CS0001

Discrete Structures 1

Module 6: Relations

Technology Driven by Innovation

OBJECTIVES:

By the end of this module, you will be able to:

- **Define** a binary relation as a subset of a Cartesian product.
- **Identify** the Domain and Range of a given relation.
- **Represent** a relation using Set-Roster, Set-Builder, Arrow Diagrams, Directed Graphs, Zero-One Matrices
- **Perform** operations on relations
- **Classify** relations by their properties

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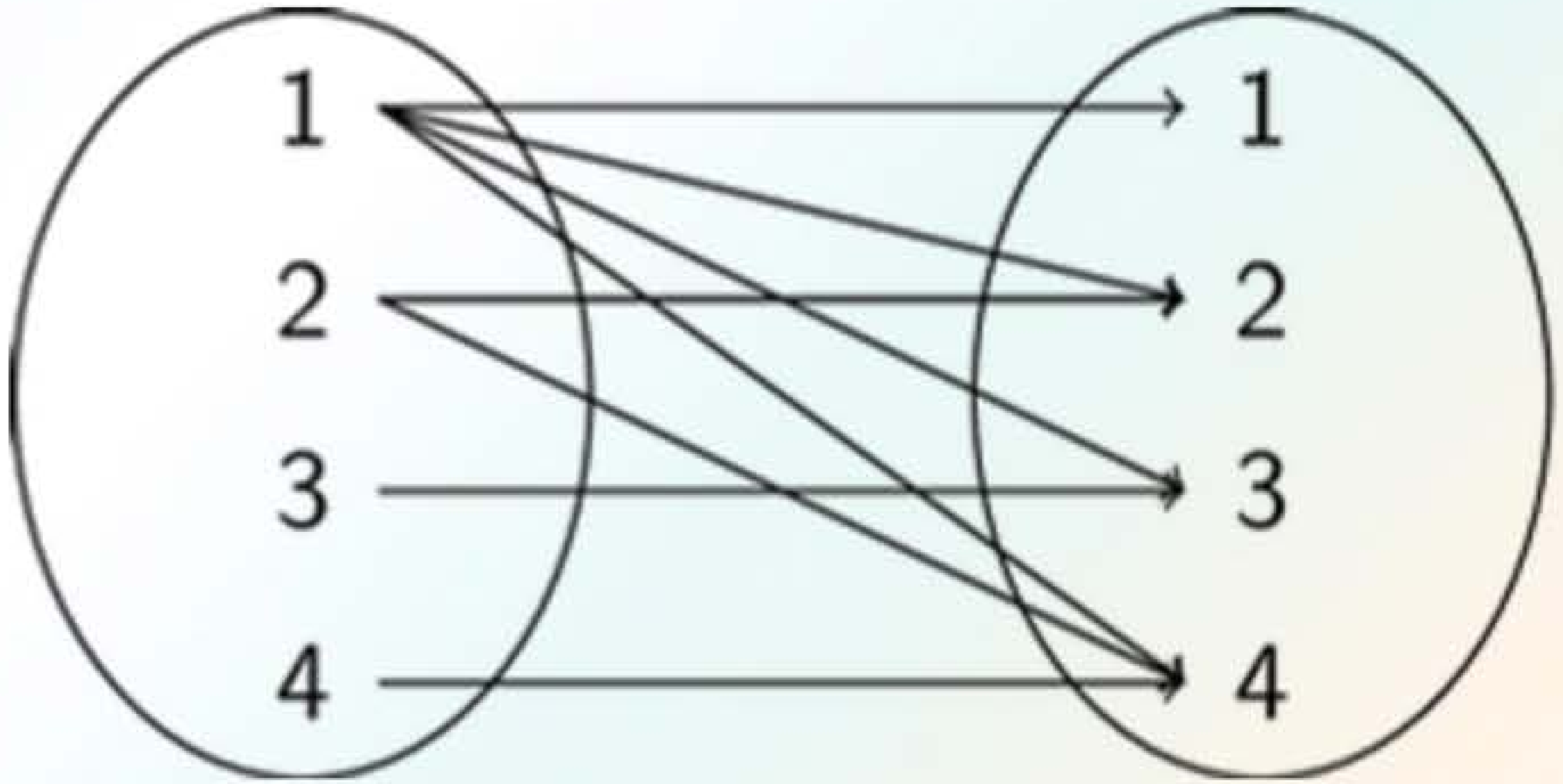
Discrete Structures 1

Subtopic 1: Introduction to Relations

RELATIONS

A Binary Relation 'R' from a **set A** to a **set B** is a set of **ordered pairs**, where the first element is from set A and the second element is from set B.



ORDERED PAIRS

- An ordered pair, written as (a, b), is a pair of elements where the **order is crucial**.
- The first element is a, and the second element is b.
- This means the pair (1, a) is **NOT** the same as the pair (a, 1).

Ordered pairs can be collected into a set. While order is crucial inside the pair (e.g., (1,a) is not (a,1)), the **order of the pairs within the set does not matter**. A set is always an unordered collection.

Example:
A = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}
B = {(2,a), (2,b), (2,c), (1,a), (1,b), (1,c)}
C = {(a,1), (b,1), (c,1), (a,2), (b,2), (c,2)}

Set A = Set B
Set A ≠ Set C

CARTESIAN PRODUCT

The Cartesian Product (**A X B**) is the **set of all possible ordered pairs** where the first element comes from **set A** and the second element comes from **set B**.

Given your two sets:

- A = {1, 2}
- B = {a, b, c}

To find the product, you systematically pair each element from **A** with every element from **B**.

Start with '1' from set A. Pair it with everything in B: (1, a), (1, b), (1, c)
Next, take '2' from set A. Pair it with everything in B: (2, a), (2, b), (2, c)

The final result is the set:
A X B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}

THINK ABOUT THIS

The Sets:

- Set A = {x, y}
- Set B = {1, 2, 3}
- Set C = {a}
- Set D = ∅ (the empty set)

Find the following Cartesian Products:

1. Find A x B
2. Find B x A
3. Find C x C
4. Find A x C
5. Find B x D

ANSWERS

Find the following Cartesian Products:

1. Find A x B

{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)}

2. Find B x A

{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)}

Note: *A x B is not equal to B x A*

3. Find C x C

{(a, a)}

Note*: The product of a set with itself is often called the "Cartesian square." Set C has one element, so the result has 1 x 1 = 1 ordered pair. The only possible pairing is the element 'a' with itself.*

ANSWERS

Find the following Cartesian Products:

4. Find A x C

{(x, a), (y, a)}

5. Find B x D

∅ (the empty set)

Note: *To create an ordered pair, we need one element from B and one from D. Set D has no elements to contribute, so it is impossible to form any pairs. The resulting set of pairs is, therefore, empty.*

DOMAIN AND RANGE

Domain: is the set of all the **first elements** in the ordered pair.
Range: is the set of all the **second elements** in the ordered pair.

Example:
R = {(1,4), (1,6), (2,9), (3,4), (3,6), (5,4), (5,6)}

To find the Domain: Just collect all the first elements from the pairs.
Domain = {1, 2, 3, 5}
(Note: Even though 1, 3, and 5 appear multiple times, we only list them once in the set.)

To find the Range: Just collect all the second elements from the pairs.
Range = {4, 6, 9}

REPRESENTING A RELATION

Method 1: Set-Builder or Set-Roster Notation

Set-Builder Notation
Defines a relation using a rule or property that the ordered pairs must satisfy. It describes 'how' to build the relation.

Example:
A = {1, 2, 3, 5}
B = {4, 6, 9}

R = {(x, y) | x ∈ A, y ∈ B, and the difference of x and y is odd}

Set-Roster Notation
Defines a relation by listing all of its ordered pairs. This shows the 'what' – the exact elements of the relation.

Example:
R = {(1,4), (1,6), (2,9), (3,4), (3,6), (5,4), (5,6)}

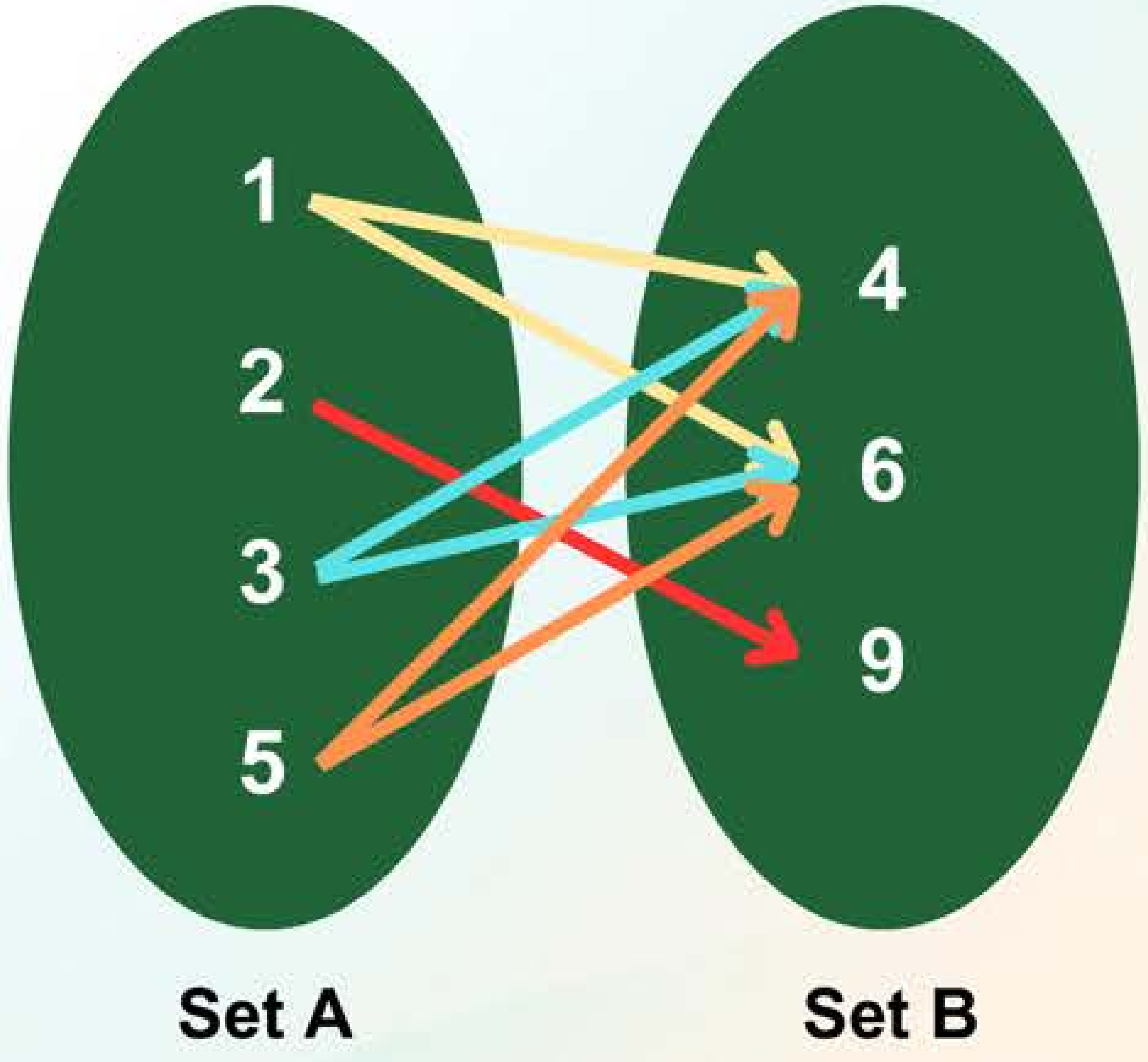
REPRESENTING A RELATION

Method 2: Arrow Diagram

An **arrow diagram** visually maps elements from the Domain to the Codomain.

It's excellent for showing relations between two sets.

- Draw two ovals for the Domain (Set A) and Codomain (Set B).
- List the elements in each set.
- Draw an arrow from **a** to **b** for every pair **(a, b)** in the relation.



R = {(1,4), (1,6), (2,9), (3,4), (3,6), (5,4), (5,6)}

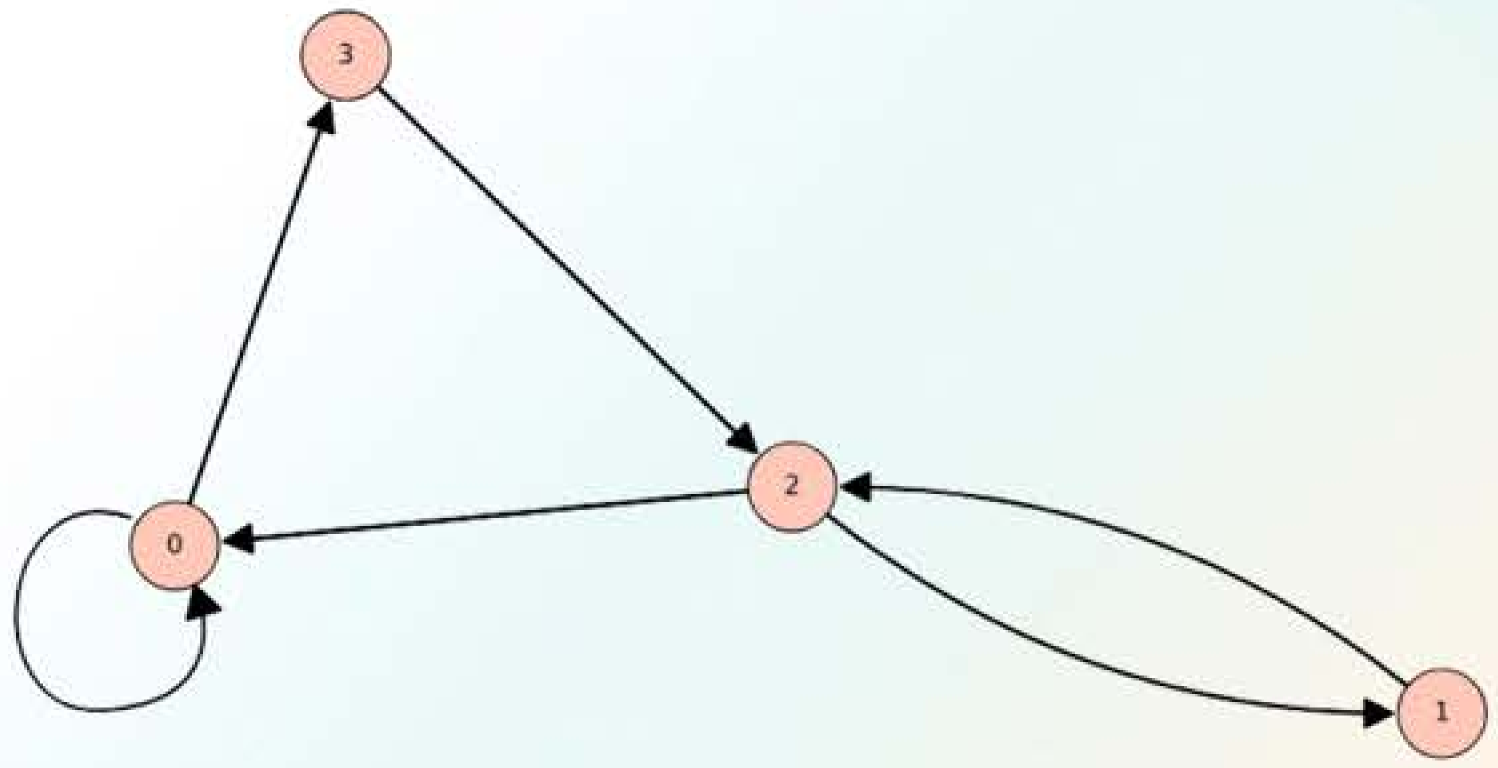
REPRESENTING A RELATION

Method 3: Directed Graph (Digraph)

A **Directed Graph** (or **Digraph**) is used to represent a relation on a single set (from Set A to itself).

It's the best way to visualize properties like loops and cycles.

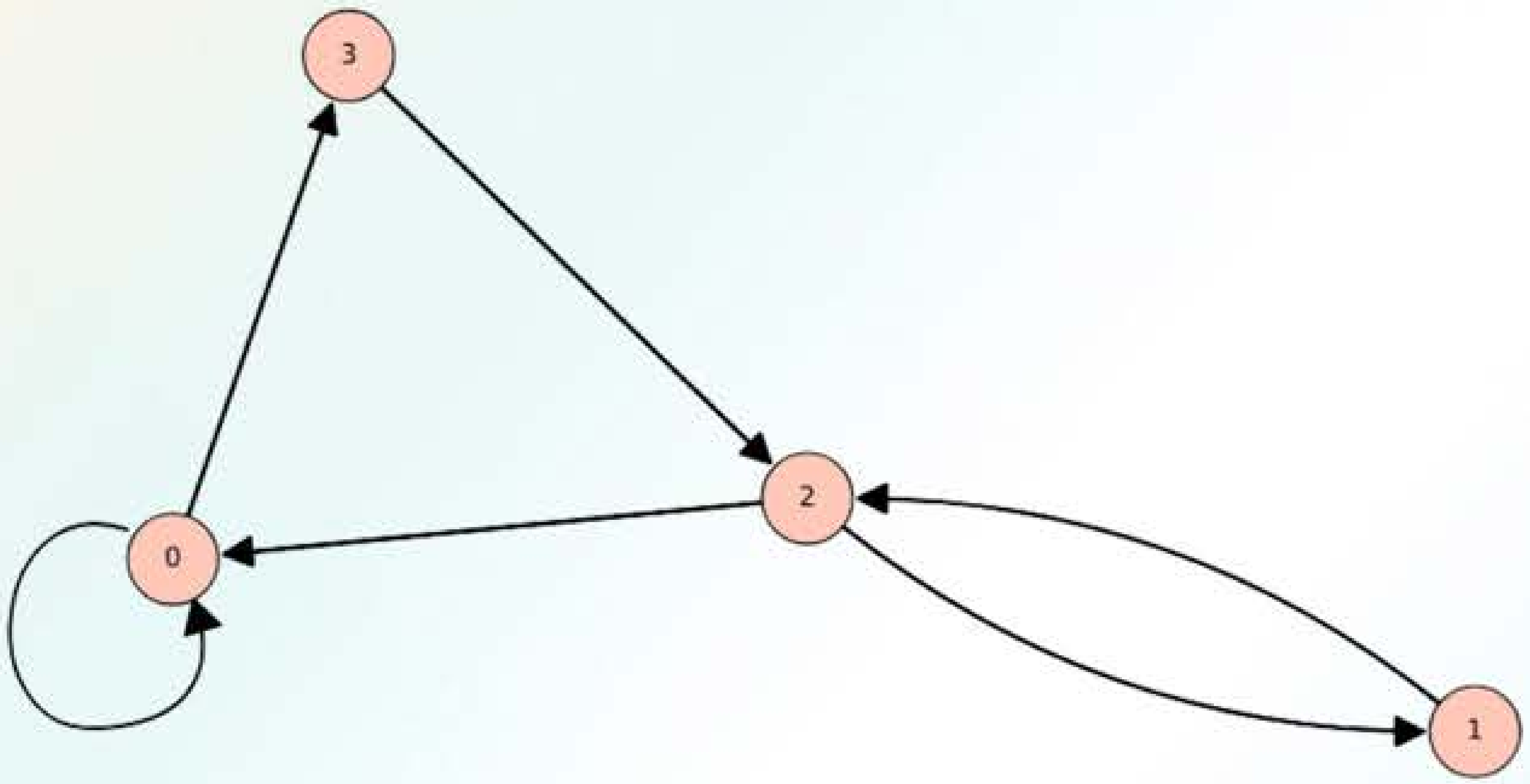
- Draw a vertex (dot) for each element in the set.
- Draw a directed edge (arrow) from **vertex a** to **b** for every pair **(a, b)** in the relation.
- A loop is an edge from a vertex to itself (e.g., (a, a)).



Example:
Set A (Vertex): {0,1,2,3}
R = {(0,0), (0,3), (1,2), (2,0), (2,1), (3,2)}

REPRESENTING A RELATION

In this example:
Set A (Vertex): {0,1,2,3}
R = {(0,0), (0,3), (1,2), (2,0), (2,1), (3,2)}



Each ordered pair (a, b) in your relation set tells you to draw one arrow.

- The first element, a, is the starting vertex (where the arrow begins).
- The second element, b, is the ending vertex (where the arrowhead points).

So,

- **(0, 0)**: An arrow starts at vertex 0 and points to vertex 0. This forms the **loop**.
- **(0, 3)**: An arrow starts at vertex 0 and points to vertex 3.
- **(1, 2)**: An arrow starts at vertex 1 and points to vertex 2.
- **(2, 0)**: An arrow starts at vertex 2 and points to vertex 0.
- **(2, 1)**: An arrow starts at vertex 2 and points to vertex 1.
- **(3, 2)**: An arrow starts at vertex 3 and points to vertex 2.

Note: More terminologies on Graph Theory will be discussed in Module 8

REPRESENTING A RELATION

Method 4: Zero-One Matrix

A matrix can represent a relation R from A to B. We create a matrix M where **rows are elements of A** and **columns are elements of B**.

M(i, j) = **1** if the **pair is in the relation**.
M(i, j) = **0** if the **pair is not in the relation**.

Example:
R = {(1,a), (1,c), (2,b), (3,a), (3,c)}
from A = {1, 2, 3} to B = {a, b, c}

	a	b	c
1	1	0	1
2	0	1	0
3	1	0	1

THINK ABOUT THIS

Work as a group and represent the sets in different methods.

- Set P = {-1, 0, 1}
- Set S = {1, 2, 3, 4, 5}
- Set K = {1, 2, 3, 4}

Task #1: A relation R1 from Set P to Set K.

- Represent R1 using an Arrow Diagram.
- Represent R1 using a Zero-One Matrix.

Task #2: The following Directed Graph represents a relation R2 on Set S.

- Represent R2 using Set-Roster Notation. (Be careful not to miss any arrows).
- Represent R2 using a Zero-One Matrix.

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Subtopic 2: Operations on Relations

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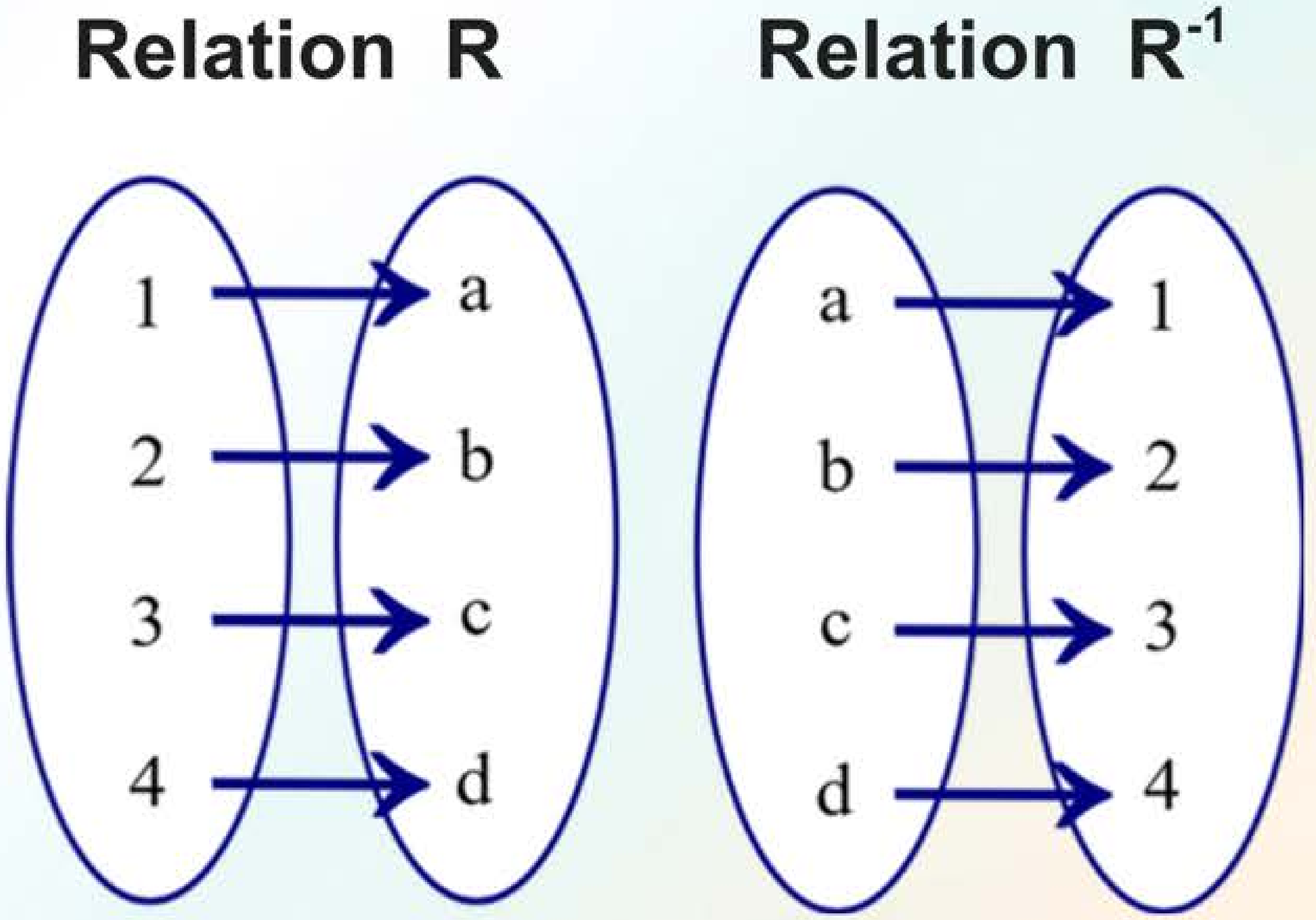
INVERSE RELATIONS

The inverse of a relation is a new relation created by **swapping the elements in each ordered pair** of the original relation.

If a relation R consists of pairs (x,y), its inverse, denoted R^{-1} consists of pairs (y,x).



EXAMPLE OF INVERSE RELATIONS

Given:
A = {1, 2, 3, 4}
B = {a, b, c}

Relation R (from A to B):
R = {(1, a), (2, c), (3, a), (4, b)}
Domain: {1, 2, 3, 4}
Range: {a, b, c}

Inverse Relation R^{-1} (from B to A)
To find the inverse, you simply reverse every ordered pair.

R^{-1} = {(a, 1), (c, 2), (a, 3), (b, 4)}
Domain of R^{-1}: {a, b, c}
Range of R^{-1}: {1, 2, 3, 4}

COMBINING RELATIONS

Given Relations on A = {1, 2, 3, 4}:
R1 = {(1,1), (2,2), (3,3)}
R2 = {(1,1), (1,2), (1,3), (1,4)}

Union R1 ∪ R2
{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)}

Intersection R1 ∩ R2
{(1,1)}

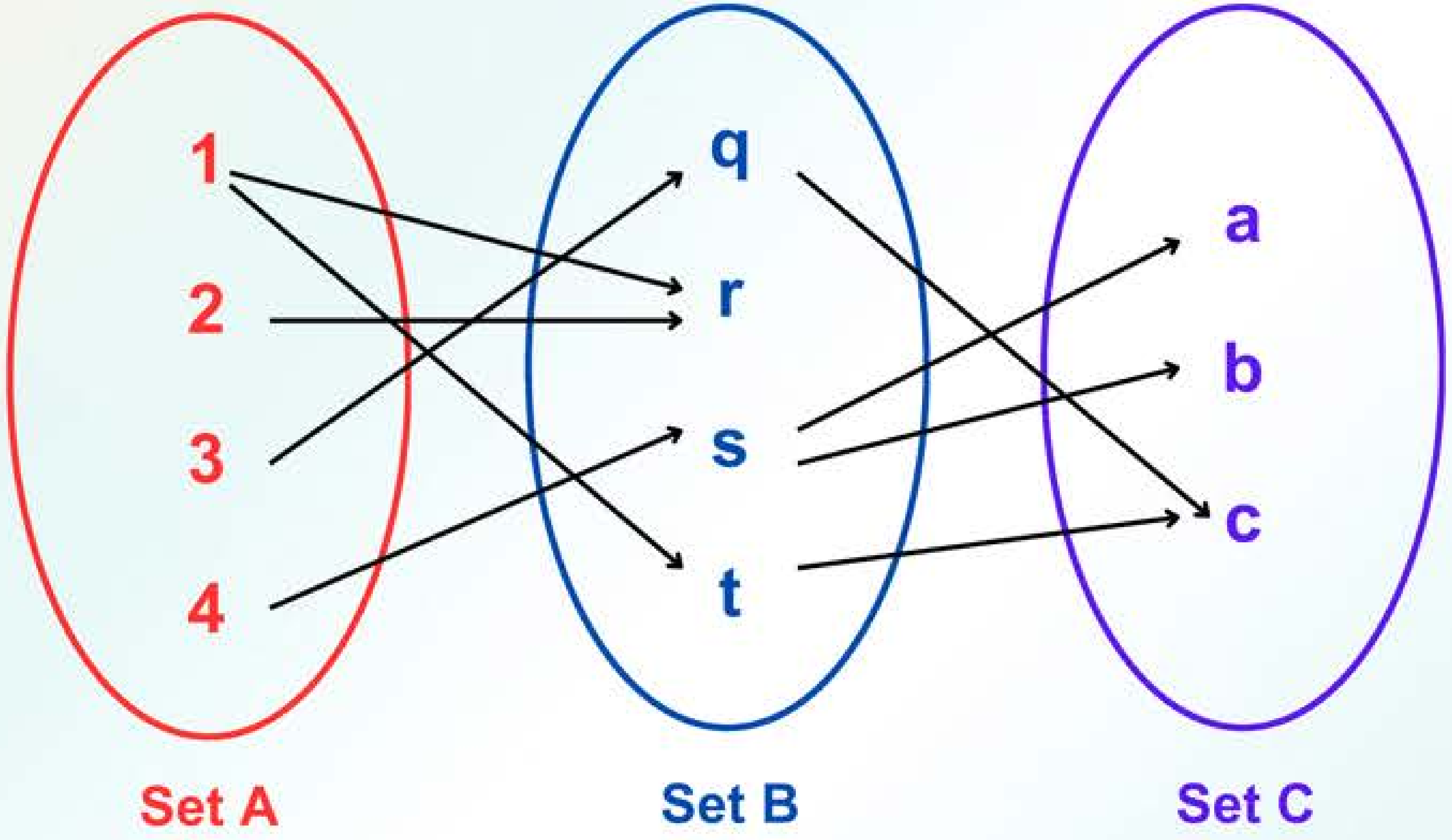
Difference R1-R2
{(2,2), (3,3)}

Relations can also be combined using simple **set operations**.

Just like sets, relations can undergo **union**, **intersection**, and **difference**, since each relation is essentially a set of ordered pairs.

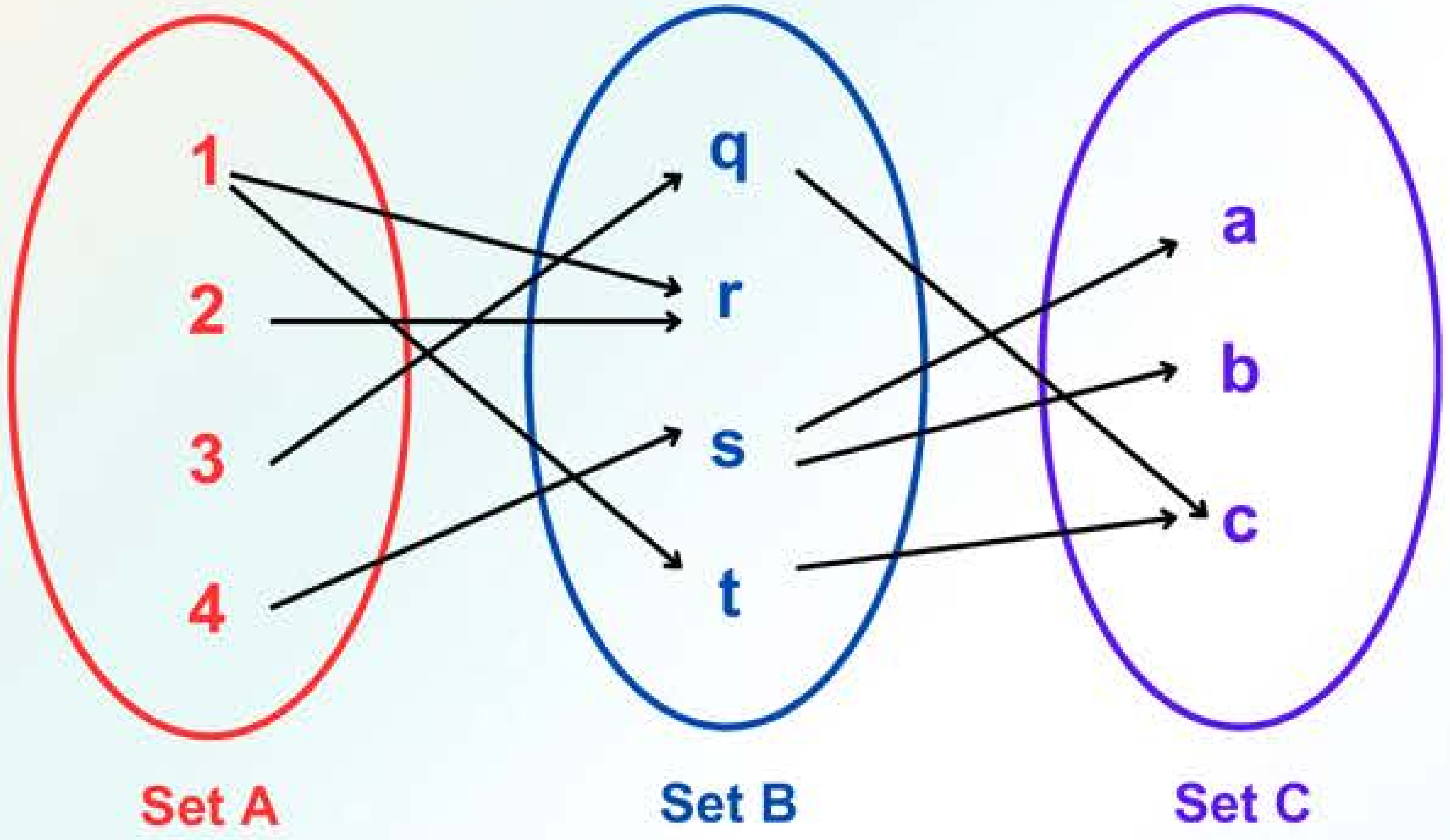
COMPOSITION OF RELATIONS

Using the arrow diagram, identify which elements from Set A are related to Set B (for relation R), and which elements from Set B are related to Set C (for relation S). Write their corresponding ordered pairs.



COMPOSITION OF RELATIONS

Using the arrow diagram, identify which elements from Set A are related to Set B (for relation R), and which elements from Set B are related to Set C (for relation S). Write their corresponding ordered pairs.



R = {(1, r), (1, t), (2, r), (3, q), (4,s)}

S = {(q, c), (s, a), (s, b), (t, c)}

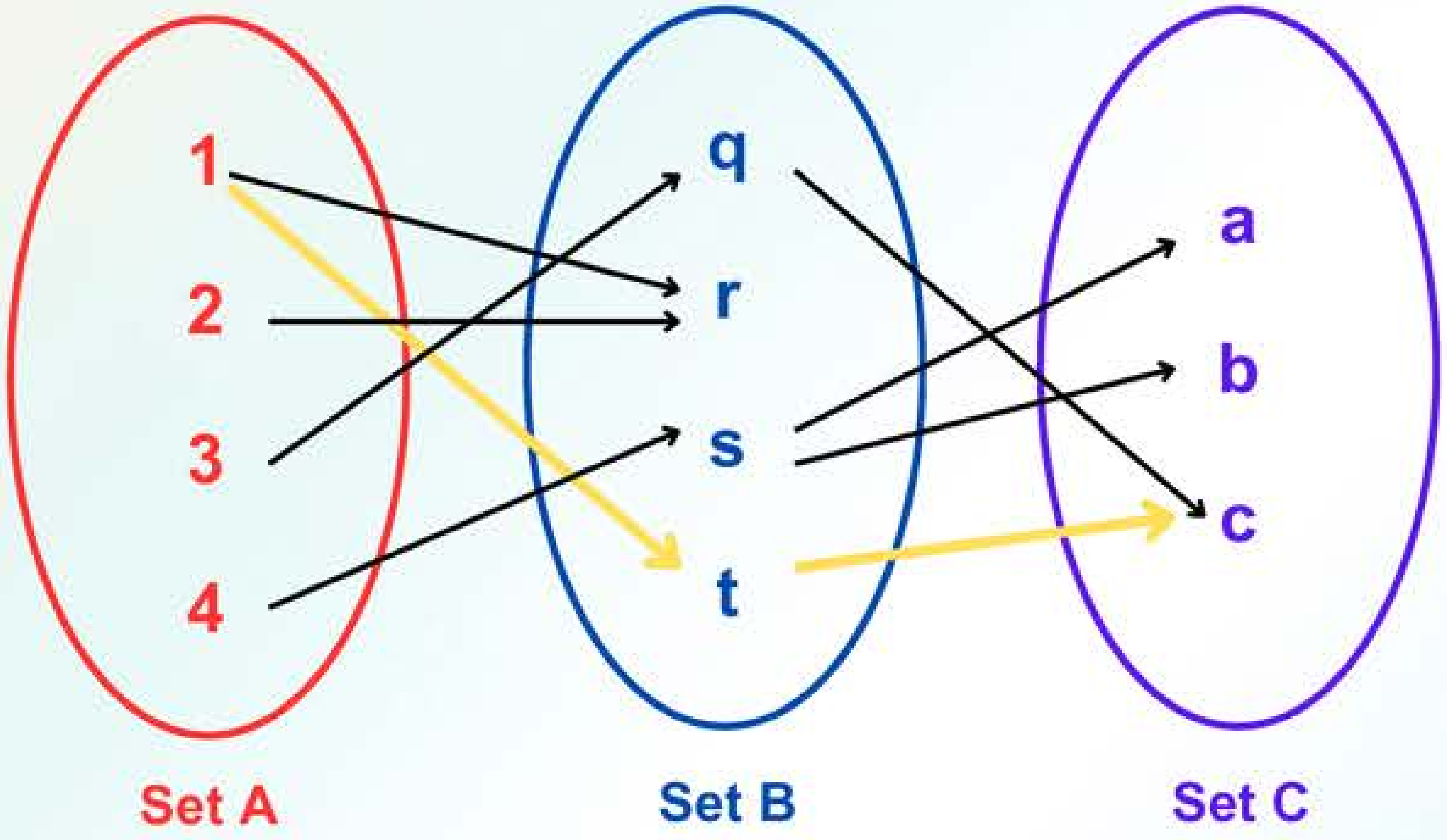
COMPOSITION OF RELATIONS

Let **R**: A → B and **S**: B → C be relations.
Composition (S ∘ R) is defined by
$\{(x, z) \mid \exists\, y \in B \text{ where } (x, y) \in R \text{ and } (y, z) \in S\}$

In simple terms, you can get from x to z if you can first get from x to some intermediate y (using relation R) and then get from that same y to z (using relation S).

COMPOSITION OF RELATIONS

To identify the **Composition (S ∘ R)** in this example, you would see that **1 relates to t**, and **t relates to c**.

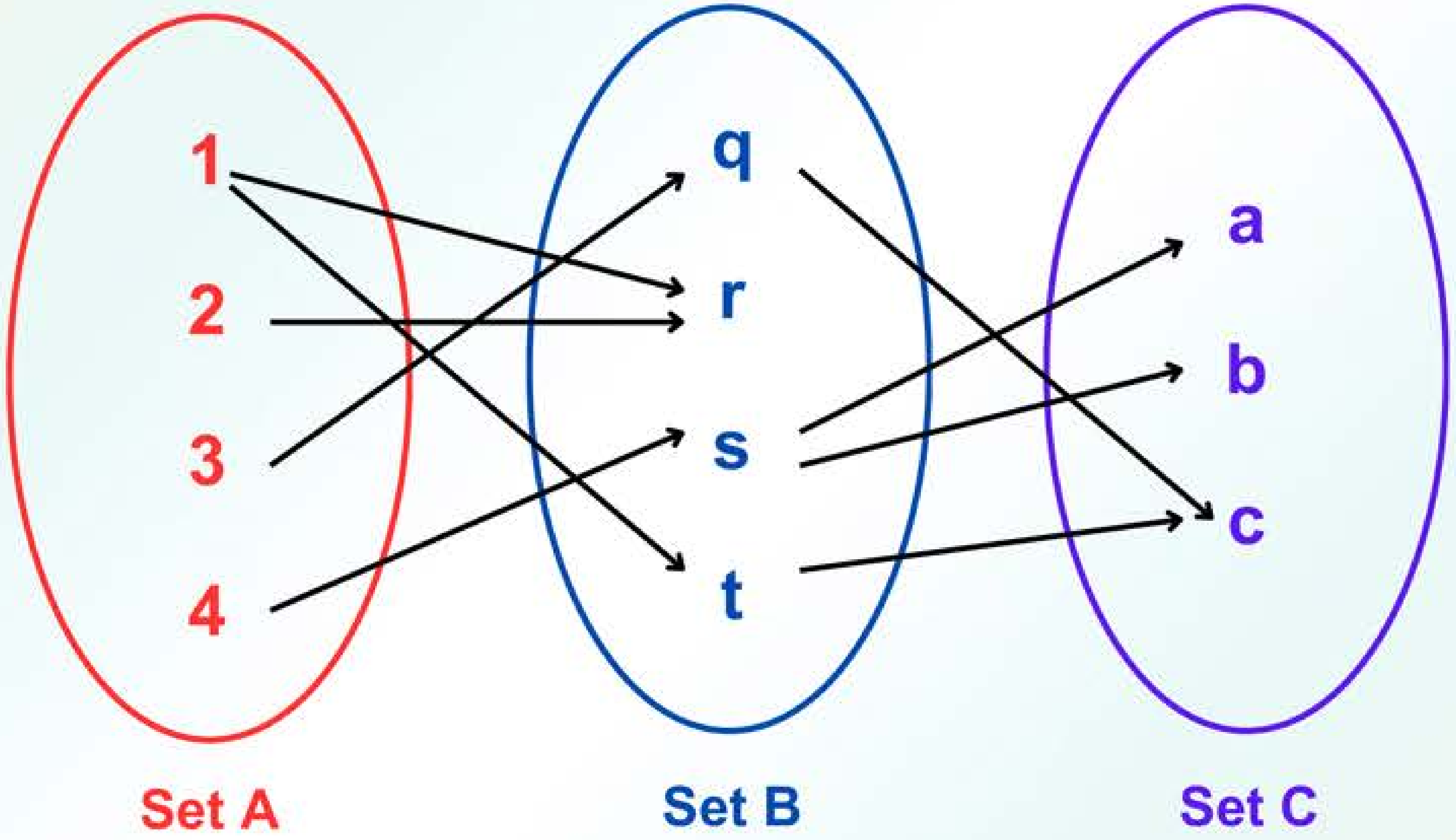


With that, we can say that **1 relates to c**. We will add them in our set of composition.

(S ∘ R) = {(1,c), ...}

Can you identify the other compositions?

COMPOSITION OF RELATIONS



$$(S \circ R) = \{(1,c), (3, c), (4, a), (4, b)\}$$

COMPOSITION OF RELATIONS

What is the composite of the relations **R and S ($S \circ R$)**, where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with

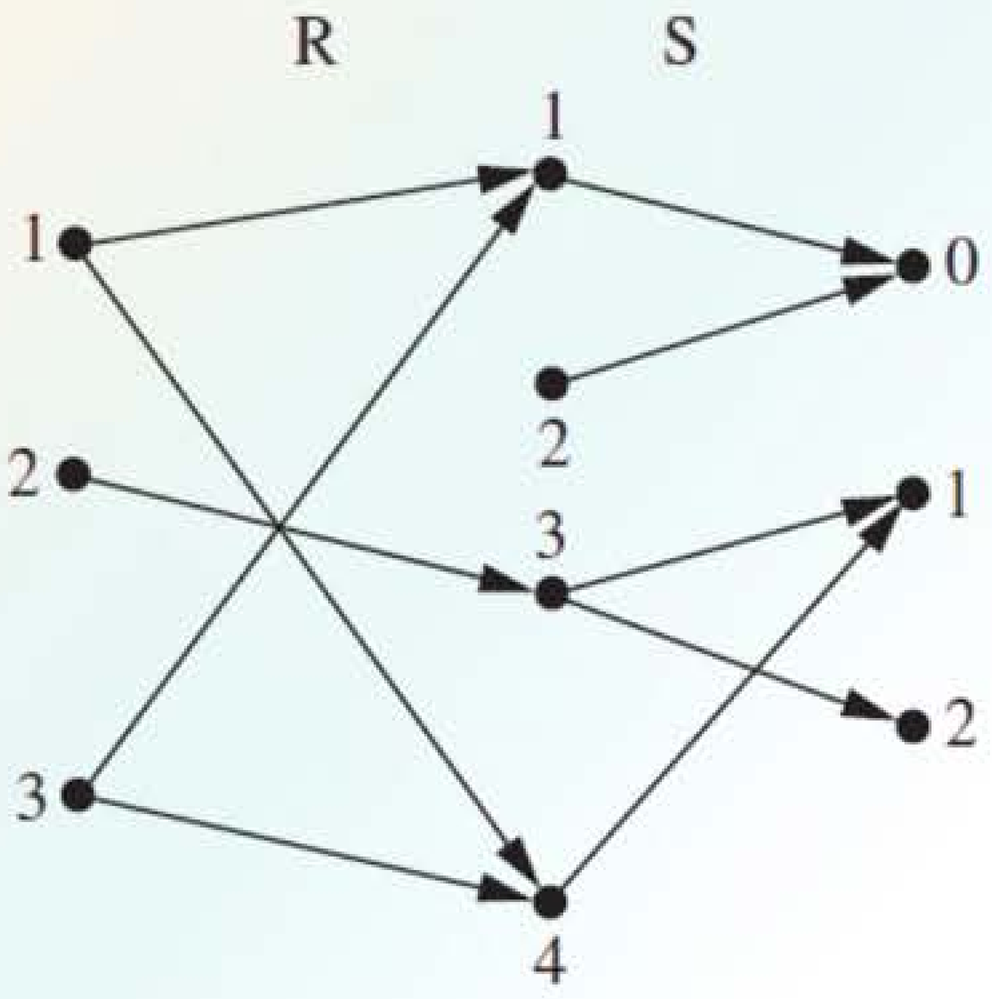
R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)} and

S is the relation from {1, 2, 3, 4} to {0, 1, 2} with

S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?

COMPOSITION OF RELATIONS

R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)}
S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}



$1 \to 1 \to 0$	$(1, 0)$
$1 \to 4 \to 1$	$(1, 1)$
$2 \to 3 \to 1$	$(2, 1)$
$2 \to 3 \to 2$	$(2, 2)$
$3 \to 1 \to 0$	$(3, 0)$
$3 \to 4 \to 1$	$(3, 1)$

S ∘ R = {(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)}

TEST YOUR KNOWLEDGE

Given:

Set A = {1, 2, 3}

Relation R = {(1, 1), (1, 2), (2, 3), (3, 1)}

Relation S = {(1, 2), (2, 1), (3, 3)}

Find the following:

1. R^{-1}
2. $R \cup S$
3. $R \cap S$
4. R - S
5. S - R
6. $S \circ R$

ANSWERS

Given:

Set A = {1, 2, 3}

Relation R = {(1, 1), (1, 2), (2, 3), (3, 1)}

Relation S = {(1, 2), (2, 1), (3, 3)}

Find the following:

1. R^{-1} = {(1, 1), (2, 1), (3, 2), (1, 3)}
2. R∪S = {(1, 1), (1, 2), (2, 1), (2, 3), (3, 1), (3, 3)}
3. R∩S = {(1, 2)}
4. R - S = {(1, 1), (2, 3), (3, 1)}
5. S - R = {(2, 1), (3, 3)}
6. S ∘ R = {(1, 1), (1, 2), (2, 3), (3, 2)}

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Subtopic 3: Properties of Relations

OUR "TEST CASE" RELATION

To understand the properties, we must test a **relation R** on a **single set A**. We will use one example relation to test all four properties.

Set A = {1, 2, 3, 4}
Relation R = {(1,1), (1,2), (2,1), (2,2), (3,3), (3,4)}

We will ask four key questions about this relation:

1. Is it Reflexive?
2. Is it Symmetric?
3. Is it Anti-symmetric?
4. Is it Transitive?

REFLEXTIVE PROPERTY

A relation R on a set A is called reflexive if **(a, a) ∈ R** for **every** element $a \in A$.

Consider the following relations on {1, 2, 3, 4}:
$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
$R_6 = \{(3, 4)\}$

Which of these relations are **reflexive**?

REFLEXTIVE PROPERTY

A relation R on a set A is called reflexive if **(a, a) ∈ R** for **every** element a ∈ A.

Consider the following relations on {1, 2, 3, 4}:
R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}
R_2 = {(1, 1), (1, 2), (2, 1)}
R_3 = {**(1, 1)**, (1, 2), (1, 4), (2, 1), **(2, 2)**, **(3, 3)**, (4, 1), **(4, 4)**}
R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}
R_5 = {**(1, 1)**, (1, 2), (1, 3), (1, 4), **(2, 2)**, (2, 3), (2, 4), **(3, 3)**, (3, 4), **(4, 4)**}
R_6 = {(3, 4)}

Which of these relations are **reflexive**?

The relations **R3** and **R5** are reflexive **because they both contain all pairs** of the form **(a, a)**, namely, (1, 1), (2, 2), (3, 3), and (4, 4). The other relations are not reflexive because they do not contain all of these ordered pairs.

SYMMETRIC PROPERTY

A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

How to test: Look at every non-loop pair. If you find even one "one-way street," the relation is not symmetric.

In digraph Test: All edges between different vertices must be "two-way arrows".

Example:

R = {(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3)}

We check every pair (and its reverse):

Checklist:

- Loops (1,1), (2,2), (3,3) are fine.
- Found (1,2). Is (2,1) in R? Yes.
- Found (2,1). Is (1,2) in R? Yes.
- Found (3,4). Is (4,3) in R? Yes.

Result: SYMMETRIC

SYMMETRIC PROPERTY

Example:

Let's define a relation R on the set A = {1, 2, 3}:
R = {(1, 1), (1, 2), (2, 1), (2, 3)}

We check every pair (and its reverse):

Checklist:

- Loops: (1, 1) is fine.
- Found (1, 2). Is (2, 1) in R? Yes.
- Found (2, 1). Is (1, 2) in R? Yes.
- Found (2, 3). Is (3, 2) in R? **No.**

Result: NOT SYMMETRIC

Reason: The **relation fails** the test because we found a **"one-way street."** The pair (2, 3) exists in the relation, **but its reverse, (3, 2), does not**.

Even though the pair (1, 2) / (2, 1) is symmetric, the entire relation is considered not symmetric **because of the single failure with (2, 3)**.

ANTISYMMETRIC PROPERTY

A relation R on a set A such that for all a, b $\in$ A, if (a, b) $\in$ R and (b, a) $\in$ R, then a = b is called **antisymmetric**.

In digraph Test: There can be no two-way arrows between **different** vertices.

Example:

A = {1, 2, 3, 4}

R = {(1,1), (1,2), (2,2), (3,3), (3,4)}

Checklist:

- Found (3,4). Its reverse (4,3) is not there. This is fine.
- Found (1,2). Its reverse (2,1) is not there. This is fine.

Result: ANTISYMMETRIC

ANTISYMMETRIC PROPERTY

A relation R on a set A such that for all a, b ∈ A, if (a, b) ∈ R and (b, a) ∈ R, then a = b is called **antisymmetric**.

In digraph Test: There can be no two-way arrows between **different** vertices.

Example:

A = {1, 2, 3}

R = {(1, 1), (1, 2), (2, 1), (3, 2)}

Checklist:

- Loop: (1, 1) is fine. (This is a "two-way arrow" but a = b, so it's allowed).
- Found (3, 2). Its reverse (2, 3) is not there. This pair is fine.
- Found (1, 2). Its reverse (2, 1) is in R? **Yes**.

Result: NOT ANTISYMMETRIC

NOT SYMMETRIC VS. ANTISYMMETRIC

Not Symmetric is a failure.
The symmetric property is broken.

Antisymmetric is a rule. It's a specific property a relation can have, just like symmetry.

Antisymmetric is NOT the same as "not symmetric".

NOT SYMMETRIC VS. ANTISYMMETRIC

The Main Difference:
The key is to see how they treat a "two-way street" (a pair like (a, b) and (b, a) where a and b are different).

Not Symmetric means: "I found at least one 'one-way street' that didn't have a return path."

- **Example:** The set R = {(1, 2), (2, 1), (3, 4)} is **Not Symmetric**. Why? Because of (3, 4). Even though (1, 2) and (2, 1) are present, the existence of (3, 4) without (4, 3) breaks the rule for the entire relation.

Antisymmetric means: "I forbid all 'two-way streets' between different elements."

- **Example:** The set R = {(1, 2), (3, 4)} is **Antisymmetric**. All its paths are one-way.
- **Example:** The set R = {(1, 2), (2, 1)} is **NOT Antisymmetric**. Why? The pair (1, 2) and its reverse (2, 1) both exist, and 1 != 2. This is the one thing this property forbids.

TRANSITIVE PROPERTY

A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

How to test: Find every "chain" (a, b) and (b, c). For each one, check if the "shortcut" (a, c) exists.

In digraph Test: If there is a path from $a \rightarrow b \rightarrow c$, there must be a direct arrow $a \rightarrow c$.

Example:

R = {(1,1), (1,2), (2,1), (2,2), (3,3), (3,4)}

TRANSITIVE PROPERTY

Example:

R = {(1,1), (1,2), (2,1), (2,2), (3,3), (3,4)}

Chain 1: (1,2) and (2,1). Needs shortcut (1,1).
Is (1,1) in R? **Yes.**

Result: TRANSITIVE

Chain 2: (2,1) and (1,2). Needs shortcut (2,2).
Is (2,2) in R? **Yes.**

(Trivial chains like (1,1) and (1,2) need (1,2), which is in R. These are fine.)

(3,4) has no pair starting with 4. **No chain, no violation.**

TRANSITIVE PROPERTY

Example #2:

Identify if it has a **TRANSITIVE PROPERTY**.
Let's define a relation R on the set **A = {1, 2, 3}**:
R = {(1, 2), (2, 3), (1, 3)}

TRANSITIVE PROPERTY

Example #2:

Identify if it has a **TRANSITIVE PROPERTY**.
Let's define a relation R on the set **A = {1, 2, 3}**:
R = {(1, 2), (2, 3), (1, 3)}

We look for any "chains" (a, b) and (b, c) and check if the "shortcut" (a, c) also exists.

- **Chain 1**: Found (1, 2) and (2, 3).
  - This forms a chain $1 \rightarrow 2 \rightarrow 3$.
  - It needs the shortcut (1, 3). Is **(1, 3)** in R? **Yes**.
- Other pairs: **(1, 3) and (2, 3) do not start any new chains** *(e.g., there are no pairs that start with 3).*

Result: TRANSITIVE

TRANSITIVE PROPERTY

Example #3:

Identify if it has a **TRANSITIVE PROPERTY**.
Let's define a relation R on the set **A = {1, 2, 3, 4}**:
R = {(1, 2), (2, 3), (1, 3), (3, 4)}

TRANSITIVE PROPERTY

Example #3:

Identify if it has a **TRANSITIVE PROPERTY**.
Let's define a relation R on the set **A = {1, 2, 3, 4}**:
R = {(1, 2), (2, 3), (1, 3), (3, 4)}

- **Chain 1**: Found (1, 2) and (2, 3).
  - This forms a chain 1 → 2 → 3.
  - It needs the shortcut (1, 3). Is **(1, 3)** in R? **Yes**.
- **Chain 2**: Found (2, 3) and (3, 4).
  - This forms the chain 2 → 3 → 4.
  - It needs the shortcut (2, 4). Is **(2, 4)** in R? **No**.

Result: NOT TRANSITIVE

TIP FROM TAMTAM:

Mastering the concepts we've just covered, especially the properties of relations, comes with practice. To help you strengthen your skills, try this:

Go back to the **examples** and **"Test Your Knowledge"** slides. Before looking at the solution, try to solve the problem again and explain why your answer is correct.

A great way to challenge yourself is to create your own small relation and test it for all the properties (Reflexive, Symmetric, Transitive, etc.). This is the best way to make the concepts stick!

KEY TAKEAWAYS

Let's review! In this module, we learned:

- ...that a **Relation** is a set of ordered pairs, specifically a subset of a Cartesian Product.
- ...how to find the **Domain** (all first elements) and **Range** (all second elements) of a relation.
- ...how to perform **Relation Operations** including **Inverse** (R^{-1}), **Composition** ($S \circ R$), **Union** ($\cup$), and **Intersection** ($\cap$).
- ...how to identify the key **Properties of Relations**: **Reflexive**, **Symmetric**, **Antisymmetric**, and **Transitive**.

Ready for the Summative Test 3?

CS0001

Discrete Structures 1

END OF MODULE 6

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