Decentralized Online Optimization

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Abstract

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1 Notations and assumptions

Define the dynamic regret as

$$\mathcal{R}_T^{DOG} = \mathbb{E}_{\xi} \sum_{i=1}^{n} \sum_{t=1}^{T} f_{i,t}(\mathbf{x}_{i,t}; \xi) - f_{i,t}(\mathbf{x}_t^*; \xi),$$

where, for any $\mathbf{x}$,

$$f_{i,t}(\mathbf{x}; \xi) := \beta \bar{f}_{i,t}(\mathbf{x}) + (1-\beta) f(\mathbf{x}; \xi),$$

and ξ is a random variable drawn from an unknown distribution. $\bar{f}_{i,t}$ is an adversary loss function, and f is a given loss function.

The budget of the dynamics is defined as

$$\sum_{t=1}^{T} \left\| \mathbf{x}_{t+1}^* - \mathbf{x}_t^* \right\| \leq D.$$

Assumption 1. *For any $i \in [n]$ and $t \in [T]$, we assume $\|\nabla f_{i,t}(\mathbf{x})\|^2 \leq G$. For any $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{X}$, we assume $\|\mathbf{x} - \mathbf{y}\|^2 \leq R$.*

Assumption 2. *For any $i \in [n]$ and $t \in [T]$, we assume the function $f_{i,t}(\mathbf{x})$ is differentiable with respect to any vector $\mathbf{x} \in \mathcal{X}$.*

2 Algorithm

Theorem 1. *Using Assumptions 1 and 2, and choosing $\eta > 0$ in Algorithm 1, we have*

$$\begin{aligned}
\mathcal{R}_T^{DOG} =& \sum_{i=1}^{n} \sum_{t=1}^{T} f_{i,t}(\mathbf{x}_{i,t}) - f_t(\mathbf{x}_t^*) \\
\leq& 13nT\beta G\eta + \left((1-\beta) \left(L + \frac{\eta L^2}{\beta} + 3\eta L^2 \right) + \frac{\beta}{\eta} \right) \frac{nTG\eta^2}{(1-\rho)^2} \\
&+ n \left(\frac{1}{\beta} + 2 \right) \left(f(\bar{\mathbf{x}}_1) - f(\bar{\mathbf{x}}_{T+1}) \right) + \frac{n}{2\eta} \left(4\sqrt{R}D + R \right).
\end{aligned}$$

Corollary 1. *Using Assumptions 1 and 2, and choosing $\eta = \frac{1}{\sqrt{\beta T}}$ in Algorithm 1, we have*

$$\mathcal{R}_T^{DOG} \leq \mathcal{O}\left(n\sqrt{\beta T} \right).$$

Algorithm 1 DOG: Decentralized Online Gradient.

Require: The learning rate η, number of iterations T, and the confusion matrix $\mathbf{W}$.

for $t = 1, 2, ..., T$ **do**

  For the i-th node with $i \in [n]$:

    Predict $\mathbf{x}_{i,t}$.

    Observe the loss function $f_{i,t}$,
    and suffer loss $f_{i,t}(\mathbf{x}_{i,t})$.

  Update:

    Query the gradient $\nabla f_{i,t}(\mathbf{x}_{i,t})$.

    $\mathbf{x}_{i,t+1} = \sum_{j=1}^{n} \mathbf{W}_{i,j}\mathbf{x}_{j,t} - \eta \nabla f_{i,t}(\mathbf{x}_{i,t})$.

Appendix

Proof details of Theorem 1:

Proof.

$$
\begin{aligned}
&\mathbb{E}\frac{1}{n}\sum_{i=1}^{n} f_{i,t}(\mathbf{x}_{i,t}) - f_t(\mathbf{x}_t^*)\\
=&\frac{1}{n}\sum_{i=1}^{n} \beta\left(\bar{f}_{i,t}(\mathbf{x}_{i,t}) - f_t(\mathbf{x}_t^*)\right) + (1-\beta)\,\mathbb{E}\left(f(\mathbf{x}_{i,t};\xi) - f(\mathbf{x}_t^*)\right)\\
=&\frac{1}{n}\sum_{i=1}^{n} \beta\left(\bar{f}_{i,t}(\mathbf{x}_{i,t}) - f_t(\mathbf{x}_t^*)\right) + (1-\beta)\left(f(\mathbf{x}_{i,t}) - f(\mathbf{x}_t^*)\right)\\
\le&\frac{1}{n}\sum_{i=1}^{n} \beta\left\langle \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \mathbf{x}_t^*\right\rangle + (1-\beta)\left\langle \nabla f(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \mathbf{x}_t^*\right\rangle\\
=&\frac{1}{n}\sum_{i=1}^{n} \beta\left(\left\langle \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\rangle + \left\langle \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} + \left\langle \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_{t+1} - \mathbf{x}_t^*\right\rangle\right\rangle\right)\\
&+\frac{1}{n}\sum_{i=1}^{n}(1-\beta)\left(\left\langle \nabla f(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\rangle + \left\langle \nabla f(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\right\rangle + \left\langle \nabla f(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_{t+1} - \mathbf{x}_t^*\right\rangle\right)\\
=&\underbrace{\frac{1}{n}\sum_{i=1}^{n} \beta\left(\left\langle \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\rangle + \left\langle \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\right\rangle\right)}_{I_1(t)}\\
&+\underbrace{\frac{1}{n}\sum_{i=1}^{n}(1-\beta)\left(\left\langle \nabla f(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\rangle + \left\langle \nabla f(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\right\rangle\right)}_{I_2(t)}\\
&+\underbrace{\frac{1}{n}\sum_{i=1}^{n}\left\langle \nabla f_{i,t}(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_{t+1} - \mathbf{x}_t^*\right\rangle}_{I_3(t)}
\end{aligned}
$$

Now, we begin to bound $I_1(t)$.

$$
I_1(t) \le \frac{\beta}{n}\sum_{i=1}^{n}\left(\frac{\eta}{2}\left\|\nabla \bar{f}_{i,t}(\mathbf{x}_{i,t})\right\|^2 + \frac{1}{2\eta}\left\|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\|^2 + \frac{\eta}{2}\left\|\nabla f_{i,t}(\mathbf{x}_{i,t})\right\|^2 + \frac{1}{2\eta}\left\|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\right\|^2\right)
$$

$$\leq \beta G\eta + \frac{\beta}{2n\eta}\sum_{i=1}^{n}\left\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\right\|^2 + \frac{\beta}{2n\eta}\sum_{i=1}^{n}\left\|\bar{\mathbf{x}}_t-\bar{\mathbf{x}}_{t+1}\right\|^2.$$

Now, we begin to bound $I_2(t)$.

$$I_2(t) = (1-\beta)\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n}\langle\nabla f(\mathbf{x}_{i,t}),\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\rangle}_{I_{22}(t)} + \underbrace{\frac{1}{n}\sum_{i=1}^{n}\langle\nabla f(\mathbf{x}_{i,t}),\bar{\mathbf{x}}_t-\bar{\mathbf{x}}_{t+1}\rangle}_{I_{23}(t)}\right).$$

For $I_{22}(t)$, we have

$$\begin{aligned}
I_{22}(t) =& \frac{1}{n}\sum_{i=1}^{n}\langle\nabla f(\mathbf{x}_{i,t}),\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\rangle\\
=& \frac{1}{n}\sum_{i=1}^{n}\langle\nabla f(\mathbf{x}_{i,t})-\nabla f(\bar{\mathbf{x}}_t),\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\rangle + \frac{1}{n}\sum_{i=1}^{n}\langle\nabla f(\bar{\mathbf{x}}_t),\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\rangle\\
\leq& \frac{L}{n}\sum_{i=1}^{n}\left\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\right\|^2 + \frac{1}{n}\sum_{i=1}^{n}\langle\nabla f(\bar{\mathbf{x}}_t),\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\rangle\\
\leq& \frac{L}{n}\sum_{i=1}^{n}\left\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\right\|^2 + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\eta}{2\beta}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 + \frac{\beta}{2\eta}\left\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\right\|^2\right).
\end{aligned}\tag{1}$$

According to Lemma 1, we have

$$\begin{aligned}
&\frac{\eta}{2}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2\\
\leq& \frac{\eta}{2}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 + \left(\frac{\eta}{2}-\frac{L\eta^2}{2}\right)\left\|\frac{1}{n}\sum_{i=1}^{n}f_{i,t}(\mathbf{x}_{i,t})\right\|^2\\
\leq& f(\bar{\mathbf{x}}_t)-f(\bar{\mathbf{x}}_{t+1}) + 4G\eta\beta^2 + \frac{\eta L^2(1-\beta)^2}{n}\sum_{i=1}^{n}\left\|\bar{\mathbf{x}}_t-\mathbf{x}_{i,t}\right\|^2.
\end{aligned}\tag{2}$$

Substituting (2) into (1), we obtain

$$\begin{aligned}
I_{22}(t) \leq& \frac{L}{n}\sum_{i=1}^{n}\left\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\right\|^2 + \left(\frac{1}{\beta}\left(f(\bar{\mathbf{x}}_t)-f(\bar{\mathbf{x}}_{t+1})\right) + 4G\eta\beta + \frac{\eta L^2(1-\beta)^2}{n\beta}\sum_{i=1}^{n}\left\|\bar{\mathbf{x}}_t-\mathbf{x}_{i,t}\right\|^2 + \frac{\beta}{2n\eta}\sum_{i=1}^{n}\left\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\right\|^2\right)\\
=& \left(\frac{L}{n}+\frac{\eta L^2(1-\beta)^2}{n\beta}+\frac{\beta}{2n\eta}\right)\sum_{i=1}^{n}\left\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\right\|^2 + \frac{1}{\beta}\left(f(\bar{\mathbf{x}}_t)-f(\bar{\mathbf{x}}_{t+1})\right) + 4G\eta\beta.
\end{aligned}$$

For $I_{23}(t)$, we have

$$\begin{aligned}
I_{23}(t) =& \frac{1}{n}\sum_{i=1}^{n}\langle\nabla f(\mathbf{x}_{i,t}),\bar{\mathbf{x}}_t-\bar{\mathbf{x}}_{t+1}\rangle\\
\leq& \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\eta}{2}\left\|\nabla f(\mathbf{x}_{i,t})\right\|^2 + \frac{1}{2\eta}\left\|\bar{\mathbf{x}}_t-\bar{\mathbf{x}}_{t+1}\right\|^2\right)\\
\leq& \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\eta}{2}\left\|\nabla f(\mathbf{x}_{i,t})-\nabla f(\bar{\mathbf{x}}_t)+\nabla f(\bar{\mathbf{x}}_t)\right\|^2 + \frac{1}{2\eta}\left\|\bar{\mathbf{x}}_t-\bar{\mathbf{x}}_{t+1}\right\|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{n} \sum_{i=1}^{n} \left(\eta \left\| \nabla f(\mathbf{x}_{i,t}) - \nabla f(\bar{\mathbf{x}}_t) \right\|^2 + \eta \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2 \right) \\
&\le \frac{1}{n} \sum_{i=1}^{n} \left(\eta L^2 \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 + \eta \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2 \right).
\end{aligned}
$$

Recall Lemma 1, and we have

$$
\begin{aligned}
I_{23}(t) =& \frac{1}{n} \sum_{i=1}^{n} \left(\eta L^2 \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 + \left(2(f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1})) + 8G\eta\beta^2 + \frac{2\eta L^2 (1-\beta)^2}{n} \sum_{i=1}^{n} \left\| \bar{\mathbf{x}}_t - \mathbf{x}_{i,t} \right\|^2 \right) \right) + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2 \\
=& \frac{\eta L^2 (1 + 2(1-\beta)^2)}{n} \sum_{i=1}^{n} \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 + 2(f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1})) + 8G\eta\beta^2 + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2 \\
\le& \frac{3\eta L^2}{n} \sum_{i=1}^{n} \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 + 2(f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1})) + 8G\eta\beta^2 + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
I_2(t) =& (1-\beta)(I_{22}(t) + I_{23}(t)) \\
\le& (1-\beta) \left(\left(\frac{L}{n} + \frac{\eta L^2 (1-\beta)^2}{n\beta} + \frac{\beta}{2n\eta} + \frac{3\eta L^2}{n} \right) \sum_{i=1}^{n} \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 + \left(\frac{1}{\beta} + 2 \right) (f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1})) \right) \\
&+ (1-\beta) \left(4G\eta\beta(1+2\beta) + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2 \right).
\end{aligned}
$$

Combine those bounds of $I_1(t)$ and $I_2(t)$. We thus have

$$
\begin{aligned}
& I_1(t) + I_2(t) \\
\le& \beta G\eta + \frac{\beta}{2n\eta} \sum_{i=1}^{n} \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 + \frac{\beta}{2n\eta} \sum_{i=1}^{n} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2 \\
&+ (1-\beta) \left(\left(\frac{L}{n} + \frac{\eta L^2 (1-\beta)^2}{n\beta} + \frac{\beta}{2n\eta} + \frac{3\eta L^2}{n} \right) \sum_{i=1}^{n} \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 + \left(\frac{1}{\beta} + 2 \right) (f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1})) \right) \\
&+ (1-\beta) \left(4G\eta\beta(1+2\beta) + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2 \right) \\
=& \left(1 + 4(1-\beta)(1+2\beta)\right) \beta G\eta + \left((1-\beta) \left(\frac{L}{n} + \frac{\eta L^2 (1-\beta)^2}{n\beta} + \frac{\beta}{2n\eta} + \frac{3\eta L^2}{n} \right) + \frac{\beta}{2n\eta} \right) \sum_{i=1}^{n} \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 \\
&+ \left(\frac{1}{\beta} + 2 \right) (f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1})) + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2 \\
\le& 13\beta G\eta + \left((1-\beta) \left(\frac{L}{n} + \frac{\eta L^2}{n\beta} + \frac{3\eta L^2}{n} \right) + \frac{\beta}{n\eta} \right) \sum_{i=1}^{n} \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 \\
&+ \left(\frac{1}{\beta} + 2 \right) (f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1})) + \frac{1}{2\eta} \left\| \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\|^2.
\end{aligned}
$$

According to 2, we have

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 \le \frac{1}{(1-\rho)^2} \sum_{t=1}^{T} \eta^2 G.
$$

Therefore,

$$
\begin{aligned}
&\sum_{t=1}^{T}(I_1(t)+I_2(t)) \\
\leq & 13T\beta G\eta + \left((1-\beta)\left(\frac{L}{n}+\frac{\eta L^2}{n\beta}+\frac{3\eta L^2}{n}\right)+\frac{\beta}{n\eta}\right)\sum_{t=1}^{T}\sum_{i=1}^{n}\left\|\mathbf{x}_{i,t}-\bar{\mathbf{x}}_t\right\|^2 \\
&+\left(\frac{1}{\beta}+2\right)\sum_{t=1}^{T}\left(f(\bar{\mathbf{x}}_t)-f(\bar{\mathbf{x}}_{t+1})\right)+\frac{1}{2\eta}\sum_{t=1}^{T}\left\|\bar{\mathbf{x}}_t-\bar{\mathbf{x}}_{t+1}\right\|^2 \\
\leq & 13T\beta G\eta + \left((1-\beta)\left(\frac{L}{n}+\frac{\eta L^2}{n\beta}+\frac{3\eta L^2}{n}\right)+\frac{\beta}{n\eta}\right)\frac{nTG\eta^2}{(1-\rho)^2} \\
&+\left(\frac{1}{\beta}+2\right)\sum_{t=1}^{T}\left(f(\bar{\mathbf{x}}_1)-f(\bar{\mathbf{x}}_{T+1})\right)+\frac{1}{2\eta}\sum_{t=1}^{T}\left\|\bar{\mathbf{x}}_t-\bar{\mathbf{x}}_{t+1}\right\|^2.
\end{aligned}
$$

Now, we begin to bound $I_3(t)$. Recall that the update rule is

$$
\mathbf{x}_{i,t+1}=\sum_{j=1}^{n}\mathbf{W}_{ij}\mathbf{x}_{j,t}-\eta\nabla f_{i,t}(\mathbf{x}_{i,t}).
$$

By taking average over $i\in[n]$ on both sides, we have

$$
\bar{\mathbf{x}}_{t+1}=\bar{\mathbf{x}}_t-\eta\left(\frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t})\right). \tag{3}
$$

Denote a new auxiliary function $h(\mathbf{z})$ as

$$
h(\mathbf{z})=\left\langle\frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t}),\mathbf{z}\right\rangle+\frac{1}{2\eta}\left\|\mathbf{z}-\bar{\mathbf{x}}_t\right\|^2.
$$

Note that (3) is equivalent to

$$
\begin{aligned}
\bar{\mathbf{x}}_{t+1}=&\operatorname*{argmin}_{\mathbf{z}\in\mathbb{R}^d}h(\mathbf{z})\\
=&\operatorname*{argmin}_{\mathbf{z}\in\mathbb{R}^d}\left\langle\frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t}),\mathbf{z}\right\rangle+\frac{1}{2\eta}\left\|\mathbf{z}-\bar{\mathbf{x}}_t\right\|^2.
\end{aligned}
$$

Furthermore, denote a new auxiliary variable $\bar{\mathbf{x}}_\tau$ as

$$
\bar{\mathbf{x}}_\tau=\bar{\mathbf{x}}_{t+1}+\tau\left(\mathbf{x}_t^*-\bar{\mathbf{x}}_{t+1}\right),
$$

where $0\leq\tau\leq 1$. According to the optimality of $\bar{\mathbf{x}}_{t+1}$, we have

$$
\begin{aligned}
0\leq & h(\bar{\mathbf{x}}_\tau)-h(\bar{\mathbf{x}}_{t+1})\\
=&\left\langle\frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t}),\bar{\mathbf{x}}_\tau-\bar{\mathbf{x}}_{t+1}\right\rangle+\frac{1}{2\eta}\left(\left\|\bar{\mathbf{x}}_\tau-\bar{\mathbf{x}}_t\right\|^2-\left\|\bar{\mathbf{x}}_{t+1}-\bar{\mathbf{x}}_t\right\|^2\right)\\
=&\left\langle\frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t}),\tau\left(\mathbf{x}_t^*-\bar{\mathbf{x}}_{t+1}\right)\right\rangle+\frac{1}{2\eta}\left(\left\|\bar{\mathbf{x}}_{t+1}+\tau\left(\mathbf{x}_t^*-\bar{\mathbf{x}}_{t+1}\right)-\bar{\mathbf{x}}_t\right\|^2-\left\|\bar{\mathbf{x}}_{t+1}-\bar{\mathbf{x}}_t\right\|^2\right)
\end{aligned}
$$

$$= \left\langle \frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t}), \tau\left(\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}\right)\right\rangle + \frac{1}{2\eta}\left(\left\|\tau\left(\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}\right)\right\|^2 + 2\left\langle \tau\left(\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}\right), \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t\right\rangle\right).$$

Dividing τ on both sides, and letting τ be close to 0, we have

$$\begin{aligned}
I_3(t) = & \left\langle \frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_{t+1} - \mathbf{x}_t^*\right\rangle \\
\leq & \frac{1}{2\eta}\left(\lim_{\tau\to 0}\tau\left\|(\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1})\right\|^2 + 2\left\langle \mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}, \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t\right\rangle\right) \\
= & \frac{1}{\eta}\left\langle \mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}, \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t\right\rangle \\
= & \frac{1}{2\eta}\left(\left\|\mathbf{x}_t^* - \bar{\mathbf{x}}_t\right\|^2 - \left\|\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}\right\|^2 - \left\|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\right\|^2\right).
\end{aligned} \tag{4}$$

Besides, we have

$$\begin{aligned}
& \left\|\mathbf{x}_{t+1}^* - \bar{\mathbf{x}}_{t+1}\right\|^2 - \left\|\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}\right\|^2 \\
= & \left\|\mathbf{x}_{t+1}^*\right\|^2 - \left\|\mathbf{x}_t^*\right\|^2 - 2\left\langle \bar{\mathbf{x}}_{t+1}, -\mathbf{x}_t^* + \mathbf{x}_{t+1}^*\right\rangle \\
= & \left(\left\|\mathbf{x}_{t+1}^*\right\| - \left\|\mathbf{x}_t^*\right\|\right)\left(\left\|\mathbf{x}_{t+1}^*\right\| + \left\|\mathbf{x}_t^*\right\|\right) - 2\left\langle \bar{\mathbf{x}}_{t+1}, -\mathbf{x}_t^* + \mathbf{x}_{t+1}^*\right\rangle \\
\leq & \left\|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\right\|\left(\left\|\mathbf{x}_{t+1}^*\right\| + \left\|\mathbf{x}_t^*\right\|\right) - 2\left\|\bar{\mathbf{x}}_{t+1}\right\|\left\|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\right\| \\
\leq & 4\sqrt{R}\left\|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\right\|. \quad \text{(due to } \left\|\mathbf{x}-\mathbf{y}\right\|^2 \leq R, \forall \mathbf{x},\mathbf{y}\in\mathcal{X})
\end{aligned}$$

Thus, telescoping $I_3(t)$ over $t\in[T]$, we have

$$\begin{aligned}
\sum_{t=1}^{T} I_3(t) \leq & \frac{1}{2\eta}\left(4\sqrt{R}\sum_{t=1}^{T}\left\|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\right\| + \left\|\bar{\mathbf{x}}_1^* - \bar{\mathbf{x}}_1\right\|^2 - \left\|\bar{\mathbf{x}}_T^* - \bar{\mathbf{x}}_{T+1}\right\|^2\right) - \frac{1}{2\eta}\sum_{t=1}^{T}\left\|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\right\| \\
\leq & \frac{1}{2\eta}\left(4\sqrt{R}\sum_{t=1}^{T} D + R\right) - \frac{1}{2\eta}\sum_{t=1}^{T}\left\|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\right\|.
\end{aligned}$$

Combining those bounds of $I_1(t)$, $I_2(t)$ and $I_3(t)$ together, we finally obtain

$$\begin{aligned}
& \mathbb{E}\sum_{t=1}^{T}\sum_{i=1}^{n} f_{i,t}(\mathbf{x}_{i,t}) - f_t(\mathbf{x}_t^*) \\
\leq & n\sum_{t=1}^{T}\left(I_1(t) + I_2(t) + I_3(t)\right) \\
\leq & 13nT\beta G\eta + \left((1-\beta)\left(L + \frac{\eta L^2}{\beta} + 3\eta L^2\right) + \frac{\beta}{\eta}\right)\frac{nTG\eta^2}{(1-\rho)^2} \\
& + n\left(\frac{1}{\beta} + 2\right)\left(f(\bar{\mathbf{x}}_1) - f(\bar{\mathbf{x}}_{T+1})\right) + \frac{n}{2\eta}\left(4\sqrt{R}D + R\right).
\end{aligned}$$

□

Lemma 1.

$$\frac{\eta}{2}\left\|\nabla f(\bar{\mathbf{x}}_t)\right\|^2 + \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\left\|\frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t})\right\|^2 \leq f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1}) + 4G\eta\beta^2 + \frac{\eta L^2(1-\beta)^2}{n}\sum_{i=1}^{n}\left\|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\right\|^2.$$

Proof.

$$
\begin{aligned}
f(\bar{\mathbf{x}}_{t+1}) \leq & f(\bar{\mathbf{x}}_t) + \langle \nabla f(\bar{\mathbf{x}}_t), \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t \rangle + \frac{L}{2} \left\| \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t \right\|^2 \\
= & f(\bar{\mathbf{x}}_t) + \left\langle \nabla f(\bar{\mathbf{x}}_t), -\frac{\eta}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\rangle + \frac{L}{2} \left\| \frac{\eta}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 \\
= & f(\bar{\mathbf{x}}_t) + \frac{\eta}{2} \left(\left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 - \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 - \left\| \frac{1}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 \right) + \frac{L}{2} \left\| \frac{\eta}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 \\
= & f(\bar{\mathbf{x}}_t) + \frac{\eta}{2} \left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 - \frac{\eta}{2} \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} \right) \left\| \frac{1}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 . \quad (4)
\end{aligned}
$$

Additionally, we have

$$
\begin{aligned}
& \left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 \\
= & \left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \left(\beta \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}) + (1-\beta) \nabla f(\mathbf{x}_{i,t}) \right) \right\|^2 \\
\leq & 2\beta^2 \left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}) \right\|^2 + 2(1-\beta)^2 \left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla f(\mathbf{x}_{i,t}) \right\|^2 \\
\leq & 2\beta^2 \left(2 \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 + 2 \left\| \frac{1}{n} \sum_{i=1}^n \nabla \bar{f}_{i,t}(\mathbf{x}_{i,t}) \right\|^2 \right) + 2(1-\beta)^2 \left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla f(\mathbf{x}_{i,t}) \right\|^2 \\
\leq & 8G\beta^2 + 2(1-\beta)^2 \left\| \nabla f(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla f(\mathbf{x}_{i,t}) \right\|^2 \\
\leq & 8G\beta^2 + \frac{2(1-\beta)^2}{n} \sum_{i=1}^n \left\| \nabla f(\bar{\mathbf{x}}_t) - \nabla f(\mathbf{x}_{i,t}) \right\|^2 \\
\leq & 8G\beta^2 + \frac{2L^2(1-\beta)^2}{n} \sum_{i=1}^n \left\| \bar{\mathbf{x}}_t - \mathbf{x}_{i,t} \right\|^2 . \quad (5)
\end{aligned}
$$

Substituting (5) into (4), we obtain

$$
f(\bar{\mathbf{x}}_{t+1}) \leq f(\bar{\mathbf{x}}_t) + \frac{\eta}{2} \left(8G\beta^2 + \frac{2L^2(1-\beta)^2}{n} \sum_{i=1}^n \left\| \bar{\mathbf{x}}_t - \mathbf{x}_{i,t} \right\|^2 \right) - \frac{\eta}{2} \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} \right) \left\| \frac{1}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 .
$$

Equivalently, we obtain

$$
\frac{\eta}{2} \left\| \nabla f(\bar{\mathbf{x}}_t) \right\|^2 + \left(\frac{\eta}{2} - \frac{L\eta^2}{2} \right) \left\| \frac{1}{n} \sum_{i=1}^n \nabla f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 \leq f(\bar{\mathbf{x}}_t) - f(\bar{\mathbf{x}}_{t+1}) + 4G\eta\beta^2 + \frac{\eta L^2(1-\beta)^2}{n} \sum_{i=1}^n \left\| \bar{\mathbf{x}}_t - \mathbf{x}_{i,t} \right\|^2 .
$$

It completes the proof.

□

Lemma 2.

$$
\frac{1}{n} \sum_{i=1}^n \sum_{t=1}^T \left\| \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \right\|^2 \leq \frac{1}{(1-\rho)^2} \sum_{t=1}^T \eta^2 G.
$$

Proof. Recall that

$$\mathbf{x}_{i,t+1} = \sum_{j=1}^{n} \mathbf{W}_{ij}\mathbf{x}_{j,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t}),$$

and

$$\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{x}}_t - \eta\left(\frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t})\right).$$

Denote

$$\begin{aligned}\mathbf{X}_t =&[\mathbf{x}_{1,t}, \mathbf{x}_{2,t}, ..., \mathbf{x}_{n,t}] \in \mathbb{R}^{d\times n},\\ \mathbf{G}_t =&[\nabla f_{1,t}(\mathbf{x}_{1,t}), \nabla f_{2,t}(\mathbf{x}_{2,t}), ..., \nabla f_{n,t}(\mathbf{x}_{n,t})] \in \mathbb{R}^{d\times n}.\end{aligned}$$

By letting $\mathbf{x}_{i,1} = \mathbf{0}$ for any $i \in [n]$, the update rule is re-formulated as

$$\mathbf{X}_{t+1} = \mathbf{X}_t\mathbf{W} - \eta\mathbf{G}_t = -\sum_{s=1}^{t}\eta\mathbf{G}_s\mathbf{W}^{t-s}.$$

Similarly, denote $\bar{\mathbf{G}}_t = \frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t})$, and we have

$$\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{x}}_t - \eta\left(\frac{1}{n}\sum_{i=1}^{n}\nabla f_{i,t}(\mathbf{x}_{i,t})\right) = -\sum_{s=1}^{t}\eta\bar{\mathbf{G}}_s. \tag{6}$$

Therefore,

$$\begin{aligned}&\sum_{i=1}^{n}\left\|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\|^2\\ \overset{①}{=}&\sum_{i=1}^{n}\left\|\sum_{s=1}^{t-1}\eta\bar{\mathbf{G}}_s - \eta\mathbf{G}_s\mathbf{W}^{t-s-1}\mathbf{e}_i\right\|^2\\ \overset{②}{=}&\left\|\sum_{s=1}^{t-1}\eta\mathbf{G}_s\mathbf{v}_1\mathbf{v}_1^{\mathrm{T}} - \eta\mathbf{G}_s\mathbf{W}^{t-s-1}\right\|_F^2\\ \overset{③}{\le}&\left(\eta\rho^{t-s-1}\left\|\sum_{s=1}^{t-1}\mathbf{G}_s\right\|_F\right)^2\\ \le&\left(\sum_{s=1}^{t-1}\eta\rho^{t-s-1}\left\|\mathbf{G}_t\right\|_F\right)^2.\end{aligned}$$

① holds due to $\mathbf{e}_i$ is a unit basis vector, whose i-th element is 1 and other elements are 0s. ② holds due to $\mathbf{v}_1 = \frac{\mathbf{1}_n}{\sqrt{n}}$. ③ holds due to Lemma 3.

According to Lemma 4, letting $a_{t-1} = \sum_{s=1}^{t-1}\rho^{t-s-1}\left\|\mathbf{G}_t\right\|_F$ and $b_{t-1} = \left\|\mathbf{G}_t\right\|_F$, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\|^2 \le&\frac{1}{n(1-\rho)^2}\sum_{t=1}^{T}\eta^2\left\|\mathbf{G}_t\right\|_F^2\\ \le&\frac{1}{(1-\rho)^2}\sum_{t=1}^{T}\eta^2 G.\end{aligned}$$

It completes the proof. $\square$

Lemma 3 (Appeared in Lemma 5 in [Tang et al., 2018])**.** *For any matrix* $\mathbf{X}_t \in \mathbb{R}^{d\times n}$*, decompose the confusion matrix* $\mathbf{W}$ *as* $\mathbf{W} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^{\mathrm{T}} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{\mathrm{T}}$*, where* $\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n] \in \mathbb{R}^{n\times n}$*,* $\mathbf{v}_i$ *is the normalized eigenvector of* λ_i*.* $\boldsymbol{\Lambda}$ *is a diagonal matrix, and* λ_i *be its i-th element. We have*

$$\left\|\mathbf{X}_t \mathbf{W}^t - \mathbf{X}_t \mathbf{v}_1 \mathbf{v}_1^{\mathrm{T}}\right\|_F^2 \le \left\|\rho^t \mathbf{X}_t\right\|_F^2,$$

where $\rho = \max\{|\lambda_2(\mathbf{W})|, |\lambda_n(\mathbf{W})|\}$*.*

Lemma 4 (Appeared in Lemma 6 in [Tang et al., 2018])**.** *Given two non-negative sequences* $\{a_t\}_{t=1}^{\infty}$ *and* $\{b_t\}_{t=1}^{\infty}$ *that satisfying*

$$a_t = \sum_{s=1}^{t} \rho^{t-s} b_s,$$

with $\rho \in [0, 1)$*, we have*

$$\sum_{t=1}^{k} a_t^2 \le \frac{1}{(1-\rho)^2} \sum_{t=1}^{k} b_s^2.$$

References

H. Tang, S. Gan, C. Zhang, T. Zhang, and J. Liu. Communication Compression for Decentralized Training. *arXiv.org*, Mar. 2018.