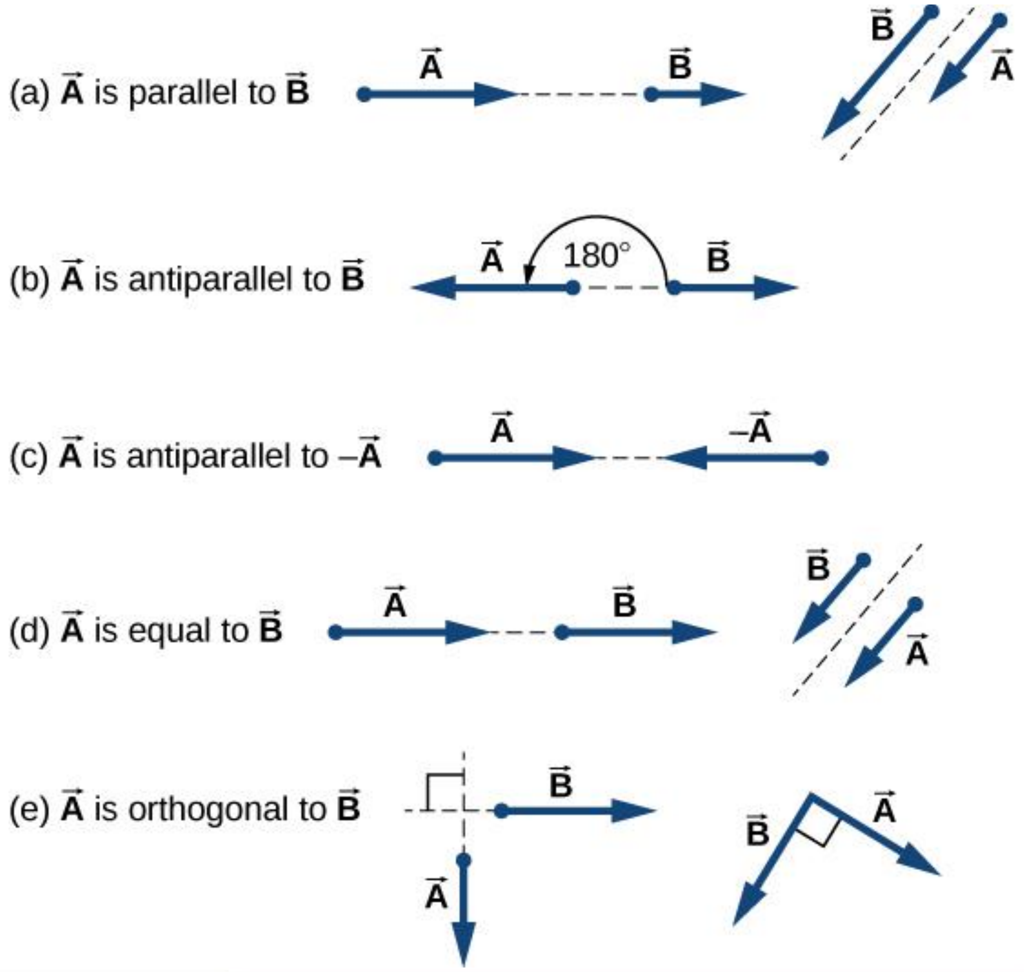
Module 2: Vectors Operation

2.1 Vector Addition

1. Add and subtract vectors written as matrix or ordered set
2. Add and subtract vectors written as ordered set
3. Add and subtract vectors using the unit vector notations
4. Add and subtract vectors written in polar form

RESULTANT VECTOR

- Combining vectors using vector addition results to one vector called resultant or resultant vector denoted by "**R**"
- The vector addition process depends on whether vectors are parallel, anti-parallel, orthogonal, etc.



(a) $\vec{\mathbf{A}} \neq \vec{\mathbf{B}}$ because $A \neq B$.

(b) $\vec{\mathbf{A}} \neq \vec{\mathbf{B}}$ because they are not parallel and $A \neq B$.

(c) $\vec{\mathbf{A}} \neq -\vec{\mathbf{A}}$ because they have different directions (even though $|\vec{\mathbf{A}}| = |-\vec{\mathbf{A}}| = A$).

(d) $\vec{\mathbf{A}} = \vec{\mathbf{B}}$ because they are parallel *and* have identical magnitudes *A* = *B.*

(e) $\vec{\mathbf{A}} \neq \vec{\mathbf{B}}$ because they have different directions (are not parallel); here, their directions differ by 90°—meaning, they are orthogonal.

VECTOR ADDITION OF A MATRIX

Given:

$$\mathbf{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Formula:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} A_x + Bx \\ A_y + By \\ A_z + B_z \end{bmatrix}$$

Examples: Find the resultant of the the following vectors:

1. $\mathbf{A} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$

$\mathbf{R} = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$ *answer*

2. $\mathbf{A} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$ $\mathbf{C} = \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}$ $\mathbf{D} = \begin{bmatrix} -1 \\ -9 \\ -3 \end{bmatrix}$

$\mathbf{R} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$ *answer*

3. $\mathbf{M} = \begin{bmatrix} 12 \\ -20 \\ 10 \end{bmatrix}$ $\mathbf{N} = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$ $\mathbf{P} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\mathbf{Q} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\mathbf{R} = \begin{bmatrix} 3 \\ -18 \\ 11 \end{bmatrix}$ *answer*

VECTOR ADDITION OF ORDERED SET

Given:

A = (Ax, Ay, Az) **B** = (Bx, By, Bz)

Formula:

R = A+B = $[(Ax + Bx) + (Ay + By) + (Az + Bz)]$

Examples: Find the resultant of the the following vectors:

1. **C** = $(3, 4, -5)$ **D** = $(0, -3, 9)$ **R** = (3, 1, 4) *answer*

2. **C** = $(3, 0)$ **D** = $(0, -3)$ **E** = $(-2, -2)$ **F** = $(7, 8)$ **G** = $(9, -1)$ **R** = (17, 2) *answer*

VECTOR ADDITION OF UNIT VECTOR FORM

Given:

$\mathbf{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ $\quad$ $\mathbf{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

Formula:

$$\mathbf{R} = \mathbf{A}+\mathbf{B} = (Ax + Bx)\hat{\imath} + (Ay + By)\hat{\jmath} + (Az + Bz)\hat{k}$$

1. Find the resultant of the vectors **A** and **B**:

$\mathbf{A} = 4\hat{\imath} - 8\hat{\jmath} + 23\hat{k}$ $\quad$ $\mathbf{B} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ $\quad$ $\mathbf{R} = 5\hat{\imath} - 5\hat{\jmath} + 21\hat{k}$ *answer*

2. Evaluate **5(2B –A)**,given the vectors in number 1

Solution:

$$5(2\mathbf{B} - \mathbf{A}) = 5\left[\, 2\left(\hat{\imath} + 3\hat{\jmath} - 2\hat{k}\right) - (4\hat{\imath} - 8\hat{\jmath} + 23\hat{k}) \,\right]$$

$$5(2\mathbf{B} - \mathbf{A}) = 5\left[\, \left(2\hat{\imath} + 6\hat{\jmath} - 4\hat{k}\right) + (-4\hat{\imath} + 8\hat{\jmath} - 23\hat{k}) \,\right]$$

$$5(2\mathbf{B} - \mathbf{A}) = 5\left[2\hat{\imath} + 6\hat{\jmath} - 4\hat{k} - 4\hat{\imath} + 8\hat{\jmath} - 23\hat{k}\,\right]$$

$$5(2\mathbf{B} - \mathbf{A}) = 5\left[-2\hat{\imath} + 14\hat{\jmath} - 27\hat{k}\,\right]$$

$$5(2\mathbf{B} - \mathbf{A}) = -10\hat{\imath} + 70\hat{\jmath} - 135\hat{k}$$ *answer*

VECTOR ADDITION OF POLAR FORM

Given:

A = 10 m/s, 30° NE

B = 25 m/s, 50° SW

note:
NE means North of East
SW means South of West

Approach:

Step 1: Determine in what quadrant the vectors lie

Step 2: Determine the components of each vector.

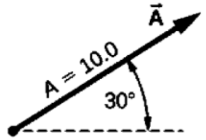
Step 3: Choose a notation (matrix, ordered set, or unit vector form) to represent the vector using its components

Step 4: Proceed with the vector addition, based on the notation selected.

Example 1: Add vectors $\boldsymbol{A} = 10\, m/s, 30°NE$ and $\boldsymbol{B} = 25\, m/s, 50°SW$

Solution:

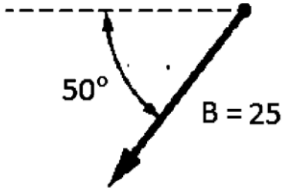
Step 1: Determine in what quadrant the vectors lie.



*Vector **A** lies in 1st Quadrant:*

x-component Ax = positive (+)

y-component Ay = positive (+)



*Vector **B** lies in 3rd Quadrant:*

x-component Bx = negative(-)

y-component By =negative(-)

Step 2: Determine the components of each vector.

Components of **A**:

Ax = A cosθ = 10 cos30° = 10(0.8660) = +8.66 m/s

Ay = A sinθ = 10 sin30° = 10(0.5000) = +5.00 m/s

Components of **B**:

Bx = -B cosθ = -25cos50° = -25(0.6428) = -16.07m/s

By = -B sinθ = -25sin50° = -25(0.7660) = -19.15 m/s

Step 3: Write the vectors in different notations.

From: Polar Form:

A = 10 m/s, 30° NE

B = 25 m/s, 50° SW

To: Ordered Set:

A = (8.66, 5.00) m/s

B = (-16.07, -19.15) m/s

Matrix:

A = $\begin{bmatrix} 8.66 \\ 5.00 \end{bmatrix}$ m/s

B = $\begin{bmatrix} -16.07 \\ -19.15 \end{bmatrix}$ m/s

Unit Vector Form:

A = 8.66$\hat{\imath}$ + 5.00$\hat{\jmath}$ m/s

B = -16.07$\hat{\imath}$ − 19.15$\hat{\jmath}$ m/s

Step 4: Proceed with the vector addition.

Ordered Set: **R** = (-7.41, -14.15) m/s

Unit vector: **R** = -7.41$\hat{\imath}$ - 14.15$\hat{\jmath}$ m/s

Matrix: **R** = $\begin{bmatrix} -7.41 \\ -14.15 \end{bmatrix}$ m/s

MAGNITUDE AND DIRECTION OF A VECTOR

- From any vector notation, use the following formula to get the vector's magnitude and direction:

Magnitude:

$$\boxed{R = \sqrt{Rx^2 + Ry^2}}$$

Direction:

$$\boxed{\theta = \tan^{-1}\frac{Ry}{Rx}}$$

Example 1: Add vectors
$\boldsymbol{A} = 10\ m/s, 30°NE$ and
$\boldsymbol{B} = 25\ m/s, 50°SW$
and find the magnitude and direction of their resultant.

To find the Resultant's magnitude and direction is:

$$\mathrm{R} = \sqrt{\mathrm{Rx}^2 + \mathrm{Ry}^2}$$

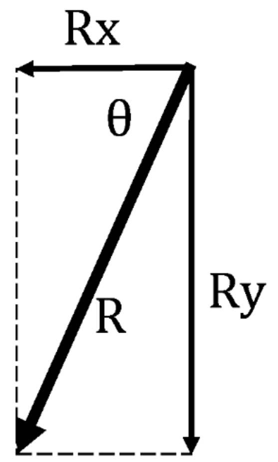
$$\mathrm{R} = \sqrt{(-7.41)^2 + (-14.15)^2}$$

$$\mathrm{R} = \sqrt{54.9081 + 200.2225}$$

$$\mathrm{R} = 15.97\ \mathrm{m/s}$$

$$\theta = \tan^{-1}\left|\frac{Ry}{Rx}\right|$$

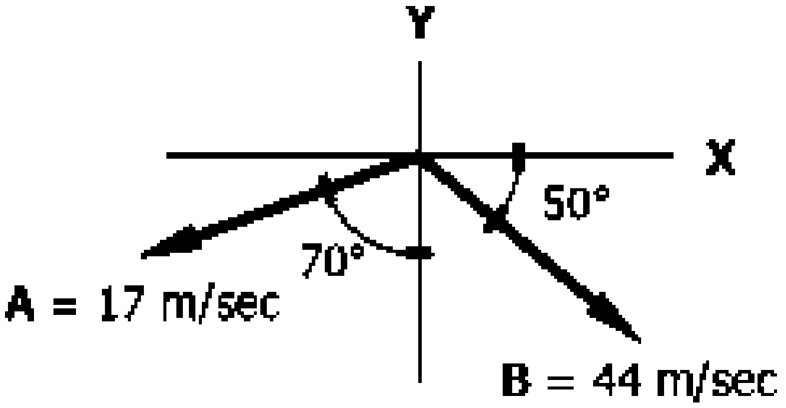
$$\theta = \tan^{-1}\frac{14.15}{7.41} = 62.35^{\mathrm{o}}$$



answer:
$\mathrm{R} = 15.97\ \mathrm{m/s}, 62.35^{\mathrm{o}}$ S of W

Find the resultant vector of vectors **A** and **B** shown, using:

A. Component method
B. Unit vector form



Solution:

A. Component method

$$R_x = \Sigma F_x$$
$$R_x = 44\cos 50^\circ - 17\sin 70^\circ$$
$$R_x = 12.31 \text{ m/sec to the right}$$

$$R = \sqrt{R_x{}^2 + R_y{}^2} = \sqrt{12.31^2 + 39.52^2}$$
$$R = 41.39 \text{ m/sec}$$

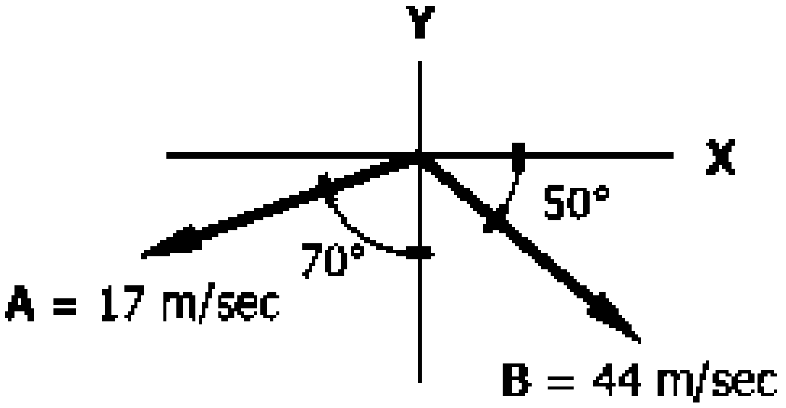
$$R_y = \Sigma F_y$$
$$R_y = -44\sin 50^\circ - 17\cos 70^\circ$$
$$R_y = -27.89 \text{ m/sec}$$
$$R_y = 39.52 \text{ m/sec downward}$$

$$\tan\theta_x = \frac{R_y}{R_x} = \frac{39.52}{12.31}$$
$$\theta_x = 72.70^\circ$$

The resultant vector R = 41.39 m/sec downward to the right at $\theta_x = 72.70°$.

Find the resultant vector of vectors **A** and **B** shown, using:

A. Component method
B. Unit vector form



Solution:

B. Unit vector form

$$\mathbf{A} = -17\sin 70^\circ\mathbf{i} - 17\cos 70^\circ\mathbf{j}$$
$$\mathbf{A} = -15.97\mathbf{i} - 5.81\mathbf{j}\ \text{m/sec}$$

$$\mathbf{B} = 44\cos 50^\circ\mathbf{i} - 44\sin 50^\circ\mathbf{j}$$

$$\mathbf{B} = 28.28\mathbf{i} - 33.70\mathbf{j}\ \text{m/sec}$$

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

$$\mathbf{R} = (-15.97 + 28.28)\mathbf{i} + (-5.81 - 33.70)\mathbf{j}$$

$$\mathbf{R} = 12.31\mathbf{i} - 39.51\mathbf{j}\ \text{m/sec}$$

$$R = \sqrt{R_x^{\,2} + R_y^{\,2}} = \sqrt{12.31^2 + (-39.51)^2}$$

$$R = 41.39\ \text{m/sec} \qquad (\textit{okay!})$$

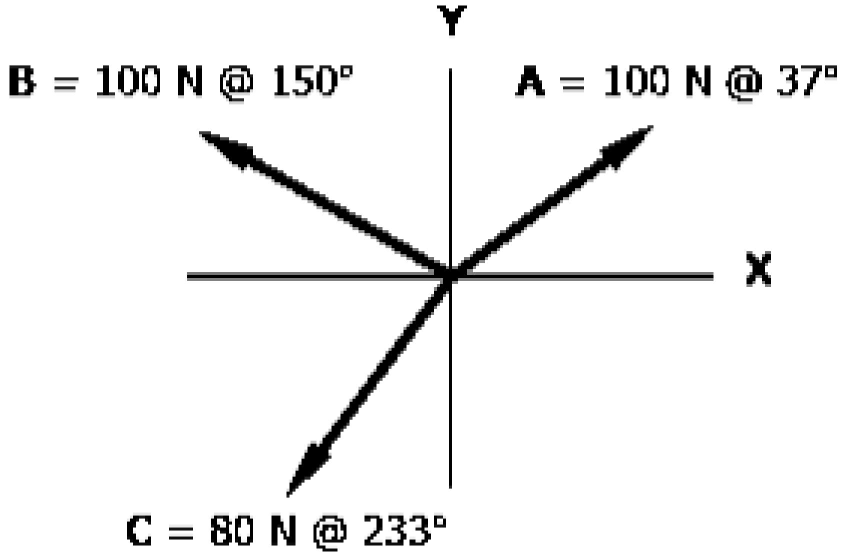
$$\tan\theta_x = \frac{R_y}{R_x} = \frac{-39.52}{12.31}$$

$$\theta_x = -72.70^\circ$$

$$\theta_x = 72.70^\circ\ \text{downward to the right} \qquad (\textit{ok!})$$

Example 3

Three vectors **A**, **B**, and **C** are shown in the figure below. Find one vector (magnitude and direction) that will have the same effect as the three vectors shown in figure.



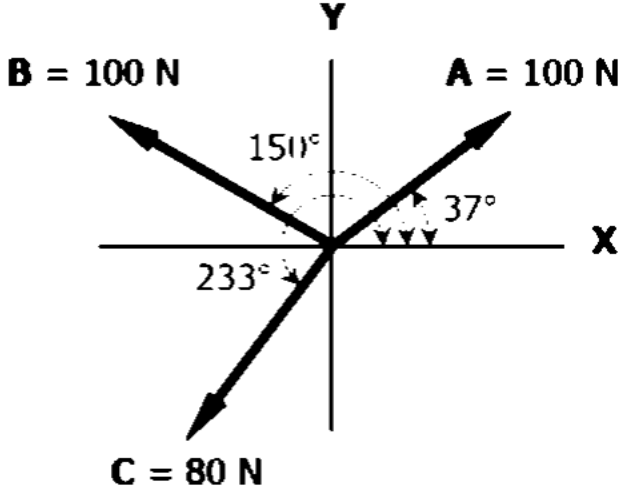
Solution:

$$R_x = 100\cos 37^\circ + 100\cos 150^\circ + 80\cos 233^\circ$$
$$R_x = -54.88\ \text{N}$$

$$R_x = 54.88\ \text{N to the left}$$

$$R_y = 100\sin 37^\circ + 100\sin 150^\circ + 80\sin 233^\circ$$

$$R_y = 46.29\ \text{N}$$



Example 3

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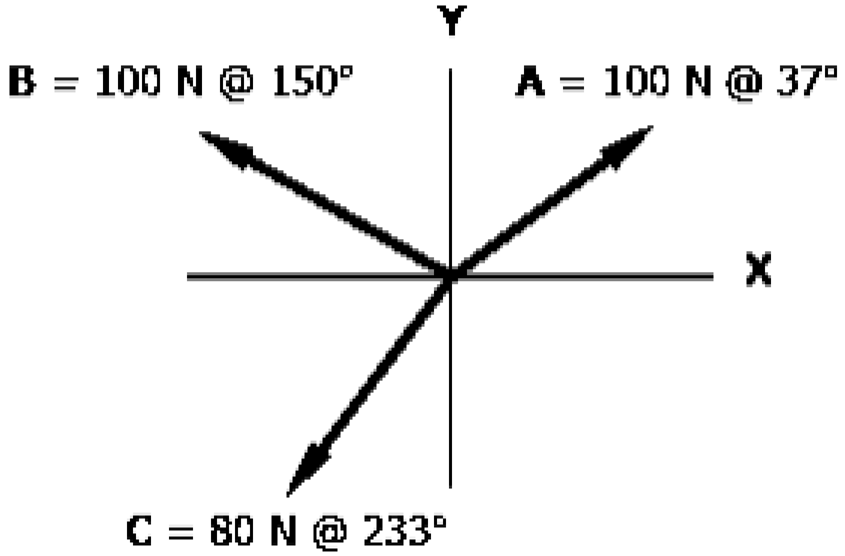


Solution (continuation):

$$R = \sqrt{R_x^{\,2} + R_y^{\,2}}$$

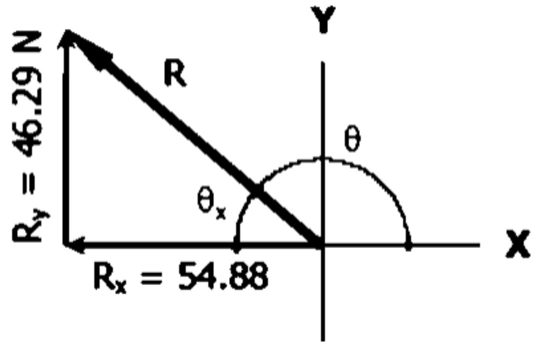
$$R = \sqrt{54.88^2 + 46.29^2}$$

$$R = 71.79 \text{ N}$$

$$\tan\theta_x = \frac{R_y}{R_x}$$

$$\tan\theta_x = \frac{46.29}{54.88}$$

$$\theta_x = 40.15^\circ$$



$$\theta = 180^\circ - \theta_x = 180^\circ - 40.15^\circ$$

$$\theta = 139.85^\circ$$

$R = 71.79$ N at 139.85° *answer*

MODULE 1: VECTORS

1.3 VECTOR PRODUCT

1. Identify the properties of dot product
2. Calculate the dot product of vectors
3. Identify the properties of cross product
4. Calculate the cross product of vectors

CHARACTERISTICS OF DOT PRODUCT

- The dot product (scalar product) of vectors **A** and **B** is defined as:

$$\mathbf{A} \bullet \mathbf{B} = \mathbf{AB}\cos\theta$$

- Angle, θ is the smallest angle between the two vectors and is always in a range of 0° to 180°
- **The result of the dot product is a scalar (positive or negative)**
- The units of the dot product will be the product of the units of the **A** and **B** vectors
- If $\mathbf{A} = (Ax, Ay, Az\,)$ and $\mathbf{B} = Bx\,, By, B_z$

$$\mathbf{A} \bullet \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

- If θ is the angle between **A** and **B**

$$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta$$

1. Find the dot product of vectors: **A** = (3, 1, 0) Newton
B = (-5, 7, 25) meter

Solution:

$$\mathbf{A}\bullet\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$$
$$\mathbf{A}\bullet\mathbf{B} = (3)(-5) + (1)(7) + (0)(25)$$
$$\mathbf{A}\bullet\mathbf{B} = -15 + 7 + 0$$
$$\mathbf{A}\bullet\mathbf{B} = -8\ \text{N.m} \qquad \textit{answer}$$

2. Find the dot product of vectors:
F = 10 N, 30° NE
G = 20 N, 65° NE

Solution:

$$\mathbf{F}\bullet\mathbf{G} = |F|\,|G|\cos\theta$$
$$\mathbf{F}\bullet\mathbf{G} = (10\ \text{N})(20\ \text{N})\cos(65-30)$$
$$\mathbf{F}\bullet\mathbf{G} = 200\cos 35^{o}$$
$$\mathbf{F}\bullet\mathbf{G} = 163.83\ \text{N}^2 \qquad \textit{answer}$$

3. Find the dot product of vectors:
D = 10 N, West
R = 20 N, North

Solution:

$\mathbf{D}\bullet\mathbf{R} = |D|\,|R| \cos\theta$

$\mathbf{D}\bullet\mathbf{R} = (10\text{ N})(20\text{ N}) \cos(0\text{-}90)$

$\mathbf{D}\bullet\mathbf{R} = 200 \cos 90^o$

$\mathbf{D}\bullet\mathbf{R} = 0$ *answer*

4. Find the angle between vectors:
A = 10 m/s, 30° NE
B = 25 m/s, 50° SW

Recall:
*Components of **A**:*
$A_x = A\cos\theta$
$A_y = A\sin\theta$

4. Find the angle between vectors:
A = 10 m/s, 30° NE
B = 25 m/s, 50° SW

Solution:

Step 1: Express the vectors in ordered set notation.
A = (8.66, 5.00) m/s
B = 16.07, 19.15) m/s

Step 2: Calculate the dot product, **A • B**

A•B = AxBx + AyBy
A•B = (8.66)(-16.07) + (5.00)(-19.15)
A•B = -139.17 - 95.75
A•B = - 234.92 m^2/s^2

Step 3: Calculate the product of magnitudes, $|A|\,|B|$

$$|A||B| = (10)(25)$$

$$|A||B| = 250$$

Step 4: Calculate the smallest angle between **A** and **B**

$$cos\theta = \frac{A \cdot B}{|A||B|}$$

$$\cos\theta = \frac{-234.92}{250} = -0.94$$

$$\theta = \cos^{-1}(-0.94)$$

$$\theta = 160^{o} \qquad \textit{answer}$$

CHARACTERISTICS OF CROSS PRODUCT

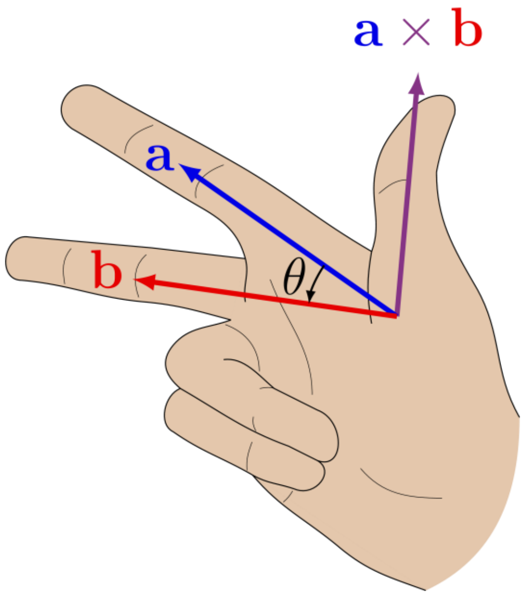
- The cross product (Vector product) of the vectors **A** and **B** is another vector **C**

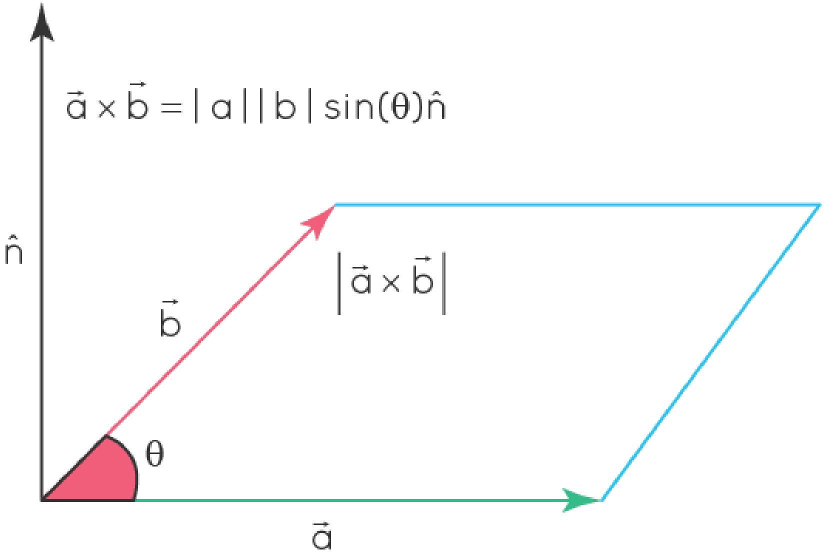
$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

- The magnitude of C is defined as:

$$\mathbf{C} = \mathbf{A\,B}\sin\theta$$

- The angle, θ is the smallest angle between **A** and **B**
- The vector **C** is perpendicular to the plane of **A** and **B** (*the direction of vector **C** is given by the right hand rule.*)





EVALUATING A CROSS PRODUCT

Matrix Notation Formula:

Vector Cross Product Formula

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \,\hat{n}$$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \qquad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

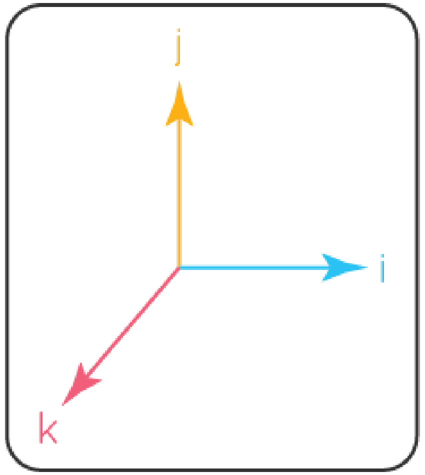
$$\vec{a} \times \vec{b} = i\,(a_2b_3 - a_3b_2) - j\,(a_1b_3 - a_3b_1) + k\,(a_1b_2 - a_2b_1)$$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \qquad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = \vec{a} \times \vec{b}$$

$$\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{c} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$



$$\vec{c} = \hat{i}\,|a_2b_3 - a_3b_2| - \hat{j}\,|a_1b_3 - a_3b_1| + \hat{k}\,|a_1b_2 - a_2b_1|$$

PROPERTIES OF A CROSS PRODUCT

$$\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$$

$$C(\mathbf{A} \times \mathbf{B}) = (C\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (C\mathbf{B})$$ *where C is scalar*

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

$$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \bullet \mathbf{C}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\mathbf{B} - (\mathbf{A} \bullet \mathbf{B})\mathbf{C}$$

1. Find the cross product of vectors:
A = (3, 1, 0)
B = 5, 7, 1)

Solution:

$$\mathbf{AxB} = \begin{vmatrix} i & j & k \\ Ax & Ay & Az \\ Bx & By & Bz \end{vmatrix}$$

$$AxB = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ -5 & 7 & 1 \end{vmatrix} = i - 3j + 26k$$

2. Find the magnitude of the cross product of vectors:
A = 10 m/s, 30° NE
B = 25 m/s, 65° NE

Solution:

C = A B sinθ

C = (10)(25) sin (65 – 30)

C = 250 sin 35°

C = 143.39 units *pointing out of the paper*

MODULE 1: VECTORS

GOT ANY QUESTIONS????

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