Proximal Online Gradient is Optimum for Dynamic Regret

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Abstract

1. Introduction

For any online algorithm $A \in \mathcal{A}$, the previous dynamic regret $\widetilde{\mathcal{R}}_T^A$ is defined by

$$\widetilde{\mathcal{R}}_T^A = \sum_{i=1}^{n} \sum_{t=1}^{T} \left(g_{i,t}(\mathbf{x}_{i,t}) - g_{i,t}(\mathbf{x}_t^*) \right), \tag{1}$$

2. Related work

Online learning has been studied for decades of years. The static regret of a sequential online convex optimization method can achieve $\mathcal{O}\left(\sqrt{T}\right)$ and $\mathcal{O}\left(\log T\right)$ bounds for convex and strongly convex loss functions, respectively (Hazan, 2016; Shalev-Shwartz, 2012). Recently, both the decentralized online learing and the dynamic regret have drawn much attention due to their wide existence in the practical big data scenarios.

2.1. Decentralized online learning

Online learning in a decentralized network has been studied in (Shahrampour and Jadbabaie, 2018; Kamp et al., 2014; Koppel et al., 2018; Zhang et al., 2018a; 2017b; Xu et al., 2015; Akbari et al., 2017; Lee et al., 2016; Nedi et al., 2015; Lee et al., 2018; Benczúr et al., 2018; Yan et al., 2013). Shahrampour and Jadbabaie (2018) studies decentralized online mirror descent, and provides $\mathcal{O}\left(n\sqrt{nTM}\right)$ dynamic regret. Here, n, T, and M represent the number of nodes in the newtork, the number of iterations, and the budget of dynamics (defined in (3)), respectively. When the Bregman divergence in the decentralized online mirror descent is chosen appropriately, the decentralized online mirror descent becomes identical to the decentralized online gradient descent. Using the same definition of dynamic regret (defined in (1)), our method obtains $\mathcal{O}\left(n\sqrt{TM}\right)$ dynamic regret for a decentralized online gradient descent, which is better than $\mathcal{O}\left(n\sqrt{nTM}\right)$ in Shahrampour and Jadbabaie (2018). The improvement of our bound benefits from a better bounded network error (see Lemma **??**). Kamp et al. (2014) studies decentralized online prediction, and presents $\mathcal{O}\left(\sqrt{nT}\right)$ static regret. It assumes that all data, used to yielded the loss, is generated from an unknown distribution. The strong assumption is not practical in the dynamic environment, and thus limits its novelity for a general online learning task. Additionally, many decentralized online optimization methods are proposed, for example, decentralized online multi-task learning (Zhang et al., 2018a), decentralized online ADMM (Xu et al., 2015), decentralized online sub-gradient descent (Akbari et al., 2017), decentralized continuous-time online saddle-point method (Lee et al., 2016), decentralized online Nesterov's primal-dual method (Nedi et al., 2015; Lee et al., 2018). Those previous methods are proved to yield $\mathcal{O}\left(\sqrt{T}\right)$ static regret, which do not have theoretical guarantee of regrets in the dynamic environment. Besides, Yan et al. (2013) provides necessary and sufficient conditions to preserve privacy for decentralized online learning methods, which is interesting to extend our method to be privacy-preserving in the future work.

2.2. Regret in dynamic environment

Dynamic regret has been widely studied for decades of years (Zinkevich, 2003; Hall and Willett, 2015; 2013; Jadbabaie et al., 2015; Yang et al., 2016; Bedi et al., 2018; Zhang et al., 2017a; Mokhtari et al., 2016; Zhang et al., 2018b; György and Szepesvári, 2016; Wei et al., 2016; Zhao et al., 2018). Zinkevich (2003) first defines the reference points $\{\mathbf{x}_t^*\}_{t=1}^T$ satisfying (3), and then proposes an online gradient descent method. The method yields $\mathcal{O}\left(\sqrt{TM}\right)$ by choosing an appropriate learning rate. The following researches achieve the sublinear dynamic regret, but extend it to different reference points. For example, Hall and Willett (2015; 2013) choose the reference points $\{\mathbf{x}_t^*\}_{t=1}^T$ satisfying $\sum_{t=1}^{T-1} \left\| \mathbf{x}_{t+1}^* - \Phi(\mathbf{x}_t^*) \right\| \leq M$, where $\Phi(\mathbf{x}_t^*)$ is the predictive optimal decision variable. When the func-

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tion Φ predicts accurately, a small M is enough to bound the dynamics. The dynamic regret is thus effectively decreased. Jadbabaie et al. (2015); Yang et al. (2016); Bedi et al. (2018); Zhang et al. (2017a); Mokhtari et al. (2016); Zhang et al. (2018b) chooses the reference points $\{\mathbf{y}_t^*\}_{t=1}^T$ with $\mathbf{y}_t^* = \operatorname{argmin}_{\mathbf{z}\in\mathcal{X}} f_t(\mathbf{z})$, where f_t is the loss function at the t-th iteration. György and Szepesvári (2016) provides a new analysis framework, which achieves $\mathcal{O}\left(\sqrt{TM}\right)$ dynamic regret for any given reference points. Besides, Zhao et al. (2018) presents that the lower bound of the dynamic regret is $\mathcal{O}\left(\sqrt{TM}\right)$. Those previous methods define the regret as (1), which is a special case of our definition. When setting $\beta = 1$, we achieve the state-of-the-art regret, that is, $\mathcal{O}\left(\sqrt{TM}\right)$.

In some literatures, the regret in a dynamic environment is measured by the number of changes of a reference point over time. It is usually denoted by shifting regret or tracking regret. (Herbster and Warmuth, 1998; György et al., 2005; Gyorgy et al., 2012; György and Szepesvári, 2016; Mourtada and Maillard, 2017; Adamskiy et al., 2016; Wei et al., 2016; Cesa-Bianchi et al., 2012; Mohri and Yang, 2018; Jun et al., 2017). Both the shifting regret and the tracking regret can be considered as a variation of the dynamic regret, and is usually studied in the setting of "learning with expert advice". But, the dynamic regret is usually studied in a general setting of online setting.

3. Notations

For any $i \in [n]$ and $t \in [T]$, the random variable $\xi_{i,t}$ is subject to a distribution D_t, that is, $\xi_{i,t} \sim D_t$. Besides, a set of random variables $\Xi_{n,T}$ and the corresponding set of distributions are defined by

$$\Xi_{n,T} = \{\xi_{i,t}\}_{1\le i\le n, 1\le t\le T}, \text{ and } \mathcal{D}_T = \{D_t\}_{1\le t\le T},$$

respectively. For math brevity, we use the notation $\Xi_{n,T} \sim \mathcal{D}_T$ to represent that $\xi_{i,t} \sim D_t$ holds for any $i \in [n]$ and $t \in [T]$. $\mathbb{E}$ represents mathematical expectation. ∂ and ∇ represent sub-gradient and gradient operators, respectively. $\|\cdot\|$ represents the ℓ_2 norm in default.

4. Problem formulation

4.1. Setup

For any online algorithm $A \in \mathcal{A}$, define its dynamic regret as

$$\mathcal{R}_T^A = \mathop{\mathbb{E}}_{\Xi_{n,T}\sim\mathcal{D}_T}\left(\sum_{i=1}^n\sum_{t=1}^T f_{i,t}(\mathbf{x}_{i,t};\xi_{i,t}) - f_{i,t}(\mathbf{x}_t^*;\xi_{i,t})\right), \tag{2}$$

where n is the number of nodes in the decentralized network. The local loss function $f_{i,t}(\mathbf{x};\xi_{i,t})$ is defined by

$$f_{i,t}(\mathbf{x};\xi_{i,t}) := \beta g_{i,t}(\mathbf{x}) + (1-\beta)h_t(\mathbf{x};\xi_{i,t})$$

with $0 < \beta < 1$, and $\xi_{i,t}$ is a random variable drawn from an unknown distribution D_t. Note that $g_{i,t}$ is an adversary loss function, which is yielded by the learning model. $h_t(\cdot;\xi_{i,t})$ is a known loss function, which depends on the random variable $\xi_{i,t}$. The expectation of $h_t(\cdot;\xi_{i,t})$ is a global model, and does not depend on the i-th node.

$\{\mathbf{x}_t^*\}_{t=1}^T$ is the sequence of reference points, and

$$\{\mathbf{x}_t^*\}_{t=1}^T \in \left\{\{\mathbf{z}_t\}_{t=1}^T : \sum_{t=1}^{T-1}\|\mathbf{z}_t - \mathbf{z}_{t+1}\| \le M\right\}.$$

Here, M is the budget of the dynamics, that is,

$$\sum_{t=1}^{T-1}\left\|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\right\| \le M. \tag{3}$$

When $M = 0$, all $\mathbf{x}_t^*$s are same, and it degenerates to the static online learning problem. When the dynamic environment changes significantly, M becomes large to model the dynamics. Besides, we denote

$$H_t(\cdot) = \mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} h_t(\cdot;\xi_{i,t}) \quad \text{for } \forall i \in [n],$$

and

$$F_{i,t}(\cdot) = \mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} f_{i,t}(\cdot;\xi_{i,t}).$$

Recall that the previous definition of the dynamic regret is (1). Using (1), the classic online learning in a decentralized network only considers the loss function, i.e., $g_{i,t}$, incurred by the learning model on every node. Comparing with it, our definition of the dynamic regret, i.e., (2), still considers the loss function, i.e., H_t. It is incurred by a global model, which is used to let the decision variables, e.g., $\mathbf{x}_{i,t}$, have some good property in practical scenarios. We present some application scenarios to explain it in Section **??**.

5. Algorithm

The decentralized online gradient method, namely DOG, is presented in Algorithm 1. At every iteration, every node needs to collect the decision variable, e.g., $\mathbf{x}_{i,t}$, from its neighbours, and then update its decision variable. Here, $\mathbf{W} \in \mathbb{R}^{n\times n}$ is the confusion matrix. It is a doubly stochastic matrix, which implies that every element of $\mathbf{W}$ is non-negative, $\mathbf{W}\mathbf{1} = \mathbf{1}$, and $\mathbf{1}^{\mathrm{T}}\mathbf{W} = \mathbf{1}^{\mathrm{T}}$. Denote $\bar{\mathbf{x}}_t = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_{i,t}$. We can verify that $\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{x}}_t - \frac{\eta}{n}\sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t};\xi_{i,t})$ (see Lemma **??**).

Algorithm 1 DOG: Decentralized Online Gradient method.

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Require: The learning rate $\eta$, number of iterations $T$, and
    the confusion matrix $\mathbf{W}$. $\mathbf{x}_{i,1} = \mathbf{0}$ for any $i \in [n]$.
1: for $t = 1, 2, ..., T$ do
2:     For the $i$-th node with $i \in [n]$:
3:     Predict $\mathbf{x}_{i,t}$.
4:     Observe the loss function $f_{i,t}$, and suffer loss
       $f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$.
5:     Update:
6:     Query a sub-gradient $\partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$.
7:     $\mathbf{x}_{i,t+1} = \sum_{j=1}^{n} \mathbf{W}_{i,j}\mathbf{x}_{j,t} - \eta \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$.
8: end for
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6. Theoretical analysis

Assumption 1. *We make the following assumptions.*

- *For any $i \in [n]$, $t \in [T]$, and $\mathbf{x}$, there exists a constant G such that*

$$\max\left\{ \mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} \left\|\nabla h_t(\mathbf{x}; \xi_{i,t})\right\|^2, \left\|\partial g_{i,t}(\mathbf{x})\right\|^2 \right\} \le G,$$

  and

$$\mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} \left\|\nabla h_t(\mathbf{x}; \xi_{i,t}) - \nabla H_t(\mathbf{x})\right\|^2 \le \sigma^2.$$

- *For any $\mathbf{x}$ and $\mathbf{y}$, we assume $\left\|\mathbf{x} - \mathbf{y}\right\|^2 \le R$.*

- *For any $i \in [n]$ and $t \in [T]$, we assume the function $f_{i,t}$ is convex, but may be non-smooth. Furthermore, we assume the function H_t has L-Lipschitz gradients. In brief, $g_{i,t}$ may be non-convex, non-smooth. H_t is smooth, but may be non-convex. $f_{i,t}$ is convex, but may be non-smooth.*

6.1. Main results

Theorem 1. *Denote constants C_0, and C_1 by*

$$C_0 := 1 + \frac{1}{2(1-\rho)^2} + 16\beta;$$

$$C_1 := \frac{L + 2\eta L^2 + 4(1-\beta)^2 L^2 \eta}{(1-\rho)^2} + 2L.$$

Using Assumption 1, and choosing $\eta > 0$ in Algorithm 1, we have

$$\mathop{\mathbb{E}}_{\Xi_{n,T}\sim\mathcal{D}_T} \sum_{t=1}^{T}\sum_{i=1}^{n} f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) - f_{i,t}(\mathbf{x}_t^*; \xi_{i,t})$$
$$\le C_0 \eta T n \beta G + (1-\beta)\eta T \sigma^2 + 4n(1-\beta)T\eta G + C_1 n T \eta^2 G + \frac{n}{2\eta}\left(4\sqrt{R}M + R\right).$$

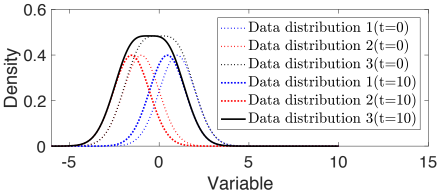


Figure 1. The illustration of the dynmaics caused by the time-varying distributions of data. Data distribution 1 and 2 are Normal distrbutions with 1 variance and $1 + \sin(t)$ mean, and 1 variance and $-1 + \sin(t)$ mean, respectively. Data distribution 3 is the sum of them, which changes over time.

Corollary 1. *Using Assumption 1, and choosing*

$$\eta = \sqrt{\frac{nM}{T\left(nG + (1-\beta)\sigma^2\right)}}$$

in Algorithm 1, we have

$$\mathcal{R}_T^{\mathrm{DOG}} \lesssim \sqrt{nMT\left(nG + (1-\beta)\sigma^2\right)}.$$

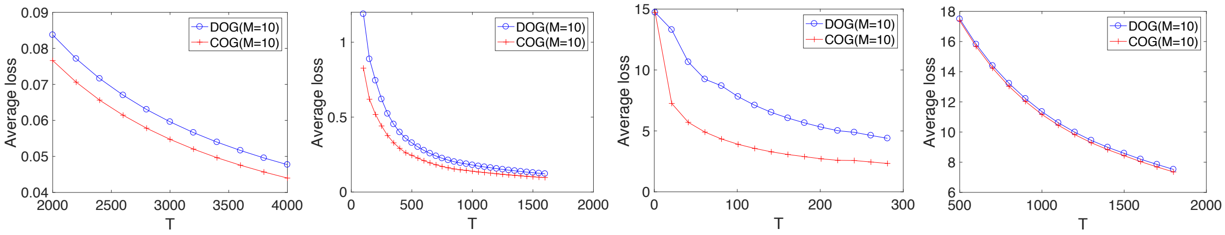
6.2. Connections with the previous results

7. Empirical studies

We conduct online logistic regression in the empirical studies. We let $f_{i,t}(\mathbf{x}; \xi_{i,t}) = \log\left(1 + \exp(-\mathbf{y}_{i,t}\mathbf{A}_{i,t}^{\mathrm{T}}\mathbf{x})\right) + \frac{\gamma}{2}\left\|\mathbf{x}\right\|^2$, where $\gamma = 10^{-3}$ is the given hyper-parameter. The compared methods include our Decentralized Online Gradient method (DOG) and the Centralized Online Gradient method (COG). The learning rate η is set to be $C\sqrt{\frac{M}{T}}$ with $10^{-2} \le C \le 20$, and we tune C for different datasets. In experiment, we use the average loss $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$, instead of the dynamic regret $\mathbb{E}_{\Xi_{n,T}\sim\mathcal{D}_{n,T}} \sum_{i=1}^{n}\sum_{t=1}^{T}\left(f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) - f_{i,t}(\mathbf{x}_t^*)\right)$ as a metric to measure the quality of a learning model. The reason is that the learning models yielded by both DOG and COG have the same reference point $\{\mathbf{x}_t^*\}_{t=1}^{T}$.

7.1. Datasets

Synthetic data We generate a data matrix $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n$, where $\mathbf{A}_i$ is placed on the i-th node, and $\mathbf{A}_i = 0.1\tilde{\mathbf{A}}_i + 0.9\hat{\mathbf{A}}_i$, where $\tilde{\mathbf{A}}_i$ represents the adversary part of data, and $\hat{\mathbf{A}}_i$ represents the stochastic part of data. $\mathbf{y}_i \in \{1, -1\}$ is the label of an instance $\hat{\mathbf{A}}_{i,t}$. The dimension of every instance is $d = 10$. Specifically, elements of $\tilde{\mathbf{A}}_i$ is sampled from the interval $[-0.5 + \sin(i), 0.5 + \sin(i)]$ randomly. Note that $\tilde{\mathbf{A}}_i$ and $\tilde{\mathbf{A}}_j$ with $i \ne j$ are drawn from different distributions. Besides, $\mathbf{y}_{i,t} \in \{1, -1\}$ is generated randomly. When $\mathbf{y}_{i,t} = 1$, $\hat{\mathbf{A}}_{i,t}$ is generated by sampling from a time-varying Normal distribution



(a) *synthetic data*, 100 nodes (b) *room-occupancy*, 5 nodes (c) *usenet2*, 5 nodes (d) *spam*, 5 nodes

Figure 2. The average loss yielded by DOG is comparable to that yielded by COG.

$\hat{\mathbf{A}}_{i,t} \sim N((1+0.5\sin(t))\cdot\mathbf{1},\mathbf{I})$. When $\mathbf{y}_{i,t} = -1$, $\hat{\mathbf{A}}_{i,t}$ is generated by sampling from another time-varying Normal distribution $\hat{\mathbf{A}}_{i,t} \sim N((-1+0.5\sin(t))\cdot\mathbf{1},\mathbf{I})$. As illustrated in Figure 1, we use those time-varying distributions of data to simulate the dynamics in the environment, which leads to the change of the optimal learning model over time. In the setting, dynamic regret is practical and necessary to measure goodness of a learning model.

Real data We use three real datasets: *room-occupancy*[1], *usenet2*[2], and a *spam*[3]. *room-occupancy* ia time-series dataset, where the dynamics exists naturally in those practical scenarios. Bothe *usenet2* and *spam* contain "concept drift" (Katakis et al., 2010), which is the source of the dynamics. Thus, the dynamic regret is practial and necessary to measure an online learning method to handle those datasets.

7.2. Results

First, we want to test whether DOG has a comparable performance with COG. We simulate a decentralized network consisting of 100 nodes to handle the synthetic data, and a network consisting of 5 nodes to handle the real data. Those nodes are connected by using a ring topology. As shown in Figure 2, both DOG and COG are effective to optimize the decision variable, and they have very similar performance.

Second, we want to varify whether DOG has a good performance with the increase of the network size. As shown in Figure 3, the performance of DOG is not sensitive to the network size, which confirms our theoretical result, that is, the average regret $\frac{1}{nT}\mathbb{E}_{\Xi_{n,T}\sim\mathcal{D}_{n,T}}\sum_{i=1}^{n}\sum_{t=1}^{T}(f_{i,t}(\mathbf{x}_{i,t};\xi_{i,t}) - f_{i,t}(\mathbf{x}_t^*))$ does not increase over the number of nodes.

Third, we want to test whether the performance of DOG is sensitive to the topology of the network. We generate four different topologies. Besides the ring topology, the *Fully connected* means all nodes are connected, where DOG degenerates to be COG. The topology *WattsStrogatz* represents a Watts-Strogatz small-world graph. There is a parameter can be tuned, e.g., 0.5 or 1 in the legend of Figure 4, to control the number of random edges. As illustrated in Figure 4, *Fully connected* has the best performance because that $\rho = 0$ in the topology, and $\rho > 0$ in other topologies.

8. Conclusion

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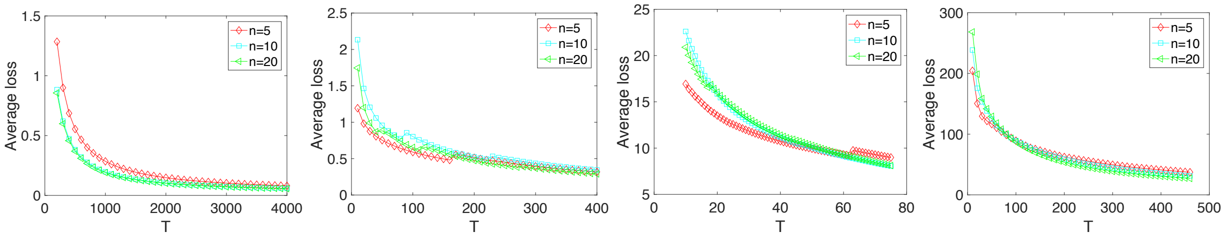
[1] https://archive.ics.uci.edu/ml/datasets/Occupancy+Detection+

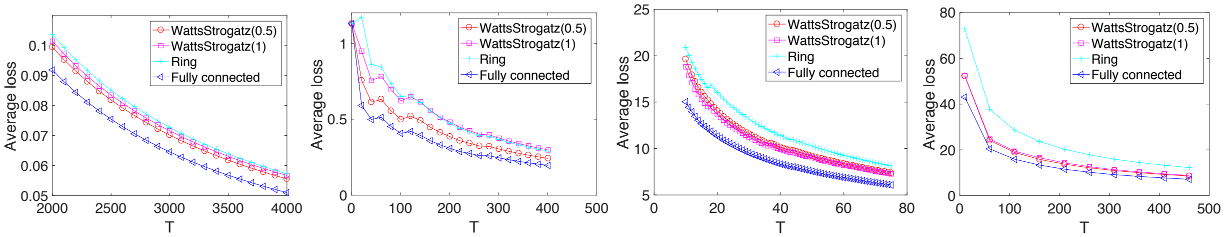
[2] http://mlkd.csd.auth.gr/concept_drift.html

[3] http://mlkd.csd.auth.gr/concept_drift.html



(a) *synthetic data*, ring topology (b) *room-occupancy*, ring topology (c) *usenet2*, ring topology (d) *spam*, ring topology

Figure 3. The average loss yielded by DOG is insensitive to the network size.



(a) *synthetic data*, 100 nodes (b) *room-occupancy*, 20 nodes (c) *usenet2*, 20 nodes (d) *spam*, 20 nodes

Figure 4. The average loss yielded by DOG is insensitive to the topology of the network.

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