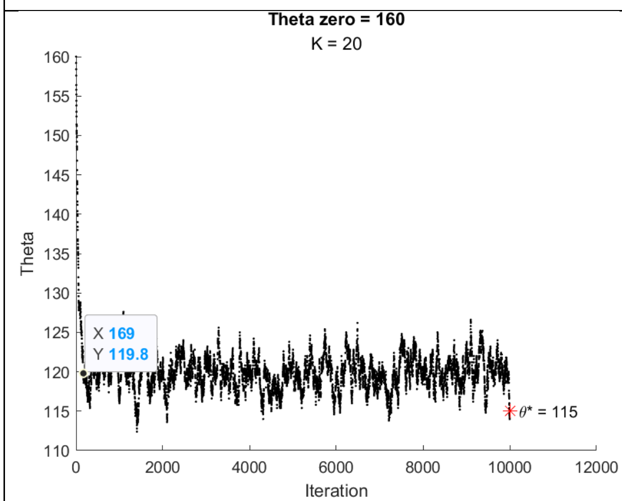
Here are results for the batch size K = 20

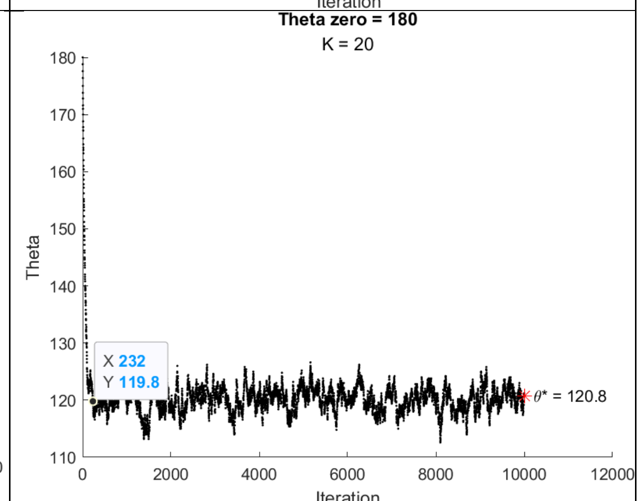


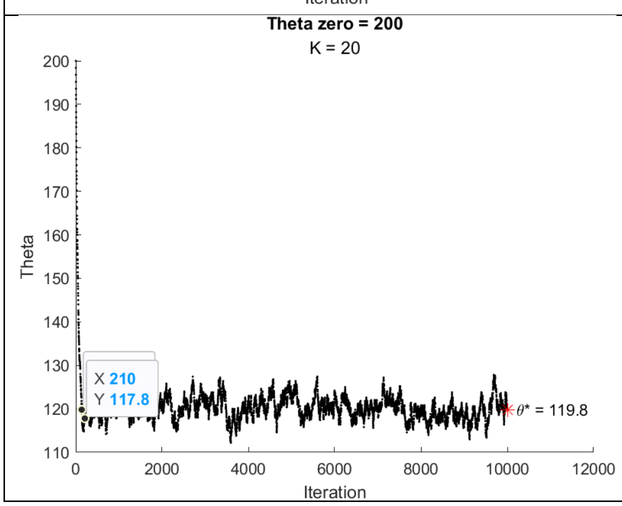








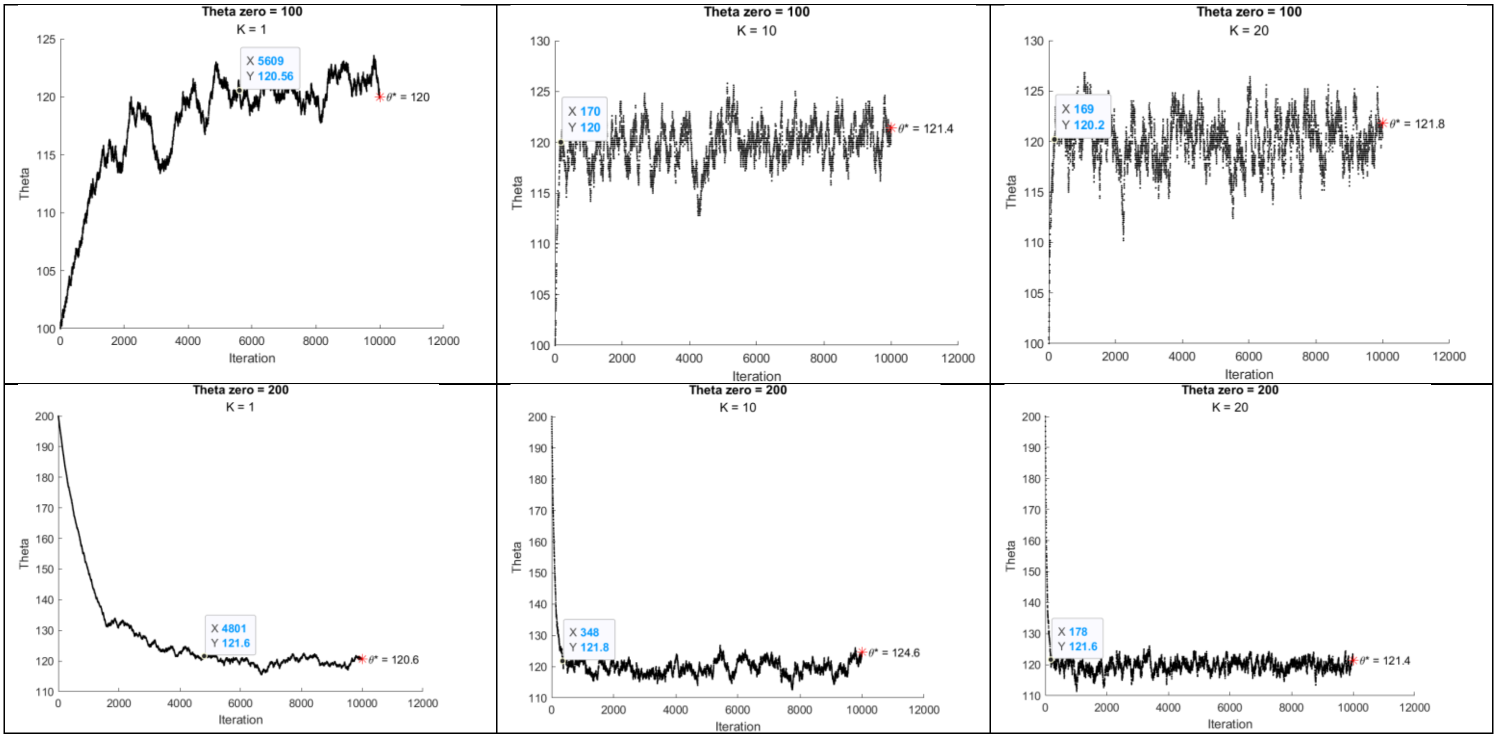




For the farest initial point, algorithm converges in 200 iterations, for the initial points that are closed to the stable point algorithm converges in 200 interactions, which is twice less compared to the batch size K=10.

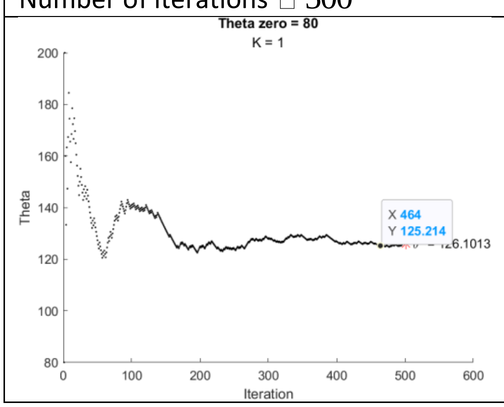
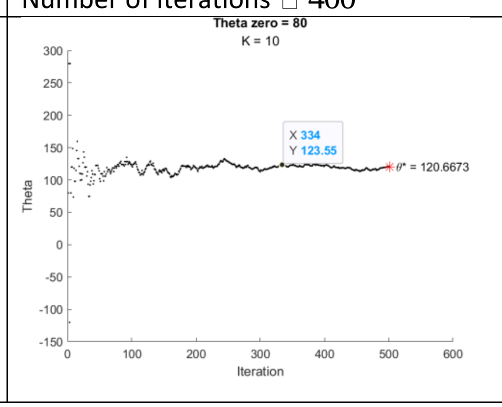
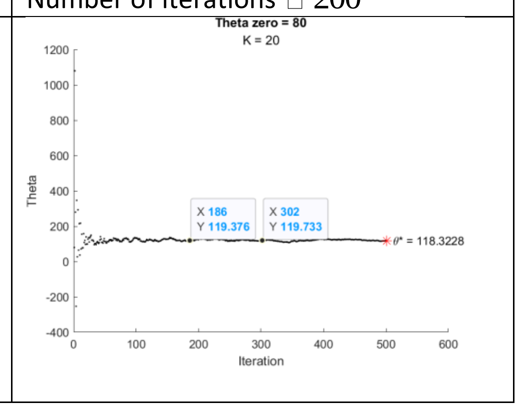
Here is comparison of the initial point 100 and 200:

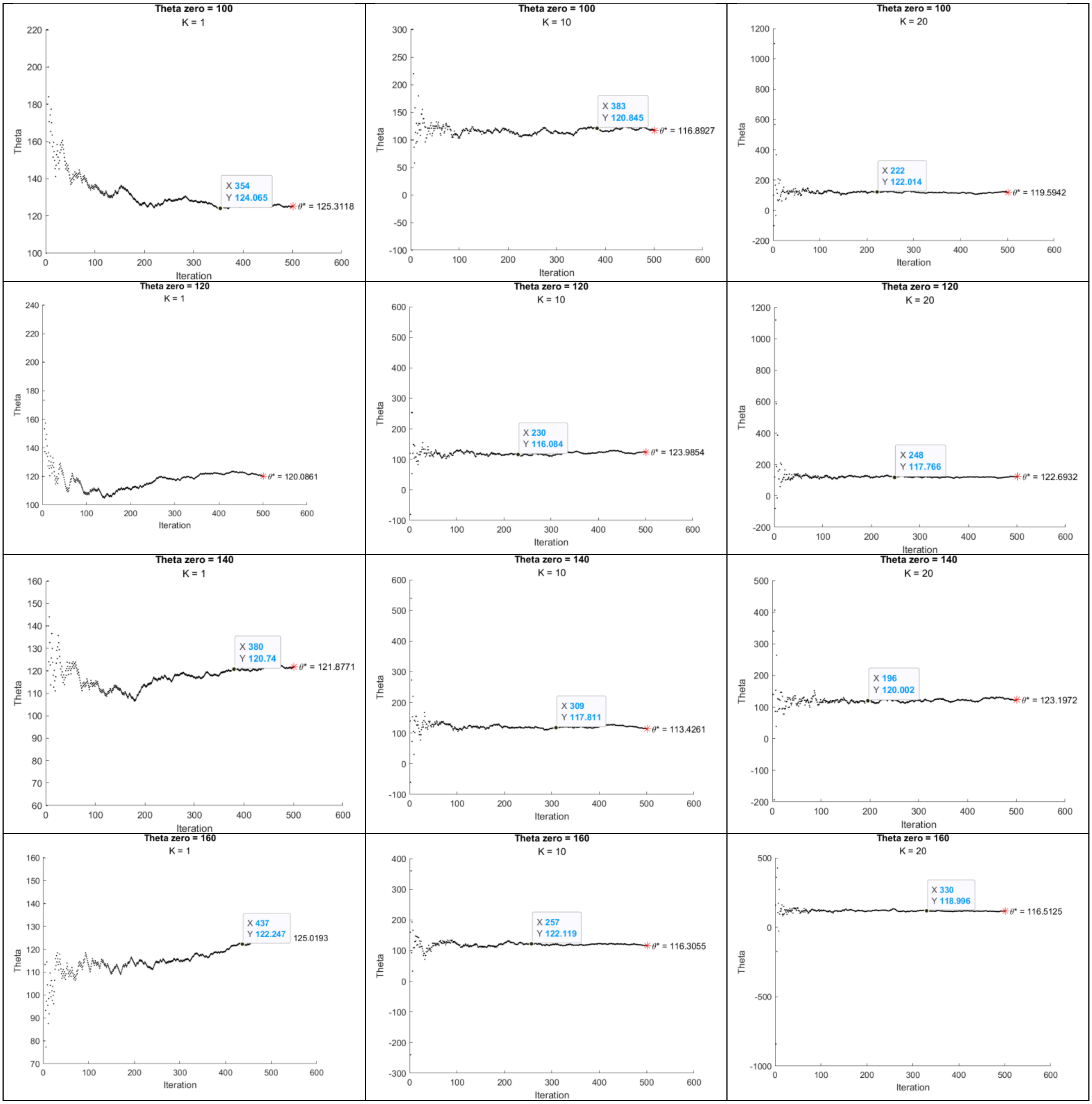
K = 1	K = 10	K = 20
Number of iterations □ 5000	Number of iterations □ 200	Number of iterations ≤ 200



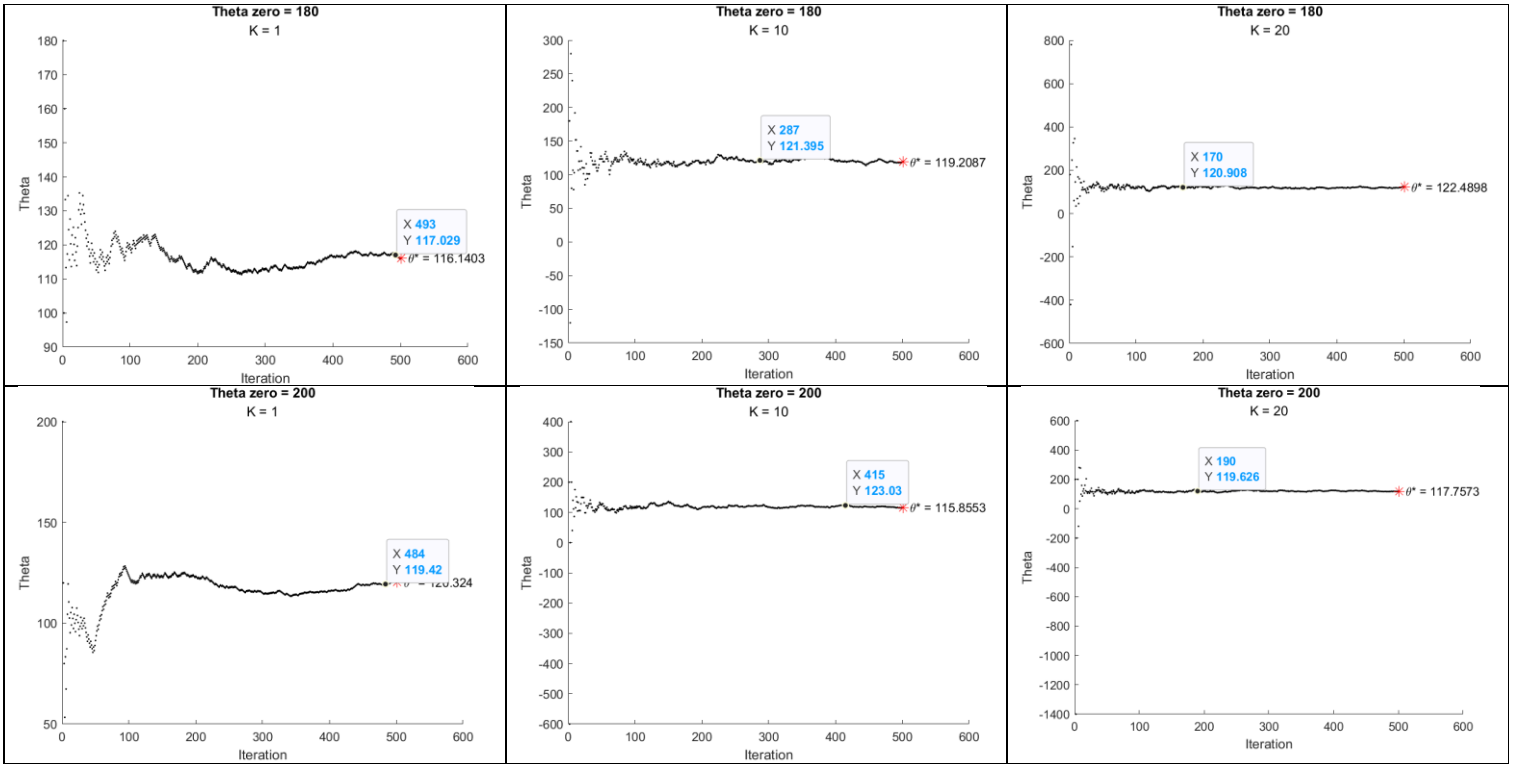
c) Now we choose the step size as decreasing in order of 1/n

The resalts are even better then the constant step size. Here is following graphs, but in general algorithm convergence to the stable point θ = 120 in 500 iterations.

K = 1	K = 10	K = 20
Number of iterations □ 500	Number of iterations □ 400	Number of iterations □ 200
		



Theta zero = 100
K = 1
X 354
Y 124.065
θ* = 125.3118
Theta zero = 100
K = 10
X 383
Y 120.845
θ* = 116.8927
Theta zero = 100
K = 20
X 222
Y 122.014
θ* = 119.5942
Theta zero = 120
K = 1
θ* = 120.0861
Theta zero = 120
K = 10
X 230
Y 116.084
θ* = 123.9854
Theta zero = 120
K = 20
X 248
Y 117.766
θ* = 122.6932
Theta zero = 140
K = 1
X 380
Y 120.74
θ* = 121.8771
Theta zero = 140
K = 10
X 309
Y 117.811
θ* = 113.4261
Theta zero = 140
K = 20
X 196
Y 120.002
θ* = 123.1972
Theta zero = 160
K = 1
X 437
Y 122.247
125.0193
Theta zero = 160
K = 10
X 257
Y 122.119
θ* = 116.3055
Theta zero = 160
K = 20
X 330
Y 118.996
θ* = 116.5125
Theta
Iteration



Algorithms itself could not be patentable, however the application of the algorithm to certain problem that has novelty in the invention of the solving this problem is patentable. So assuming that his algorithm has novelty of creating the device for fitness industry, we will denote the steps of this algorithm as following.

The person has an device that measure instant heart rate for the 5 consecutive times we get our $\{\xi_k\}$ batch, then the feedback function is updating the speed of the machine in the following. If measured hart rate is lover then desirable heart rate alpha, we add speed to fitness machine, if the hart rate of the person higher than desirable we lower the speed of the fitness machine.

EXERCISE 6.3. Consider the supply/demand problem of Example 6.3. The demand function $D(\theta) = \theta^{-\eta}, \eta > 0$ is known. However the supply function $S(\theta)$ is only known to be an increasing function of θ that is analytic (infinitely continuously differentiable). Instead, a complex simulation model is used to produce statistically independent unbiased estimates ξ_n such that

$$\mathbb{E}[\xi_n \mid \theta_n] = S(\theta_n); \quad \mathsf{Var}[\xi_n] = 1. \tag{6.11}$$

(a) Show that

$$\theta_{n+1} = \theta_n + \epsilon Y_n, \quad Y_n = D(\theta_n) - \xi_n$$

satisfies the assumptions of Theorem 6.1.

(b) Use $d\eta = 5$ for the demand function. Your economics guru has estimated that $\theta^* \approx 1$ and $S'(\theta^*) \approx 4.5$. With this information, apply Theorem 6.1 to identify the values of a, σ^2 for the (approximate) limit Orstein Uhlenbeck process $U(t)$, and find T such that $e^{-aT} \approx 0.0001$.

(c) Show that $\epsilon \approx 0.0005$ yields a precision of 0.01 (half width of the approximate confidence interval after T/ϵ iterations, with confidence level $\alpha = 0.05$).

(d) In this part of the problem you will generate the random observations $\{\xi_n\}$ and run the stochastic approximation. Conditional on θ_n, let $\xi_n \sim$LN(m, v^2) have a lognormal distribution. First find the parameters for the m and v such that (6.11) holds, with $S(\theta) = \theta^s, s = 4.3$. Next, run the algorithm and discuss your results.

Given: $D(\theta) = \theta^{-y}$, $y > 0$

$S(\theta)$ is an incresing function of θ

(a) $E[\xi_n \mid \theta_n] = S(\theta_n)$ $\quad Var[\xi_n] = 1$

In Theorem 6.1 it states that all the conditions of Theorem 5.3 holds

$\{\xi_n\}$ is an observation of supply given theta. Transition propabilities are continuous and differentiable function of θ. Furthermore, the transition probabilities are positive due to expectation of random variable $\{\xi_n\}$ by given price is an increasing differential function. The process is ergodic MC so there exist unique stationary measure. Since a price is always a number so θ is tight.

$Y_n = D(\theta_n) - \xi_n$ has no bias term, the $Var = 1$, so it is bounded so Theorem 5.3 holds

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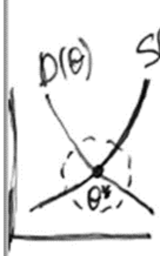
A6) Symmetrix Matrix σ = Martingale will be $\delta M_n^{\epsilon} = Y_n^{\epsilon} - g(\xi_{n-1}^{\epsilon}, \theta_n^{\epsilon})$. Since the price θ is one dimation (just a number) so σ is also one dimention and then it is symmetric. From here, there is a neighborhood of θ_n that converges to θ^* (or in our case $\theta^* = 1$). Neighborhood of θ_n is between production cost (it can't be a 0 by the definition) and it is increasing to make a profit ($\uparrow \infty$)



A7

$$g(\theta,\xi) = g(\theta^*,\xi) + \nabla_\theta g(\theta^*,\xi)(\theta-\theta^*) + p_1(\theta,\xi)$$

Error term should satisfy $E[p_1(\theta,\xi_n^\epsilon)] = O(\|\theta-\theta^*\|^2)$ with $n \to \infty$, $\epsilon \to 0$

Since $D(\theta)$ and $S(\theta)$ are monotonic functions that are decreasing and icreasing, so $G(\theta)$ should be differentiable over all θ that we have and tail of errors will go to 0. Therefor, the next power of derivatives should be decreasing (to go to zero).

A8 Since we have unbiased estimator, $g = E[\xi_n|\theta_n]$

For example let's use (b) values

$$A = \nabla G(\theta) = -(S'(\theta^*) - D'(\theta^*)) = -(4,5+5) = -9$$

we have negative eign value, so A is Hurwitz so A8 holds.

(b) Given: $\gamma = 5$, $\theta^* \approx 1$ and $S'(\theta^*) \approx 4.5$

Find: a, $\underline{\sigma^2}$, T such that $e^{-aT} \approx 0.0001$

Since we have a constant variace $Var[\xi_n] = 1$

then $Var = \sigma^2$ according to Example 6.1, page 153

and then $\sigma^2 = 1$

From page 153, proof of Theorem 6.1 and under stability condition $\underline{a<0}$

$$a = -G'(\theta^*) = -(S'(\theta^*) - D'(\theta^*)) =$$

$$= -(4.5 + 5) = -9.5.$$

$$\boxed{\begin{array}{l}\text{Use Power Rule} \\ D = \theta^{-5} \qquad D' = -\frac{5}{\theta^6} \qquad \text{or } D' = -5\end{array}}$$

$$e^{aT} = e^{-(-9.5)T} \approx 0.0001 \implies T = -0.9695$$

(c) Given : $\epsilon \approx 0.0005$ $p = 0.01$

$\frac{T}{\epsilon}$ with confidence level $a = 0.05$

According to example 6.1 on page 154, with given ϵ one can stop at N such that $e^{-N\epsilon} \leq p$ for a given prexision p_1. In our case we have to choose such N that would satisfy

$$e^{-N \cdot 0.0005} \leq 0.01 \Rightarrow -N\epsilon \leq \ln p \Rightarrow$$

$$\Rightarrow N \geq -\frac{\ln p}{\epsilon} \Rightarrow N \geqslant \frac{\ln 0.01}{0.0005}$$

or $N \geqslant 9210$

(d) Given $\xi_n \sim \ln(m, v^2)$ conditional on θ

$S(\theta) = \theta^s$ where $s = 4.3$

Will use $D(\theta) = \theta^{-5}$ from (a)

$$\sigma^2 = Var[\xi_n] = 1 \quad , \quad \mu = E[\xi_n | \theta_n] = S(\theta_n) = \theta^s$$

The mean of the log-normal distribution is

$$m = e^{\mu + \frac{\sigma^2}{2}} = exp(\theta^s + \tfrac{1}{2}\sigma^2) =$$

$$= exp(\theta^s + \tfrac{1}{2})$$

$$v^2 = exp(2(\sigma^2 + S(\theta))) - exp(\sigma^2 + 2S(\theta))$$

$$= exp(2(1 + \theta^2)) - exp(1 + 2\theta^s)$$

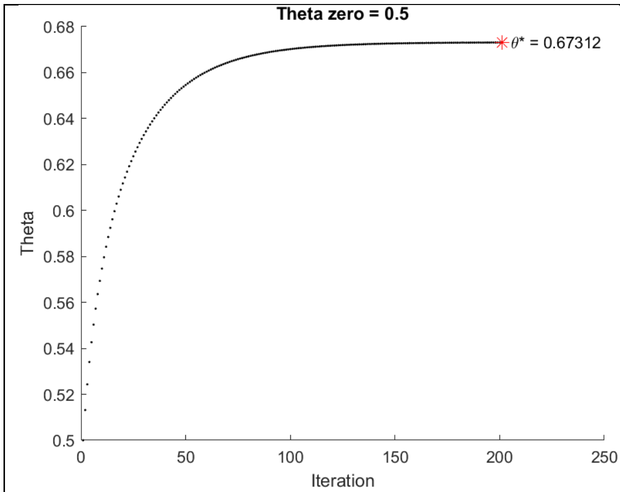
For $\theta^* = 1$

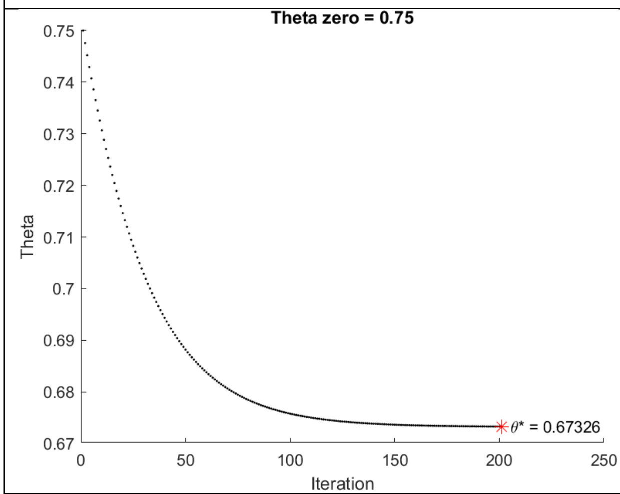
$$m = exp(1^{4.3} + \tfrac{1}{2}) = 4.48$$

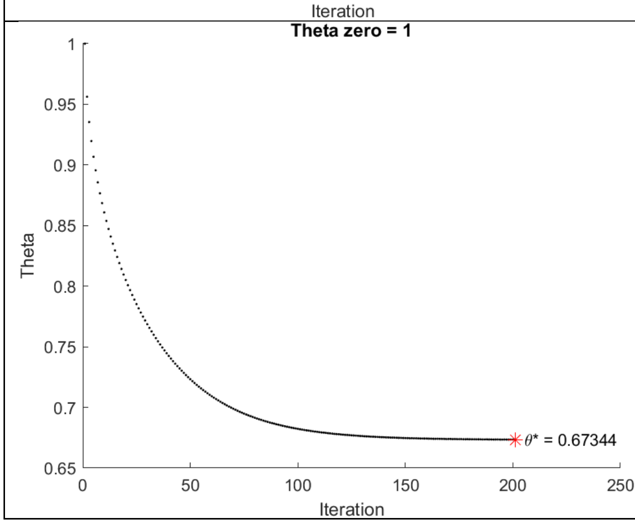
$$v^2 = exp(2(1 + 1^2)) + exp(1 + 2 \cdot 1^{4.3})$$

$$= 34.5$$

After we run algorithm we obtained that stable θ is 0.67, it shown on the graphs. Due to assumption that θ is close to the 1 we tried initial value for θ close to the one, i.e. 0.5, 0.75, 1.







On the graphs we can see that the algorithm converges really fast within less then 200 iterations,

EXERCISE 6.6. An investor wishes to divide her capital in two assets $\{S_i(t), i = 1, 2\}$, which she will access at maturity time T. Assume that her total capital is 1 (by changing monetary units if necessary). Although not known precisely, the corresponding means (μ_1, μ_2) and variances (σ_1^2 and σ_2^2) satisfy $\mu_1 > \mu_2$ and $\sigma_1^2 >> \sigma_2^2$. In order to balance her profit and risk, the investor wishes to find the proportion θ that solves the problem:

$$\max_{\theta} \quad \mathbb{E}[X(\theta)]$$
$$\text{s.t. } \mathbb{E}[X(\theta)^2] \leq B,$$

where $\mathbb{E}[S_1(T)^2] > B > \mathbb{E}[S_2(T)^2]$ and $X(\theta) = \theta S_1(T) + (1-\theta) S_2(T)$. Although the exact distributions of the two assets are not known, it is possible to use historical observations or simulations to produce consecutive samples $\xi_n = (\xi_{n,1}, \xi_{n,2}) \stackrel{\mathcal{L}}{=} (S_1(T), S_2(T))$.

(a) Argue that the optimal value θ^* of the above problem must satisfy $0 < \theta^* < 1$, and that the constraint will be active at the optimum.

(b) For a particular value $x = (x_1, x_2)$, let $\phi(x, \theta) = -\theta x_1 - (1-\theta)x_2$ so that $J(\theta) = \mathbb{E}[\phi(S_1(T), S_2(T); \theta)]$ is the function that we wish to minimise. Write the Langrangian of the problem and show that it is a convex NLP, so that the Arrow-Hurwicz algorithm for the deterministic problem converges to the optimal solution. Specify the vector field $G(\theta, \lambda)$ $(\lambda > 0)$ for the corresponding limit ODE.

(c) Consider the stochastic version of the Arrow-Hurwicz algorithm using one-sided finite differences: for each integer n let $\xi_n = (\xi_{n,1}, \xi_{n,2})$ and $\xi'_n = (\xi'_{n,1}, \xi'_{n,2})$ have the same distribution as $(S_1(T).S_2(T))$ and suppose that these samples are statistically independent, that is, $\xi_n \perp \xi'_n$ for all n. The algorithm is:

$$\begin{aligned}\theta_{n+1} &= \theta_n - \frac{\epsilon_n}{c_n}\Big(\phi(\xi'_n, \theta_n + c_n) - \phi(\xi_n, \theta_n) + \lambda_n\left(\phi^2(\xi'_n, \theta_n + c_n) - \phi^2(\xi_n, \theta_n)\right)\Big) \\ \lambda_{n+1} &= \big(\lambda_n + \epsilon_n(\phi^2(\xi_n, \theta_n) - B\big)_+ \qquad (6.12)\end{aligned}$$

with $\epsilon_n = \mathcal{O}(n^{-\gamma}), c_n = \mathcal{O}(n^{-c})$. Use Theorem 6.2 to establish that the fastest convergence is achieved at $c = 1/4$ and gives $\kappa = 1/2 < 1$.

(d) Let $\Delta_n \stackrel{\text{def}}{=} \xi_{n,1} - \xi_{n,2}$. Show that $J'(\theta) = -\mathbb{E}[\Delta_n]$ and that $g'(\theta) = 2\mathbb{E}[\theta\Delta_n^2 + \xi_{n,2}\Delta_n]$. Use Theorem 4.1 to show that the stochastic Arrow-Hurwitz algorithm

$$\begin{aligned}\theta_{n+1} &= \theta_n - \epsilon_n\left(-\Delta_n + 2\lambda_n(\theta_n\Delta_n^2 + \xi_{n,2}\Delta_n)\right) \\ \lambda_{n+1} &= \left(\lambda_n + \epsilon_n(\theta_n\Delta_n + \xi_{n,2})^2 - B\right)_+\end{aligned}$$

converges to the solution of the optimization problem θ^*. What do you need to assume on the step size sequence $\{\epsilon_n\}$? Use Theorem 6.2 to find the convergence rate κ assuming that $\epsilon = n^{-\gamma}$. Specify the values of β and δ. Next, find the asymptotic efficiency and the corresponding rate, using Theorem 6.4.

(e) Assume now that the same observation is used at iteration n in order to calculate $\phi(\xi_n, \theta_n \pm c_n)$. [Notice that in this model the noise is exogenous (the value of the two assets is independent of the proportion that the investor decides to allocate), so that it is straightforward to implement common random numbers (CRN) by using the same random variable $\xi_n = \xi'_n$ a.s.]

$$\begin{aligned}\theta_{n+1} &= \theta_n - \frac{\epsilon_n}{2c_n}\Big(\phi(\xi_n, \theta_n + c_n) - \phi(\xi_n, \theta_n - c_n) + \lambda_n(\phi^2(\xi_n, \theta_n + c_n) - \phi^2(\xi_n, \theta_n - c_n))\Big) \\ \lambda_{n+1} &= \big(\lambda_n + \epsilon_n(\phi^2(\xi_n, \theta_n) - B\big)_+ \qquad (6.13)\end{aligned}$$

Show that this implementation can achieve $\kappa = 1$ and find the value of c that achieves this maximal rate.[*Hint:* Use Taylor series for $\phi(\xi, \theta)$ for a fixed value of ξ to express the vector field in terms of the stochastic derivative $\phi'(\xi, \theta)$, and establish that the variance term is $V_n = \mathcal{O}(1)$.]

a. The full capital is equal to one. So then the optimal proportion (θ) of the customer capital should be between 0 and 1.

b.

(b) $J(\theta) = \mathbb{E}[\phi\, S_1(T), S_2(T); \theta] =$

$= \mathbb{E}[-\theta S_1(T) - (1-\theta) S_2(T)] =$

$= -\mathbb{E}[\theta S_1(T)] - \mathbb{E}[(1-\theta) S_2(T)] =$

$= \cancel{\mathbb{E}} -\theta\, \mathbb{E}[S_1(T)] - (1-\theta)\mathbb{E}[S_2(T)] = \theta\mu_1 - (1-\theta)\mu_2$

let's call $\mathbb{E}(S_1(T)) = \mu_1$

$\mathbb{E}[S_2(T)] = \mu_2$

Condition:

$g(\theta) = \mathbb{E}[X(\theta)^2] - B \le 0$

~~Var(X(θ))~~ $X(\theta)^2 = \theta^2 S_1^2(T) + 2\theta(1-\theta) S_1(T) S_2(T) +$

$+ (1-\theta)^2 S_2^2(T)$

$\mathbb{E}(X(\theta)^2) = \theta^2 \overbrace{\mathbb{E}[S_1^2(T)]}^{> B} + 2\theta(1-\theta)\, \mathbb{E}(S_1(T) S_2(T)) +$

$+ (1-\theta)^2 \underbrace{\mathbb{E}[S_2^2(T)]}_{< B}$

$\mathrm{Var}(X(\theta)) = \cancel{\theta^2}\, \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 =$

$\mathcal{L} = -\theta\, \mathbb{E}(S_1) - (1-\theta)\, \mathbb{E}(S_2) + \lambda(\theta^2 \mathbb{E}(S_1^2) +$

$+ 2\theta(1-\theta)\, \mathbb{E}(S_1 \cdot S_2) + (1-\theta)^2 \mathbb{E}[S_2^2] - B)$

$\frac{\partial \mathcal{L}}{\partial \theta} = -\mathbb{E}(S_1) - (-1)\mathbb{E}(S_2) + \lambda(2\theta\, \mathbb{E}(S_1^2) +$

$+ 2(1-\theta)\mathbb{E}(S_1 \cdot S_2) + 2\theta(-1)\mathbb{E}(S_1 \cdot S_2) +$

$+ 2(1-\theta)(-1)\mathbb{E}(S_2^2) - B) =$

$= -\mathbb{E}(S_1) + \mathbb{E}(S_2) + \lambda(2\theta\, \mathbb{E}(S_1^2) - 2(1-\theta)\mathbb{E}(S_2^2) +$

$+ (1 - 4\theta)\mathbb{E}(S_1 \cdot S_2) - B)$

$\frac{\partial^2 \mathcal{L}}{\partial \theta^2} = \lambda[2\underbrace{\mathbb{E}(S_1^2)}_{> B} + 2\underbrace{\mathbb{E}(S_2^2)}_{< B} - 4\, \mathbb{E}(S_1 S_2) - B] =$

$= \lambda 2\, [\underbrace{(\mathbb{E}(S_1^2) - B)}_{> 0} + \underbrace{(\mathbb{E}(S_2^2) - B)}_{< 0} + B -$

$- 2\mathbb{E}(S_1 \cdot S_2)]$

if $\mathbb{E}(S_1) > \mathbb{E}(S_2)$

then $\mathbb{E}(S_1 \cdot S_2) < \mathbb{E}(X)\mathbb{E}(Y)$

all expression

$< \mathbb{E}(S_1^2) + \mathbb{E}(S_2^2) - 2\mathbb{E}(S_1)\mathbb{E}(S_2) - B \Rightarrow$

$\mathbb{E}(S_1^2) + \mathbb{E}(S_2^2) - 2\mathbb{E}(S_1 S_2) - B <$

$\mathbb{E}(S_1^2) > B$

$B > \mathbb{E}(S_2^2)$

$2\,\mathbb{E}(S_1)\,\mathbb{E}(S_2) = 2\mu_1\mu_2 > B$

Thus $\frac{\partial^2 L}{\partial \theta^2} < 0$ $\Rightarrow$ NLP $\to$ convex

$g''(\theta) < 0$

So Arrow-Hurwicz algorith for deterministic problem

Thus $\nabla_\theta L = \nabla_\theta J(\theta) + \lambda \nabla_\theta g(\theta) \Rightarrow$

$\Rightarrow$ ~~G(θ)~~ ~~∇_θ~~

~~G(θ) = ∇θ L~~

$$G(\theta) = \begin{pmatrix} -(\nabla L(\theta,\lambda)) \\ g(\theta) \end{pmatrix} =$$

$$= \begin{pmatrix} -\mu_1 + \mu_2 + \lambda\{2\theta\sigma_1^2 - 2(1-\theta)\sigma_2^2 + (1+4\theta)\mathbb{E}(S_1S_2)\} \\ \theta^2\mathbb{E}(S_1^2) + 2(1-\theta)\theta\,\mathbb{E}(S_1S_2) + (1-\theta)^2\mathbb{E}[S_2^2] - B \end{pmatrix}$$