CS0001

Discrete Structures 1

Module 3: Predicate Logic and Rules of Inference

Technology Driven by Innovation

OBJECTIVE:

By the end of this module, you will be able to:

- **Define** predicates and use quantifiers ($\forall$, $\exists$) to create propositions from propositional functions.
- **Translate** complex English statements involving quantifiers into formal logical expressions.
- **Define** a valid argument and differentiate it from a fallacy.
- **Apply** the rules of inference (such as Modus Ponens and Modus Tollens) to construct and validate formal proofs.

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Subtopic 1: Predicates and Quantifiers (Universal and Existential)

PREDICATES

Statements involving variables such as:

- $x > 3$
- $x = y + 3$
- $x + y = z$
- computer x is under attack by an intruder
- computer x is functioning properly

Are found in mathematical assertions, in computer programs, and in system specifications.

PREDICATES

Statements involving variables such as:

- $x > 3$
- $x = y + 3$
- $x + y = z$
- computer x is under attack by an intruder
- computer x is functioning properly

Do they have truth values?

PREDICATES

Statements involving variables such as:

- $x > 3$
- $x = y + 3$
- $x + y = z$
- computer x is under attack by an intruder
- computer x is functioning properly

Do they have truth values? NONE.

PREDICATES

In this example:

- $x > 3$

x is greater than 3 - has two parts

1st part: variable *x* is the **subject of the statement**
2nd part: *is greater than 3* is the **predicate**

We can also denote x is greater than 3 as P(x), where it becomes a propositional function. If a value is assigned to x, then it can have a truth value.

QUANTIFICATION

When the variables in a propositional function are assigned values, the **resulting statement becomes a proposition** with a certain truth value.

However, there is a way called **Quantification**, which means you are **creating a proposition** from a **propositional function**.

QUANTIFICATION

In English, the words:

- all
- some
- many
- none
- few

are used in quantifications.

QUANTIFIERS

TABLE 1 Quantifiers.		
Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

There are two types of quantification:

- **Universal Quantification** - tells us that a predicate is true for EVERY element under consideration.
- **Existential Quantification** - tells us that there is one or more element under consideration for which the predicate is true.

UNIVERSAL QUANTIFICATION

The universal quantification of P(x) is the statement:

P(x) for all values of x in the domain.

The notation **$\forall$x P(x)** denotes the universal quantification of P(x). Here $\forall$ **is called the universal quantifier**.
We read it as “for all”.

If there is an element which is false, we call it a counterexample.

UNIVERSAL QUANTIFICATION

Example #1:

Let $P(x)$ be the statement "$x + 1 > x$".
What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Solution:

Because $P(x)$ is true for all real numbers x, the quantification **$\forall x\, P(x)$ is true**.

UNIVERSAL QUANTIFICATION

Example #2:

Let $Q(x)$ be the statement "$x < 2$".
What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution:

Because $Q(x)$ is not true for all real numbers x, the quantification **$\forall x\, Q(x)$ is false**.

THINK ABOUT THIS:

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement: **$x^2 < 10$** and the domain consists of the positive integers not exceeding 4?

THINK ABOUT THIS:

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement: **$x^2 < 10$** and the domain consists of the positive integers not exceeding 4?

The positive integers not exceeding 4 are: 1, 2, 3, 4, so:

$P(1)$: $1^2 < 10$

$P(2)$: $2^2 < 10$

$P(3)$: $3^2 < 10$

$P(4)$: $4^2 < 10$

THINK ABOUT THIS:

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement: **$x^2 < 10$** and the domain consists of the positive integers not exceeding 4?

The positive integers not exceeding 4 are: 1, 2, 3, 4, so:

$P(1)$: $1^2 < 10$

$P(2)$: $2^2 < 10$

$P(3)$: $3^2 < 10$

$P(4)$: $4^2 < 10$ IS FALSE

Hence, the truth value is FALSE.

EXISTENTIAL QUANTIFICATION

The existential quantification of P(x) is the statement:

There exists an element x in the domain such that P(x).

The notation **$\exists x$ P(x)** denotes the existential quantification of P(x). Here $\exists$ **is called the existential quantifier**.
We read it as “for some”, “there exists”, “for at least one”.

EXISTENTIAL QUANTIFICATION

Example #1:

Let P(x) denote the statement **x > 3**. What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

Solution:

Because $x > 3$ is sometimes true - for instance, when $x = 4$, the existential quantification of P(x), $\exists x\, P(x)$ is **TRUE**.

EXISTENTIAL QUANTIFICATION

Example #2:

Let Q(x) denote the statement **$x = x + 2$**. What is the truth value of the quantification $\exists x\ Q(x)$, where the domain consists of all real numbers?

Solution:
Because Q(x) is false for every real number x, the existential quantification $\exists x\ Q(x)$ is **FALSE**.

TRANSLATING FROM ENGLISH INTO LOGICAL EXPRESSIONS

Statement: "All birds can fly."

This can be interpreted as: **"If something is a bird, then it can fly."**

This translates to: "For every x is a bird (Bx), then x can fly (Fx).

Logical Expression: $\forall x(Bx \rightarrow Fx)$

TRANSLATING FROM ENGLISH INTO LOGICAL EXPRESSIONS

Statement: "Some dog is blue."

We are directly asserting that **there exists at least one dog** that satisfies both conditions: **being a dog** and **being blue.**

The sentence is stating that both conditions hold true for that specific dog:

- Dx: x is a dog.
- Bx: x is blue.

TRANSLATING FROM ENGLISH INTO LOGICAL EXPRESSIONS

Statement: "Some dog is blue."

We are directly asserting that **there exists at least one dog** that satisfies both conditions: **being a dog** and **being blue.**

$(\exists x)(Dx \land Bx)$ - translates to "There exists some x such that x is a dog and x is blue."

Therefore, for some x, it is simultaneously true that x is a dog and x is blue. This leads us to use conjunction ($\land$), meaning "both conditions are true for x."

TRANSLATING FROM ENGLISH INTO LOGICAL EXPRESSIONS

Examples:

- All lions are fierce.
- Some lions do not drink coffee.
- Some fierce creatures do not drink coffee.

We can express these statements as:

- $\forall x\ (P(x) \rightarrow Q(x))$
- $\exists x\ (P(x) \land \neg R(x))$
- $\exists x\ (Q(x) \land \neg R(x))$

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Subtopic 2: Arguments

ARGUMENTS

An **argument** is an assertion that a given set of propositions P1, P2, ... Pn called premises yields another proposition Q, called the conclusion. It is denoted by:

$P1, P2, ..., Pn \vdash Q$

The **turnstile symbol** (⊢) is used in formal logic to mean that something is provable or logically implies another statement.

VALID ARGUMENT

An argument P1, P2, ..., Pn $\vdash$ Q is said to be valid if Q is true whenever all the premises P1, P2, ... Pn, are true.

Example:
P1: "If it rains, the ground will be wet." ($R \rightarrow W$)
P2: "It is raining." (R)

Conclusion:
Q: "The ground is wet." (W)

VALID ARGUMENT

Formal representation: P1, P2 $\vdash$ Q
Premise: P1 = (R $\rightarrow$ W), P2 = R
Conclusion: Q = W

The argument is **valid** because if both premises are **true** (i.e., if it rains, the ground will be wet, and it is currently raining), the conclusion Q (the ground is wet) must also be **true**.

This example demonstrates that the conclusion Q follows logically from the premises.

Therefore, the argument P1, P2 $\vdash$ Q is valid.

INVALID ARGUMENT

An argument which is not valid is called an **invalid argument** or a **fallacy**.

Also, note that an argument $P1, P2, ..., Pn \vdash Q$ is valid if and only if the proposition: $(P1 \land P2 \land ... \land Pn) \rightarrow Q$ is a tautology.

INVALID ARGUMENT

Example:
P1: "If it rains, the ground will be wet." ($R \rightarrow W$)
P2: "The ground is wet." (W)

Conclusion:
Q: "It is raining." (R)

Formal representation: $P1, P2 \vdash Q$
Premise: $P1 = (R \rightarrow W)$, $P2 = W$
Conclusion: $Q = R$

INVALID ARGUMENT

Example:
P1: "If it rains, the ground will be wet." ($R \rightarrow W$)
P2: "The ground is wet." (W)

Conclusion:
Q: "It is raining." (R)

This argument is **invalid** because just knowing that the ground is wet does not necessarily mean that it is raining. The ground could be wet for other reasons (e.g., sprinklers). So, the conclusion Q(it is raining) doesn't logically follow from the premises.

TAUTOLOGY TEST

R	W	P1 $(R \rightarrow W)$	P1 ∧ P2 $((R \rightarrow W) \wedge W)$	(P1 ∧ P2) → Q $((R \rightarrow W) \wedge W) \rightarrow R$

Determine the truth value:
P1: "If it rains, the ground will be wet." $(R \rightarrow W)$
P2: "The ground is wet." (W)

Conclusion:
Q: "It is raining." (R)

TAUTOLOGY TEST

		P1	P1 ∧ P2	(P1 ∧ P2) → Q
R	**W**	**(R → W)**	**((R → W) ∧ W)**	**((R → W) ∧ W) → R**
F	T	T	T	**F**

Determine the truth value:
P1: "If it rains, the ground will be wet." (R → W)
P2: "The ground is wet." (W)

Conclusion:
Q: "It is raining." (R)

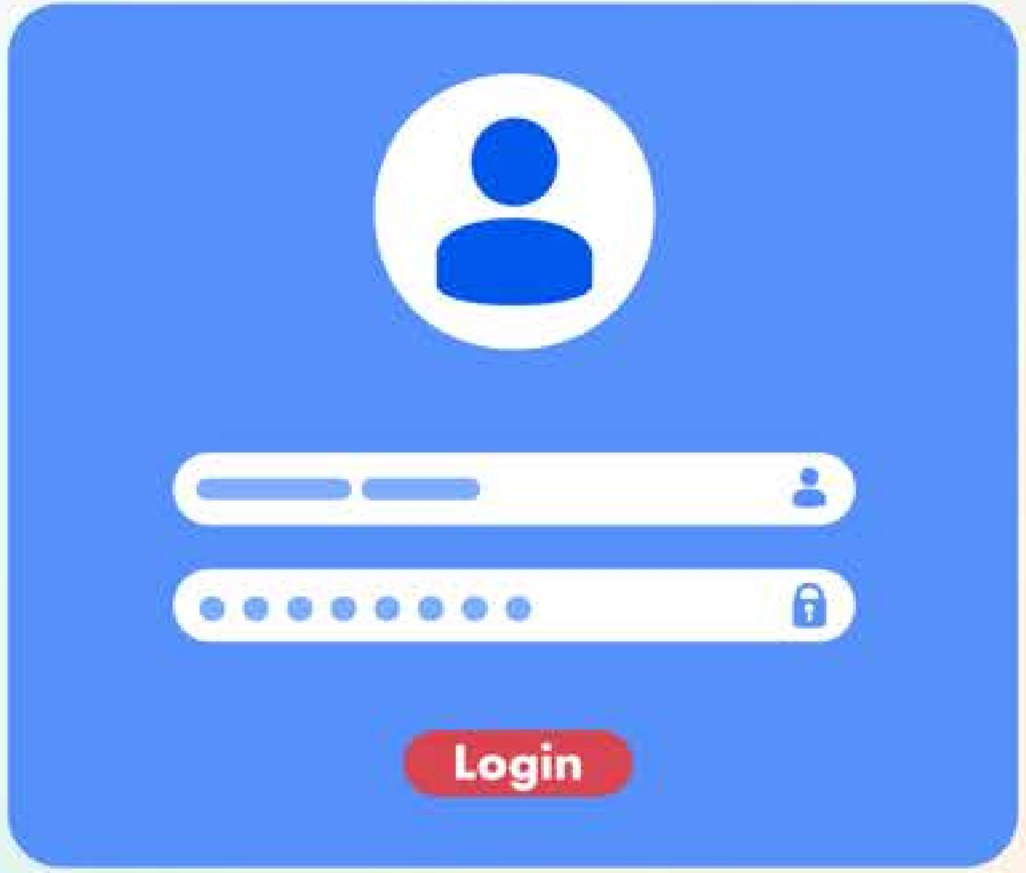
The argument is a **fallacy** because the proposition:
((R → W) ∧ W) → R
is not a tautology, meaning it is not always true. Therefore, P1, P2 ⊢ Q is **invalid**.

ARGUMENTS

Consider the following argument involving propositions and determine if the argument is **valid** or **not**:

If you have a current password, then you can log onto the network.
You have a current password.

Therefore,
You can log onto the network.



ARGUMENTS

If you have a current password, then you can log onto the network.
You have a current password.

Therefore,
You can log onto the network.

Can be written as:

$p \rightarrow q$

p

$\therefore q$

$p \rightarrow q$
p
$\therefore q$

This argument is valid because its form is valid.

ARGUMENTS AND INFERENCE

We can always use a truth table to show that an argument form is valid. We can show this if the premises are true, the conclusion must be true.

R	W	$(R \to W)$	$((R \to W) \land W)$	$((R \to W) \land W) \to R$
T	T	T	T	T

However, this can be a tedious approach.

THINK ABOUT THIS

What if the argument form involves 10 different propositional variables. How many different rows will it require to prove that it is valid?

$$2^{10} = 1024$$

Fortunately, we do not have to resort to truth tables. Instead, we can establish the validity of some relatively simple argument forms called **Rules of Inference**.

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Subtopic 3: Rules of Inference

INFERENCE

refers to the **logical process of deriving a conclusion from a set of premises** using valid logical steps.

The rules of inference provide a framework for making these logical steps, ensuring that the conclusions drawn from true premises are also true. Essentially, inference is **how we reason** from given information to deduce new information.

RULES OF INFERENCE

Rules of Inference

Modus Ponens

$$\begin{array}{l} p \\ p \to q \\ \hline q \end{array}$$

Modus Tollens

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

Addition

$$\begin{array}{l} p \\ \hline p \vee q \end{array}$$

Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline q \vee r \end{array}$$

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

Simplification

$$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

Conjunction

$$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$$

RULES OF INFERENCE

Rules of inference are patterns of logically valid deductions from hypotheses to conclusions.
We will review "inference rules" (i.e., correct & fallacious), and "proof methods".

RULES OF INFERENCE

- *Inference Rule* –
  - Pattern establishing that if we know that a set of *hypotheses* are all true, then a certain related *conclusion* statement is true.

$$\begin{array}{l} \quad \textit{Hypothesis 1} \\ \underline{\textit{Hypothesis 2} \ \ldots} \\ \therefore \textit{conclusion} \end{array}$$

"$\therefore$" means "therefore"

RULES OF INFERENCE

- Each logical inference rule corresponds to an implication that is a tautology.
- *Hypothesis 1* Inference rule

  Hypothesis 2 …

  ∴ *conclusion*
- Corresponding tautology:

  ((*Hypoth. 1*) ∧ (*Hypoth. 2*) ∧ …) → *conclusion*

RULES OF INFERENCE

Rule of Addition

$$\begin{array}{l} p \\ \therefore p \lor q \end{array}$$

Hypothesis: It is raining. (P)

Conclusion: It is either raining or the sun is shining. $(P \lor Q)$

Rule of Simplification

$$\begin{array}{l} p \land q \\ \therefore p \end{array}$$

Hypothesis: It is both sunny and warm. $(P \land Q)$

Conclusion: It is sunny. (P)

* *Commutative Law can be applied*

RULES OF INFERENCE

Rule of Conjunction

$$\begin{array}{l} p \\ q \\ \therefore p \wedge q \end{array}$$

Hypothesis 1: It is sunny. (P)

Hypothesis 2: It is warm. (Q)

Conclusion: It is sunny and it is warm. $(P \wedge Q)$

* *Commutative Law can be applied*

RULES OF INFERENCE

Modus Ponens (a.k.a. *law of detachment*)

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

"The mode of affirming"

Hypothesis 1: If it is raining, then the ground is wet. (P-> Q)
Hypothesis 2: It is raining. (P)
Conclusion: Therefore, the ground is wet. (Q)

Modus Tollens

$$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$$

"The mode of denying"

Hypothesis 1: If it is sunny, then John will go for a run. (P-> Q)
Hypothesis 2: John did not go for a run. (-Q)
Conclusion: Therefore, it is not sunny. (-P)

RULES OF INFERENCE

Hypothetical Syllogism

- $$p \to q$$
  $$q \to r$$
  $$\therefore p \to r$$

Hypothesis 1: If it is sunny, then John will go for a run. (P-> Q)
Hypothesis 2: If John goes for a run, then he will feel energized.(Q-> R)
Conclusion: Therefore, if it is sunny, then John will feel energized. ($P \to R$)

Disjunctive Syllogism

- $$p \lor q$$
  $$\neg p$$
  $$\therefore q$$

Hypothesis 1: It is either sunny or it is raining. ($P \lor Q$)
Hypothesis 2: It is not sunny. ($\neg P$)
Conclusion: Therefore, it is raining. (Q)

RULES OF INFERENCE

Resolution

- $p \lor q$
  $-q \lor r$
  $\therefore p \lor r$

Hypothesis 1: It is sunny or it is cloudy. ($P \lor Q$)
Hypothesis 2: It is not cloudy or it is rainy. ($\neg Q \lor R$)
Conclusion: Therefore, it is sunny or it is rainy. ($P \lor R$)

RECAP: LOGICAL EQUIVALENCES

Identity	*Name*
$\bar{\bar{x}}=x$	Law of the double complement
$x+x=x$ $x\cdot x=x$	Idempotent laws
$x+0=x$ $x\cdot 1=x$	Identity laws
$x+1=1$ $x\cdot 0=0$	Domination laws
$x+y=y+x$ $xy=yx$	Commutative laws
$x+y(y+z)=(x+y)+z$ $x(yz)=(xy)z$	Associative laws
$x+yz=(x+y)+(x+z)$ $x(y+z)=xy+xz$	Distributive laws
$\overline{(xy)}=\bar{x}+\bar{y}$ $\overline{(x+y)}=\bar{x}\bar{y}$	De Morgan's laws
$x+xy=x$ $x(x+y)=x$	Absorption laws
$x+\bar{x}=1$	Unit property
$x\bar{x}=0$	Zero property

You have now learned two powerful sets of rules: the **Rules of Inference** from this module and the **Rules of Replacement** (also known as Logical Equivalences) from Module 2.

A formal proof uses both of these tools together to build a valid argument from the premises to the conclusion.

COMMON FALLACIES

- A *fallacy* is an inference rule or other proof method that is not logically valid.
  - May yield a false conclusion!
- Fallacy of *affirming the conclusion*:
  - "$p \rightarrow q$ is true, and q is true, so p must be true." (No, because $\mathbf{F} \rightarrow \mathbf{T}$ is true.)
- Fallacy of *denying the hypothesis*:
  - "$p \rightarrow q$ is true, and p is false, so q must be false." (No, again because $\mathbf{F} \rightarrow \mathbf{T}$ is true.)

CHALLENGE

Hypotheses:
If Jason will help Joey review Rules of Inference, then Joey will get a high score at the Summative Assessment. If Joey will get a high score a the Summative Assessment, then Joey can pass Discrete Structures. Jason will help Joey review Rules of Inference.

Conclusion:
Therefore, Joey can pass Discrete Structures.

Prove that the argument is VALID.

SOLUTION:

Hypotheses:
If Jason will help Joey review Rules of Inference, then Joey will get a high score at the Summative Assessment. If Joey will get a high score a the Summative Assessment, then Joey can pass Discrete Structures. Jason will help Joey review Rules of Inference.

Conclusion:
Therefore, Joey can pass Discrete Structures.

Let:
P = Jason will help Joey review Rules of Inference
Q = Joey will get a high score at the Summative Assessment
R = Joey can pass Discrete Structures

Arguments:

1. P -> Q
2. Q -> R
3. P

$\therefore$ R

Solution:
(4) P->R (HS, 1,2)
(5) R (MP, 3,4)
VALID

To prove that the argument is valid, the solution should contain the new arguments that will support the conclusion, together with the corresponding rules of inference.

TRY

Hypotheses:
Either Miryl will get a perfect score at the final exam, or she will fail Discrete Structures 1. She will not fail Discrete Structures 1.

Conclusion:
Therefore, Miryl will get a perfect score at the final exam.

Prove that the argument is VALID.

SOLUTION:

Hypotheses:
Either Miryl will get a perfect score at the final exam, or she will fail Discrete Structures 1. She will not fail Discrete Structures 1.

Conclusion:
Therefore, Miryl will get a perfect score at the final exam.

Let:
P = Miryl will get a perfect score at the final exam.
Q = She will fail Discrete Structures 1.

Arguments:
1. $P \vee Q$
2. $- Q$

$\therefore$ P

Solution:
(3) P (DS, 1,2)
VALID

DS - Disjunctive Syllogism

TRY

Hypotheses:
Trevor is allergic to nuts. If Trevor is allergic to nuts, then he can’t eat cashews and hazelnuts.

Conclusion:
Therefore, he can’t eat hazelnuts.

Prove that the argument is VALID.

SOLUTION:

Hypotheses:
Trevor is allergic to nuts. If Trevor is allergic to nuts, then he can't eat cashews and hazelnuts.

Conclusion:
Therefore, he can't eat hazelnuts.

Let:
A = Trevor is allergic to nuts
B = He can't eat cashews
C = He can't eat hazelnuts

Premises:
1. A
2. A -> B ^ C

$\therefore$ C

Solution 1:
(3) B^C (MP, 1,2)
(4) C (S, 3)
VALID

Solution 2:
(3) B^C (MP, 1,2)
(4) C^B (COMMUTATIVE, 3)
(5) C (S, 4)
VALID

TIP FROM TAMTAM:

Mastering the concepts we've just covered, especially proving the validity of arguments using Rules of Inference, comes with practice. To help you strengthen your skills, a **supplementary presentation deck** is available for this module.

It contains many more examples, practice challenges, and detailed step-by-step solutions. If you feel you need more review or would simply like more hands-on practice, please be sure to check out that supplemental deck in our course materials.

KEY TAKEAWAYS

- **Predicates** are statements with variables; Quantifiers (∀, ∃) turn them into true/false propositions by making a claim about a domain.
- A **valid argument** is a sequence of premises that logically guarantees the conclusion. Its structure is a tautology.
- **Rules of Inference** are our primary tools for building valid proofs step-by-step, allowing us to move beyond using giant truth tables.
- **Modus Ponens** ($p, p \to q \therefore q$) and **Modus Tollens** ($\neg q, p \to q \therefore \neg p$) are two of the most fundamental rules for building logical arguments.

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END OF MODULE 3

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