

# Gossip Online Learning: Exchanging Local Models to Track Dynamics

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**Abstract**

## 1 Notations

For any  $i \in [n]$  and  $t \in [T]$ , the random variable  $\xi_{i,t}$  is subject to a distribution  $D_{i,t}$ , that is,  $\xi_{i,t} \sim D_{i,t}$ . Besides, a set of random variables  $\Xi_{n,T}$  and the corresponding set of distributions are defined by

$$\Xi_{n,T} = \{\xi_{i,t}\}_{1 \leq i \leq n, 1 \leq t \leq T}, \text{ and } \mathcal{D}_{n,T} = \{D_{i,t}\}_{1 \leq i \leq n, 1 \leq t \leq T},$$

respectively. For math brevity, we use the notation  $\Xi_{n,T} \sim \mathcal{D}_{n,T}$  to represent that  $\xi_{i,t} \sim D_{i,t}$  holds for any  $i \in [n]$  and  $t \in [T]$ .  $\mathbb{E}$  represents mathematical expectation.  $\partial$  and  $\nabla$  represent sub-gradient and gradient operators, respectively.  $\|\cdot\|$  represents the  $\ell_2$  norm in default.

## 2 Problem formulation

### 2.1 Setup

For any online algorithm  $A \in \mathcal{A}$ , define its dynamic regret as

$$\mathcal{R}_T^A = \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \left( \sum_{i=1}^n \sum_{t=1}^T f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) - f_{i,t}(\mathbf{x}_t^*; \xi_{i,t}) \right), \quad (1)$$

where  $n$  is the number of nodes in the decentralized network. The local loss function  $f_{i,t}(\mathbf{x}; \xi_{i,t})$  is defined by

$$f_{i,t}(\mathbf{x}; \xi_{i,t}) := \beta g_{i,t}(\mathbf{x}) + (1 - \beta) h_t(\mathbf{x}; \xi_{i,t})$$

with  $0 < \beta < 1$ , and  $\xi_{i,t}$  is a random variable drawn from an unknown distribution  $D_{i,t}$ .  $g_{i,t}$  is an adversary loss function.  $h_t(\cdot, \xi_{i,t})$  is a given loss function depending on the random variable  $\xi_{i,t}$ .  $\{\mathbf{x}_t^*\}_{t=1}^T$  is the sequence of reference points, and

$$\{\mathbf{x}_t^*\}_{t=1}^T \in \left\{ \{\mathbf{z}_t\}_{t=1}^T : \sum_{t=1}^{T-1} \|\mathbf{z}_t - \mathbf{z}_{t+1}\| \leq M \right\}.$$

Here,  $M$  is the budget of the dynamics, that is,

$$\sum_{t=1}^{T-1} \|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\| \leq M. \quad (2)$$

Besides, we denote

$$H_t(\cdot) = \mathbb{E}_{\xi_{i,t} \sim D_{i,t}} h_t(\cdot; \xi_{i,t}),$$

and

$$F_{i,t}(\cdot) = \mathbb{E}_{\xi_{i,t} \sim D_{i,t}} f_{i,t}(\cdot; \xi_{i,t}).$$

For every iteration, classic online learning in a decentralized network only considers the loss function, i.e.,  $F_{i,t}$ , incurred by the learning model on every local node. Comparing with it, our definition of the dynamic regret, i.e., (1), still considers the loss function, i.e.,  $H_t$ , incurred by a global model. The global model is used to let the decision variables, e.g.,  $\mathbf{x}_{i,t}$ , have some good property in practical scenarios. We present some application scenarios to explain it in Section 2.2.

## 2.2 Application scenarios

**Communication efficient online learning.** Suppose we want to conduct online learning in a decentralized network. At every iteration, a node has to broadcast the local model to its neighbours, and the communication efficiency needs to be considered. In the case,  $g_{i,t}(\mathbf{x})$  represents the loss incurred by the learning model, and  $h_t(\mathbf{x}; \xi_{i,t})$  represents the loss incurred by some quantization method to guarantee the communication efficiency. A small  $\beta$  means a strong guarantee for the communication efficiency.

Suppose we want to conduct online classification by using logistic regression model. Given an instance  $\mathbf{a}_{i,t} \in \mathbb{R}^d$  and its label  $\mathbf{y}_{i,t} \in \{1, -1\}$ . In the case,  $g_{i,t}(\mathbf{x}) = \log(1 + \exp(-\mathbf{y}_{i,t} \mathbf{a}_{i,t}^T \mathbf{x}))$ . We let  $h_t(\mathbf{x}; \xi_{i,t}) = \lambda_t \|\mathbf{Q}\mathbf{x}\|_1$ <sup>1</sup>. Here,  $\lambda_t$  with  $\lambda_t > 0$  is a given hyper-parameter.  $\mathbf{Q} \in \mathbb{R}^{(d-1) \times d}$  is a special matrix:

$$\mathbf{Q} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{bmatrix}.$$

Here,  $h_t(\mathbf{x}; \xi_{i,t})$  induces the difference between elements of  $\mathbf{x}$  to be sparse. Thus, it is able to transmit  $\mathbf{x}$  by using few different elements, and improve the communication efficiency. When  $\lambda_t$  is a constant, and does not depend on  $t$ ,  $h_t(\mathbf{x}; \xi_{i,t})$  plays a role of a regularizer.

**Online learning with privacy protection.** Suppose we want to conduct online learning on a decentralized network. But, there is a hacker who can sniff at the network, and obtains the transmitted data packages. To protect the privacy, we use a randomization encryption method to protect the local model before transmitting it in the network. In the case,  $g_{i,t}(\mathbf{x}_{i,t})$  represents the loss incurred by the learning model.  $h_t(\mathbf{x}_{i,t}; \xi_{i,t})$  represents the loss incurred by some randomization encryption method, e.g., objective perturbation [Chaudhuri et al., 2011, Wang et al., 2017], to protect the privacy. A small  $\beta$  means a strong guarantee for the data privacy.

Similarly, suppose we want to conduct online classification by using logistic regression model. Given an instance  $\mathbf{a}_{i,t} \in \mathbb{R}^d$  and its label  $\mathbf{y}_{i,t} \in \{1, -1\}$ . In the case,  $g_{i,t}(\mathbf{x}) = \log(1 + \exp(-\mathbf{y}_{i,t} \mathbf{a}_{i,t}^T \mathbf{x}))$ . We use the objective perturbation strategy [Chaudhuri et al., 2011, Wang et al., 2017] to protect the privacy. Specifically, we let  $h_t(\mathbf{x}; \xi_{i,t}) = \mathbf{x}^T \xi_{i,t}$ , where  $\xi_{i,t}$  is random noise, whose density is

$$v(\mathbf{x}) = \frac{1}{\lambda} \exp(-\delta_{i,t} \|\mathbf{x}\|).$$

Here,  $\lambda$  is a given hyper-parameter,  $\delta_{i,t}$  is a known function of the constant  $\epsilon_{i,t}$  for  $\epsilon_{i,t}$ -differential privacy [Dwork and Roth, 2014].

**Online recommendation with unreliable features.** Suppose we want to decide whether to recommend music to Bob by using a public dataset consisting of historical browser records on Youtube. But,

<sup>1</sup>In the case, the random variable  $\xi_{i,t}$  is not necessary, which is a special case.

some values of features in those records are not reliable. For example, Alice’s browser record is in the public dataset. But Alice does not want to let others know her real birthday and age. She submits random numbers for such information when signing up as an Youtube user. Note that those unreliable values, e.g., Alice’s age and birthday, usually do not change, which is modeled by an unknown distribution. But, other reliable values, e.g., Alice’s preference to music, may change over time, which is a classic setting for an online learning problem. In the case,  $g_{i,t}(\mathbf{x}_{i,t})$  represents the loss incurred by those reliable features in the learning model, e.g., preference to music.  $h_t(\mathbf{x}_{i,t}; \xi_{i,t})$  represents the loss incurred by those unreliable features in the learning model, e.g., age and birthday. A small  $\beta$  means significant attention on those unreliable features.

Suppose we still use logistic regression to decide whether to recommend music to Bob. Without loss of generality, features corresponding to those unreliable values are denoted by the beginning  $s$  features. Given a user’s behavior record  $\mathbf{a}_{i,t}$  and its label  $\mathbf{y}_{i,t} \in \{1, -1\}$ . In the case,  $g_{i,t}(\mathbf{x}) = \log \left( 1 + \exp \left( -\mathbf{y}_{i,t} \mathbf{a}_{i,t}^T \hat{\mathbf{I}} \mathbf{x} \right) \right)$ , where  $\hat{\mathbf{I}}$  is yielded by letting the first  $s$  diagonal elements of an identity matrix be 0s.  $\xi_{i,t} = \check{\mathbf{I}} \mathbf{a}_{i,t} \mathbf{y}_{i,t}^T$ , and  $h_t(\mathbf{x}; \xi_{i,t}) = \log \left( 1 + \exp \left( -\xi_{i,t}^T \mathbf{x} \right) \right)$ , where  $\check{\mathbf{I}}$  is yielded by letting the last  $(d - s)$  diagonal elements of an identity matrix be 0s. Here,  $\xi_{i,t}$  is drawn from an unknown distribution, that is,  $\xi_{i,t} \sim D_{i,t}$ , and  $D_{i,t}$  usually changes insignificant over  $t$ , or does not change over  $t$ .

### 3 Algorithm

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**Algorithm 1** DOG: Decentralized Online Gradient method.

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**Require:** The learning rate  $\eta$ , number of iterations  $T$ , and the confusion matrix  $\mathbf{W}$ .  $\mathbf{x}_{i,1} = \mathbf{0}$  for any  $i \in [n]$ .

- 1: **for**  $t = 1, 2, \dots, T$  **do**
    - For the  $i$ -th node with  $i \in [n]$ :
    - 2:     Predict  $\mathbf{x}_{i,t}$ .
    - 3:     Observe the loss function  $f_{i,t}$ ,  
and suffer loss  $f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$ .
    - Update:
    - 4:     Query a sub-gradient  $\partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$ .
    - 5:      $\mathbf{x}_{i,t+1} = \sum_{j=1}^n \mathbf{W}_{i,j} \mathbf{x}_{j,t} - \eta \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$ .
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The decentralized online gradient method, namely DOG, is presented in Algorithm 1. At every iteration, every node needs to collect the decision variable, e.g.,  $\mathbf{x}_{i,t}$ , from its neighbours, and then update its decision variable. Here,  $\mathbf{W} \in \mathbb{R}^{n \times n}$  is the confusion matrix. It is a doubly stochastic matrix, which implies that every element of  $\mathbf{W}$  is non-negative,  $\mathbf{W}\mathbf{1} = \mathbf{1}$ , and  $\mathbf{1}^T \mathbf{W} = \mathbf{1}^T$ .

### 4 Theoretical analysis

**Assumption 1.** *We make the following assumptions.*

- For any  $i \in [n]$ ,  $t \in [T]$ , and  $\mathbf{x}$ , there exists a constant  $G$  such that

$$\max \left\{ \mathbb{E}_{\xi_{i,t} \sim D_{i,t}} \|\nabla h_t(\mathbf{x}; \xi_{i,t})\|^2, \|\partial g_{i,t}(\mathbf{x})\|^2 \right\} \leq G,$$

and

$$\mathbb{E}_{\xi_{i,t} \sim D_{i,t}} \|\nabla h_t(\mathbf{x}; \xi_{i,t}) - \nabla H_t(\mathbf{x})\|^2 \leq \sigma^2.$$

- For any  $\mathbf{x}$  and  $\mathbf{y}$ , we assume  $\|\mathbf{x} - \mathbf{y}\|^2 \leq R$ .

- For any  $i \in [n]$  and  $t \in [T]$ , we assume the function  $f_{i,t}$  is convex, but may be non-smooth. Furthermore, we assume the function  $H_t$  has  $L$ -Lipschitz gradients. In brief,  $g_{i,t}$  may be non-convex, non-smooth.  $H_t$  is smooth, but may be non-convex.  $f_{i,t}$  is convex, but may be non-smooth.

**Theorem 1.** Denote

$$C_0 := \frac{1}{\sqrt{\beta^2 + \eta}} + 4;$$

$$C_1 := \frac{\beta}{2\eta} + L + \frac{\sqrt{\beta^2 + \eta}}{2\eta} + 2\eta L^2 + C_0(1 - \beta)^2 L^2 \eta.$$

Using Assumption 1, and choosing  $\eta > 0$  in Algorithm 1, we have

$$\begin{aligned} & \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \sum_{i=1}^n f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) - f_t(\mathbf{x}_t^*; \xi_{i,t}) \\ & \leq \eta T (n\beta G + (1 - \beta)\sigma^2) + n(1 - \beta)C_0 \left( \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T (H_t(\bar{\mathbf{x}}_t) - H_t(\bar{\mathbf{x}}_{t+1})) \right) \\ & \quad + (1 - \beta) \frac{nT\eta^2 G C_1}{(1 - \rho)^2} + n(1 - \beta)C_0 \left( 4T\beta^2 \eta G + \frac{TGL\eta^2}{2} \right) + \frac{n}{2\eta} (4\sqrt{R}M + R). \end{aligned}$$

**Corollary 1.** Recall that

$$C_0 = \frac{1}{\sqrt{\beta^2 + \eta}} + 4.$$

Using Assumption 1, and choosing

$$\eta = \sqrt{\frac{nM}{T(n\beta G + (1 - \beta)\sigma^2)}}$$

in Algorithm 1, we have

$$\mathcal{R}_T^{\text{DOG}} \lesssim \sqrt{nMT(\beta nG + (1 - \beta)\sigma^2)} + n(1 - \beta)C_0 \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T (H_t(\bar{\mathbf{x}}_t) - H_t(\bar{\mathbf{x}}_{t+1})).$$

## 5 Empirical studies

### 5.1 Communication efficient online logistic regression

## References

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## Appendix

### Proof to Theorem 1:

*Proof.*

$$\begin{aligned}
& \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{1}{n} \sum_{i=1}^n f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) - f_{i,t}(\mathbf{x}_t^*; \xi_{i,t}) \\
& \leq \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{1}{n} \sum_{i=1}^n \langle \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}), \mathbf{x}_{i,t} - \mathbf{x}_t^* \rangle \\
& = \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{1}{n} \sum_{i=1}^n \beta \langle \partial g_{i,t}(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \mathbf{x}_t^* \rangle + (1 - \beta) \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{1}{n} \sum_{i=1}^n \langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \mathbf{x}_{i,t} - \mathbf{x}_t^* \rangle \\
& = \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{1}{n} \sum_{i=1}^n \beta (\langle \partial g_{i,t}(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle + \langle \partial g_{i,t}(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \rangle + \langle \partial g_{i,t}(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_{t+1} - \mathbf{x}_t^* \rangle) \\
& \quad + \frac{1}{n} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \sum_{i=1}^n (1 - \beta) (\langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle + \langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \rangle) \\
& \quad + \frac{1}{n} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \sum_{i=1}^n (1 - \beta) (\langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \bar{\mathbf{x}}_{t+1} - \mathbf{x}_t^* \rangle) \\
& = \underbrace{\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{1}{n} \sum_{i=1}^n \beta (\langle \partial g_{i,t}(\mathbf{x}_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle + \langle \partial g_{i,t}(\mathbf{x}_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \rangle)}_{I_1(t)} \\
& \quad + \underbrace{\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{1}{n} \sum_{i=1}^n (1 - \beta) (\langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle + \langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \rangle)}_{I_2(t)} \\
& \quad + \underbrace{\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\langle \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}), \bar{\mathbf{x}}_{t+1} - \mathbf{x}_t^* \right\rangle}_{I_3(t)}
\end{aligned}$$

Now, we begin to bound  $I_1(t)$ .

$$\begin{aligned}
I_1(t) & \stackrel{\textcircled{1}}{\leq} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{\beta}{n} \sum_{i=1}^n \left( \frac{\eta}{2} \|\partial g_{i,t}(\mathbf{x}_{i,t})\|^2 + \frac{1}{2\eta} \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \frac{\eta}{2} \|\partial g_{i,t}(\mathbf{x}_{i,t})\|^2 + \frac{1}{2\eta} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \right) \\
& \leq \beta G \eta + \frac{\beta}{2n\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \frac{\beta}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2.
\end{aligned}$$

① holds due to  $\langle \mathbf{a}, \mathbf{b} \rangle \leq \frac{\eta}{2} \|\mathbf{a}\|^2 + \frac{1}{2\eta} \|\mathbf{b}\|^2$  holds for any  $\eta > 0$ .

Now, we begin to bound  $I_2(t)$ .

$$I_2(t) = (1 - \beta) \left( \underbrace{\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{1}{n} \sum_{i=1}^n \langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle}_{J_1(t)} + \underbrace{\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\langle \frac{1}{n} \sum_{i=1}^n \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\rangle}_{J_2(t)} \right).$$

For  $J_1(t)$ , we have

$$\begin{aligned}
J_1(t) &= \frac{1}{n} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \sum_{i=1}^n \langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle \\
&= \frac{1}{n} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \sum_{i=1}^n \langle \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}) - \nabla H_t(\bar{\mathbf{x}}_t), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle + \frac{1}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \langle \nabla H_t(\bar{\mathbf{x}}_t), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle \\
&= \frac{1}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \langle \nabla H_t(\mathbf{x}_{i,t}) - \nabla H_t(\bar{\mathbf{x}}_t), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle + \frac{1}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \langle \nabla H_t(\bar{\mathbf{x}}_t), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle \\
&\stackrel{\textcircled{1}}{\leq} \frac{L}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \frac{1}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \langle \nabla H_t(\bar{\mathbf{x}}_t), \mathbf{x}_{i,t} - \bar{\mathbf{x}}_t \rangle \\
&\stackrel{\textcircled{2}}{\leq} \frac{L}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \frac{1}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \left( \frac{\eta}{2\nu} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + \frac{\nu}{2\eta} \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \right) \\
&\leq \frac{L}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \frac{\eta}{2\nu} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + \frac{\nu}{2\eta n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2. \quad (3)
\end{aligned}$$

① holds due to  $H_t$  has  $L$ -Lipschitz gradients. ② holds because that  $\langle \mathbf{a}, \mathbf{b} \rangle \leq \frac{\nu}{2} \|\mathbf{a}\|^2 + \frac{1}{2\nu} \|\mathbf{b}\|^2$  holds for any  $\nu > 0$ .

For  $J_2(t)$ , we have

$$\begin{aligned}
J_2(t) &= \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\langle \frac{1}{n} \sum_{i=1}^n \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}), \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1} \right\rangle \\
&\leq \frac{\eta}{2} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\| \frac{1}{n} \sum_{i=1}^n \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}) \right\|^2 + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \\
&\leq \frac{\eta}{2} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\| \frac{1}{n} \sum_{i=1}^n (\nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}) - \nabla H_t(\mathbf{x}_{i,t}) + \nabla H_t(\mathbf{x}_{i,t})) \right\|^2 + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \\
&\leq \eta \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\| \frac{1}{n} \sum_{i=1}^n (\nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}) - \nabla H_t(\mathbf{x}_{i,t})) \right\|^2 + \eta \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left\| \frac{1}{n} \sum_{i=1}^n \nabla H_t(\mathbf{x}_{i,t}) \right\|^2 \\
&\quad + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \\
&\stackrel{\textcircled{1}}{\leq} \frac{\eta}{n} \sigma^2 + \eta \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left\| \frac{1}{n} \sum_{i=1}^n (\nabla H_t(\mathbf{x}_{i,t}) - \nabla H_t(\bar{\mathbf{x}}_t) + \nabla H_t(\bar{\mathbf{x}}_t)) \right\|^2 + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \\
&\leq \frac{\eta}{n} \sigma^2 + 2\eta \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left\| \frac{1}{n} \sum_{i=1}^n (\nabla H_t(\mathbf{x}_{i,t}) - \nabla H_t(\bar{\mathbf{x}}_t)) \right\|^2 \\
&\quad + 2\eta \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \\
&\leq \frac{\eta}{n} \sigma^2 + \frac{2\eta}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\nabla H_t(\mathbf{x}_{i,t}) - \nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\
&\quad + 2\eta \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2
\end{aligned}$$

$$\stackrel{\textcircled{2}}{\leq} \frac{\eta}{n} \sigma^2 + \frac{2\eta L^2}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + 2\eta \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2.$$

① holds due to

$$\begin{aligned} & \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\| \frac{1}{n} \sum_{i=1}^n (\nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}) - \nabla H_t(\mathbf{x}_{i,t})) \right\|^2 \\ &= \frac{1}{n^2} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left( \sum_{i=1}^n \mathbb{E}_{\xi_{i,t} \sim D_{i,t}} \|\nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}) - \nabla H_t(\mathbf{x}_{i,t})\|^2 \right) \\ & \quad + \frac{1}{n^2} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left( 2 \sum_{i=1}^n \sum_{j=1, j \neq i}^n \left\langle \mathbb{E}_{\xi_{i,t} \sim D_{i,t}} \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}) - \nabla H_t(\mathbf{x}_{i,t}), \mathbb{E}_{\xi_{j,t} \sim D_{j,t}} \nabla h_t(\mathbf{x}_{j,t}; \xi_{j,t}) - \nabla H_t(\mathbf{x}_{j,t}) \right\rangle \right) \\ &= \frac{1}{n^2} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \mathbb{E}_{\xi_{i,t} \sim D_{i,t}} \|\nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t}) - \nabla H_t(\mathbf{x}_{i,t})\|^2 + 0 \\ &\leq \frac{1}{n} \sigma^2. \end{aligned}$$

② holds due to  $H_t$  has  $L$  Lipschitz gradients.

Therefore, we obtain

$$\begin{aligned} & I_2(t) \\ &= (1 - \beta)(J_1(t) + J_2(t)) \\ &= (1 - \beta) \left( \frac{L}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \frac{\eta}{2\nu} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + \frac{\nu}{2\eta n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \right) \\ & \quad + (1 - \beta) \left( \frac{\eta}{n} \sigma^2 + \frac{2\eta L^2}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \right) \\ & \quad + (1 - \beta) \left( 2\eta \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \right) \\ &\leq (1 - \beta) \left( \frac{L}{n} + \frac{\nu}{2n\eta} + \frac{2\eta L^2}{n} \right) \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \left( \frac{\eta}{2\nu} + 2\eta \right) (1 - \beta) \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\ & \quad + \frac{\eta(1 - \beta)\sigma^2}{n} + \frac{1 - \beta}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2. \end{aligned}$$

Combine those bounds of  $I_1(t)$  and  $I_2(t)$ . We thus have

$$\begin{aligned} & I_1(t) + I_2(t) \\ &\leq \beta G\eta + \frac{\beta}{2n\eta} \sum_{i=1}^n \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \frac{\beta}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \\ & \quad + (1 - \beta) \left( \frac{L}{n} + \frac{\nu}{2n\eta} + \frac{2\eta L^2}{n} \right) \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 + \left( \frac{\eta}{2\nu} + 2\eta \right) (1 - \beta) \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\ & \quad + \frac{\eta(1 - \beta)\sigma^2}{n} + \frac{1 - \beta}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \\ &= \eta \left( \beta G + \frac{(1 - \beta)\sigma^2}{n} \right) + (1 - \beta) \left( \frac{\beta}{2n\eta} + \frac{L}{n} + \frac{\nu}{2n\eta} + \frac{2\eta L^2}{n} \right) \sum_{i=1}^n \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \end{aligned}$$

$$+ \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 + \left( \frac{\eta}{2\nu} + 2\eta \right) (1 - \beta) \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2.$$

Therefore, we have

$$\begin{aligned} & \sum_{t=1}^T (I_1(t) + I_2(t)) \\ & \leq \eta T \left( \beta G + \frac{(1 - \beta)\sigma^2}{n} \right) + (1 - \beta) \left( \frac{\beta}{2n\eta} + \frac{L}{n} + \frac{\nu}{2n\eta} + \frac{2\eta L^2}{n} \right) \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{i=1}^n \sum_{t=1}^T \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \\ & \quad + \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 + \left( \frac{\eta}{2\nu} + 2\eta \right) (1 - \beta) \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{t=1}^T \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2. \end{aligned}$$

Now, we begin to bound  $I_3(t)$ . Recall that the update rule is

$$\mathbf{x}_{i,t+1} = \sum_{j=1}^n \mathbf{W}_{ij} \mathbf{x}_{j,t} - \eta \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}).$$

According to Lemma 3, we have

$$\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{x}}_t - \eta \left( \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right). \quad (4)$$

Denote a new auxiliary function  $\phi(\mathbf{z})$  as

$$\phi(\mathbf{z}) = \left\langle \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}), \mathbf{z} \right\rangle + \frac{1}{2\eta} \|\mathbf{z} - \bar{\mathbf{x}}_t\|^2.$$

It is trivial to verify that (4) satisfies the first-order optimality condition of the optimization problem:  $\min_{\mathbf{z} \in \mathbb{R}^d} \phi(\mathbf{z})$ , that is,

$$\nabla \phi(\bar{\mathbf{x}}_{t+1}) = \mathbf{0}.$$

We thus have

$$\begin{aligned} \bar{\mathbf{x}}_{t+1} &= \operatorname{argmin}_{\mathbf{z} \in \mathbb{R}^d} \phi(\mathbf{z}) \\ &= \operatorname{argmin}_{\mathbf{z} \in \mathbb{R}^d} \left\langle \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}), \mathbf{z} \right\rangle + \frac{1}{2\eta} \|\mathbf{z} - \bar{\mathbf{x}}_t\|^2. \end{aligned}$$

Furthermore, denote a new auxiliary variable  $\bar{\mathbf{x}}_\tau$  as

$$\bar{\mathbf{x}}_\tau = \bar{\mathbf{x}}_{t+1} + \tau (\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}),$$

where  $0 < \tau \leq 1$ . According to the optimality of  $\bar{\mathbf{x}}_{t+1}$ , we have

$$\begin{aligned} 0 &\leq \phi(\bar{\mathbf{x}}_\tau) - \phi(\bar{\mathbf{x}}_{t+1}) \\ &= \left\langle \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}), \bar{\mathbf{x}}_\tau - \bar{\mathbf{x}}_{t+1} \right\rangle + \frac{1}{2\eta} (\|\bar{\mathbf{x}}_\tau - \bar{\mathbf{x}}_t\|^2 - \|\bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t\|^2) \\ &= \left\langle \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}), \tau (\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}) \right\rangle + \frac{1}{2\eta} (\|\bar{\mathbf{x}}_{t+1} + \tau (\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}) - \bar{\mathbf{x}}_t\|^2 - \|\bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t\|^2) \end{aligned}$$



$$= \left\langle \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}), \tau (\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}) \right\rangle + \frac{1}{2\eta} \left( \|\tau (\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1})\|^2 + 2 \langle \tau (\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}), \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t \rangle \right).$$

Note that the above inequality holds for any  $0 < \tau \leq 1$ . Divide  $\tau$  on both sides, and we have

$$\begin{aligned} I_3(t) &= \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\langle \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}), \bar{\mathbf{x}}_{t+1} - \mathbf{x}_t^* \right\rangle \\ &\leq \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left( \lim_{\tau \rightarrow 0^+} \tau \|\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}\|^2 + 2 \langle \mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}, \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t \rangle \right) \\ &= \frac{1}{\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \langle \mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}, \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t \rangle \\ &= \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left( \|\mathbf{x}_t^* - \bar{\mathbf{x}}_t\|^2 - \|\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}\|^2 - \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \right). \end{aligned} \quad (5)$$

Besides, we have

$$\begin{aligned} &\|\mathbf{x}_{t+1}^* - \bar{\mathbf{x}}_{t+1}\|^2 - \|\mathbf{x}_t^* - \bar{\mathbf{x}}_{t+1}\|^2 \\ &= \|\mathbf{x}_{t+1}^*\|^2 - \|\mathbf{x}_t^*\|^2 - 2 \langle \bar{\mathbf{x}}_{t+1}, -\mathbf{x}_t^* + \mathbf{x}_{t+1}^* \rangle \\ &= (\|\mathbf{x}_{t+1}^*\| - \|\mathbf{x}_t^*\|) (\|\mathbf{x}_{t+1}^*\| + \|\mathbf{x}_t^*\|) - 2 \langle \bar{\mathbf{x}}_{t+1}, -\mathbf{x}_t^* + \mathbf{x}_{t+1}^* \rangle \\ &\leq \|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\| (\|\mathbf{x}_{t+1}^*\| + \|\mathbf{x}_t^*\|) + 2 \|\bar{\mathbf{x}}_{t+1}\| \|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\| \\ &\leq 4\sqrt{R} \|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\|. \end{aligned}$$

The last inequality holds due to our assumption, that is,  $\|\mathbf{x}_{t+1}^*\| = \|\mathbf{x}_{t+1}^* - \mathbf{0}\| \leq \sqrt{R}$ ,  $\|\mathbf{x}_t^*\| = \|\mathbf{x}_t^* - \mathbf{0}\| \leq \sqrt{R}$ , and  $\|\bar{\mathbf{x}}_{t+1}\| = \|\bar{\mathbf{x}}_{t+1} - \mathbf{0}\| \leq \sqrt{R}$ .

Thus, telescoping  $I_3(t)$  over  $t \in [T]$ , we have

$$\begin{aligned} &\sum_{t=1}^T I_3(t) \\ &\leq \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \left( 4\sqrt{R} \sum_{t=1}^T \|\mathbf{x}_{t+1}^* - \mathbf{x}_t^*\| + \|\bar{\mathbf{x}}_1^* - \bar{\mathbf{x}}_1\|^2 - \|\bar{\mathbf{x}}_T^* - \bar{\mathbf{x}}_{T+1}\|^2 \right) - \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2 \\ &\leq \frac{1}{2\eta} (4\sqrt{R}M + R) - \frac{1}{2\eta} \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t+1}\|^2. \end{aligned}$$

Here,  $M$  the budget of the dynamics, which is defined in (2).

Combining those bounds of  $I_1(t)$ ,  $I_2(t)$  and  $I_3(t)$  together, we finally obtain

$$\begin{aligned} &\mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \sum_{i=1}^n f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) - f_t(\mathbf{x}_t^*; \xi_{i,t}) \\ &\leq n \sum_{t=1}^T (I_1(t) + I_2(t) + I_3(t)) \\ &\leq \eta T (n\beta G + (1-\beta)\sigma^2) + (1-\beta) \left( \frac{\beta}{2\eta} + L + \frac{\nu}{2\eta} + 2\eta L^2 \right) \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{i=1}^n \sum_{t=1}^T \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \\ &\quad + n \left( \frac{\eta}{2\nu} + 2\eta \right) (1-\beta) \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{t=1}^T \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + \frac{n}{2\eta} (4\sqrt{R}M + R) \end{aligned}$$

$$\begin{aligned}
& \stackrel{\textcircled{1}}{\leq} \eta T (n\beta G + (1-\beta)\sigma^2) + n(1-\beta) \left( \frac{1}{\nu} + 4 \right) \left( \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T (H_t(\bar{\mathbf{x}}_t) - H_t(\bar{\mathbf{x}}_{t+1})) \right) \\
& \quad + (1-\beta) \left( \frac{\beta}{2\eta} + L + \frac{\nu}{2\eta} + 2\eta L^2 + \left( \frac{1}{\nu} + 4 \right) (1-\beta)^2 L^2 \eta \right) \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 \\
& \quad + n(1-\beta) \left( \frac{1}{\nu} + 4 \right) \left( 4T\beta^2 \eta G + \frac{TGL\eta^2}{2} \right) + \frac{n}{2\eta} (4\sqrt{R}M + R) \\
& \stackrel{\textcircled{2}}{\leq} \eta T (n\beta G + (1-\beta)\sigma^2) + n(1-\beta) \left( \frac{1}{\nu} + 4 \right) \left( \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T (H_t(\bar{\mathbf{x}}_t) - H_t(\bar{\mathbf{x}}_{t+1})) \right) \\
& \quad + (1-\beta) \left( \frac{\beta}{2\eta} + L + \frac{\nu}{2\eta} + 2\eta L^2 + \left( \frac{1}{\nu} + 4 \right) (1-\beta)^2 L^2 \eta \right) \frac{nT\eta^2 G}{(1-\rho)^2} \\
& \quad + n(1-\beta) \left( \frac{1}{\nu} + 4 \right) \left( 4T\beta^2 \eta G + \frac{TGL\eta^2}{2} \right) + \frac{n}{2\eta} (4\sqrt{R}M + R).
\end{aligned}$$

① holds due to Lemma 2. That is, we have

$$\begin{aligned}
& \frac{\eta}{2} \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{t=1}^T \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\
& \leq \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T (H_t(\bar{\mathbf{x}}_t) - H_t(\bar{\mathbf{x}}_{t+1})) + 4T\beta^2 \eta G + \frac{(1-\beta)^2 L^2 \eta}{n} \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{t=1}^T \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 + \frac{TGL\eta^2}{2}.
\end{aligned} \tag{6}$$

② holds due to Lemma 4

$$\mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{i=1}^n \sum_{t=1}^T \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \leq \frac{nT\eta^2 G}{(1-\rho)^2}.$$

Letting  $\nu = \sqrt{\beta^2 + \eta}$ , we have

$$\begin{aligned}
& \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \sum_{i=1}^n f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) - f_t(\mathbf{x}_t^*; \xi_{i,t}) \\
& \leq \eta T (n\beta G + (1-\beta)\sigma^2) + n(1-\beta) \left( \frac{1}{\sqrt{\beta^2 + \eta}} + 4 \right) \left( \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T (H_t(\bar{\mathbf{x}}_t) - H_t(\bar{\mathbf{x}}_{t+1})) \right) \\
& \quad + (1-\beta) \left( \frac{\beta}{2\eta} + L + \frac{\sqrt{\beta^2 + \eta}}{2\eta} + 2\eta L^2 + \left( \frac{1}{\sqrt{\beta^2 + \eta}} + 4 \right) (1-\beta)^2 L^2 \eta \right) \frac{nT\eta^2 G}{(1-\rho)^2} \\
& \quad + n(1-\beta) \left( \frac{1}{\sqrt{\beta^2 + \eta}} + 4 \right) \left( 4T\beta^2 \eta G + \frac{TGL\eta^2}{2} \right) + \frac{n}{2\eta} (4\sqrt{R}M + R).
\end{aligned}$$

It completes the proof. □

**Lemma 1.** Using Assumption 1, we have

$$\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})\|^2 \leq G.$$

*Proof.*

$$\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})\|^2$$

$$\begin{aligned}
&= \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\beta \partial g_{i,t}(\mathbf{x}_{i,t}) + (1-\beta) \nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t})\|^2 \\
&\leq \beta \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\partial g_{i,t}(\mathbf{x}_{i,t})\|^2 + (1-\beta) \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\nabla h_t(\mathbf{x}_{i,t}; \xi_{i,t})\|^2 \\
&\leq G.
\end{aligned}$$

It completes the proof.  $\square$

**Lemma 2.** Using Assumption 1, and setting  $\eta > 0$  in Algorithm 1, we have

$$\begin{aligned}
&\frac{\eta}{2} \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{t=1}^T \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\
&\leq \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T (H_t(\bar{\mathbf{x}}_t) - H_t(\bar{\mathbf{x}}_{t+1})) + 4T\beta^2\eta G + \frac{(1-\beta)^2 L^2 \eta}{n} \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{t=1}^T \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 + \frac{TGL\eta^2}{2}.
\end{aligned} \tag{7}$$

*Proof.*

$$\begin{aligned}
&\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} H_t(\bar{\mathbf{x}}_{t+1}) \\
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} H_t(\bar{\mathbf{x}}_t) + \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \langle \nabla H_t(\bar{\mathbf{x}}_t), \bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t \rangle + \frac{L}{2} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\bar{\mathbf{x}}_{t+1} - \bar{\mathbf{x}}_t\|^2 \\
&= \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} H_t(\bar{\mathbf{x}}_t) + \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\langle \nabla H_t(\bar{\mathbf{x}}_t), -\frac{\eta}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right\rangle + \frac{L}{2} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\| \frac{\eta}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right\|^2 \\
&= \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} H_t(\bar{\mathbf{x}}_t) + \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left\langle \nabla H_t(\bar{\mathbf{x}}_t), -\frac{\eta}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}) \right\rangle + \frac{L}{2} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\| \frac{\eta}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right\|^2.
\end{aligned} \tag{8}$$

Besides, we have

$$\begin{aligned}
&\mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left\langle \nabla H_t(\bar{\mathbf{x}}_t), -\frac{\eta}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}) \right\rangle \\
&= \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \left( \left\| \nabla H_t(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 - \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 - \left\| \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}) \right\|^2 \right) \\
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \left( \left\| \nabla H_t(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n (\beta \partial g_{i,t}(\mathbf{x}_{i,t}) + (1-\beta) \nabla h_t(\mathbf{x}_{i,t})) \right\|^2 \right) - \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \left( 2\beta^2 \left\| \nabla H_t(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \partial g_{i,t}(\mathbf{x}_{i,t}) \right\|^2 + 2(1-\beta)^2 \left\| \nabla H_t(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \nabla h_t(\mathbf{x}_{i,t}) \right\|^2 \right) \\
&\quad - \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \left( 2\beta^2 \left\| \nabla H_t(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \partial g_{i,t}(\mathbf{x}_{i,t}) \right\|^2 + \frac{2(1-\beta)^2}{n} \sum_{i=1}^n \|\nabla H_t(\bar{\mathbf{x}}_t) - \nabla h_t(\mathbf{x}_{i,t})\|^2 \right) \\
&\quad - \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \left( 2\beta^2 \left\| \nabla H_t(\bar{\mathbf{x}}_t) - \frac{1}{n} \sum_{i=1}^n \partial g_{i,t}(\mathbf{x}_{i,t}) \right\|^2 + \frac{2(1-\beta)^2 L^2}{n} \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 \right) - \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \left( 4\beta^2 \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 + 4\beta^2 \left\| \frac{1}{n} \sum_{i=1}^n \partial g_{i,t}(\mathbf{x}_{i,t}) \right\|^2 + \frac{2(1-\beta)^2 L^2}{n} \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 \right) \\
&\quad - \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\
&\stackrel{\textcircled{1}}{\leq} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \left( 8\beta^2 G + \frac{2(1-\beta)^2 L^2}{n} \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 \right) - \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \frac{\eta}{2} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2. \tag{9}
\end{aligned}$$

① holds due to

$$\begin{aligned}
\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 &= \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \\
&= \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left\| \mathbb{E}_{\xi_{i,t} \sim D_{i,t}} \nabla h_t(\bar{\mathbf{x}}_t; \xi_{i,t}) \right\|^2 \\
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left( \mathbb{E}_{\xi_{i,t} \sim D_{i,t}} \|\nabla h_t(\bar{\mathbf{x}}_t; \xi_{i,t})\|^2 \right), \quad \forall i \in [n] \\
&\leq G,
\end{aligned}$$

and

$$\mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left\| \frac{1}{n} \sum_{i=1}^n \partial g_{i,t}(\mathbf{x}_{i,t}) \right\|^2 \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \|\partial g_{i,t}(\mathbf{x}_{i,t})\|^2 \leq G.$$

According to Lemma 1, we have

$$\mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \|\partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})\|^2 \leq G. \tag{10}$$

Substituting (9) and (10) into (8), and telescoping  $t \in [T]$ , we obtain

$$\begin{aligned}
&\mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T H_t(\bar{\mathbf{x}}_{t+1}) \\
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} H_t(\bar{\mathbf{x}}_t) + \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \left\langle \nabla H_t(\bar{\mathbf{x}}_t), -\frac{\eta}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}) \right\rangle + \frac{L}{2} \mathbb{E}_{\Xi_{n,t} \sim \mathcal{D}_{n,t}} \left\| \frac{\eta}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right\|^2 \\
&\leq \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} H_t(\bar{\mathbf{x}}_t) + \left( \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \left( 8\beta^2 G + \frac{2(1-\beta)^2 L^2}{n} \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 \right) - \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \right) + \frac{GL\eta^2}{2} \\
&= \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} H_t(\bar{\mathbf{x}}_t) + \left( 4\eta\beta^2 G + \frac{(1-\beta)^2 L^2 \eta}{n} \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 - \mathbb{E}_{\Xi_{n,t-1} \sim \mathcal{D}_{n,t-1}} \frac{\eta}{2} \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \right) + \frac{GL\eta^2}{2}.
\end{aligned}$$

Telescoping over  $t \in [T]$ , we have

$$\begin{aligned}
&\frac{\eta}{2} \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{t=1}^T \|\nabla H_t(\bar{\mathbf{x}}_t)\|^2 \tag{11} \\
&\leq \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T (H_t(\bar{\mathbf{x}}_t) - H_t(\bar{\mathbf{x}}_{t+1})) + 4T\beta^2 \eta G + \frac{(1-\beta)^2 L^2 \eta}{n} \mathbb{E}_{\Xi_{n,T-1} \sim \mathcal{D}_{n,T-1}} \sum_{t=1}^T \sum_{i=1}^n \|\bar{\mathbf{x}}_t - \mathbf{x}_{i,t}\|^2 + \frac{TGL\eta^2}{2}.
\end{aligned}$$

It completes the proof.  $\square$

**Lemma 3.** Denote  $\bar{\mathbf{x}}_t = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_{i,t}$ . We have

$$\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{x}}_t - \eta \left( \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right).$$

*Proof.* Denote

$$\begin{aligned} \mathbf{X}_t &= [\mathbf{x}_{1,t}, \mathbf{x}_{2,t}, \dots, \mathbf{x}_{n,t}] \in \mathbb{R}^{d \times n}, \\ \mathbf{G}_t &= [\nabla f_{1,t}(\mathbf{x}_{1,t}; \xi_{1,t}), \nabla f_{2,t}(\mathbf{x}_{2,t}; \xi_{2,t}), \dots, \nabla f_{n,t}(\mathbf{x}_{n,t}; \xi_{n,t})] \in \mathbb{R}^{d \times n}. \end{aligned}$$

Recall that

$$\mathbf{x}_{i,t+1} = \sum_{j=1}^n \mathbf{W}_{ij} \mathbf{x}_{j,t} - \eta \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}).$$

Equivalently, we re-formulate the update rule as

$$\mathbf{X}_{t+1} = \mathbf{X}_t \mathbf{W} - \eta \mathbf{G}_t.$$

Since the confusion matrix  $\mathbf{W}$  is doubly stochastic, we have

$$\mathbf{W} \mathbf{1} = \mathbf{1}.$$

Thus, we have

$$\begin{aligned} \bar{\mathbf{x}}_{t+1} &= \frac{1}{n} \sum_{i=1}^n \mathbf{x}_{i,t+1} \\ &= \mathbf{X}_{t+1} \frac{\mathbf{1}}{n} \\ &= \mathbf{X}_t \mathbf{W} \frac{\mathbf{1}}{n} - \eta \mathbf{G}_t \frac{\mathbf{1}}{n} \\ &= \mathbf{X}_t \frac{\mathbf{1}}{n} - \eta \mathbf{G}_t \frac{\mathbf{1}}{n} \\ &= \bar{\mathbf{x}}_t - \eta \left( \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right). \end{aligned}$$

It completes the proof. □

**Lemma 4.** Using Assumption 1, and setting  $\eta > 0$  in Algorithm 1, we have

$$\mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{i=1}^n \sum_{t=1}^T \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \leq \frac{nT\eta^2 G}{(1-\rho)^2}.$$

*Proof.* Recall that

$$\mathbf{x}_{i,t+1} = \sum_{j=1}^n \mathbf{W}_{ij} \mathbf{x}_{j,t} - \eta \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}),$$

and according to Lemma 3, we have

$$\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{x}}_t - \eta \left( \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right).$$

Denote

$$\begin{aligned}\mathbf{X}_t &= [\mathbf{x}_{1,t}, \mathbf{x}_{2,t}, \dots, \mathbf{x}_{n,t}] \in \mathbb{R}^{d \times n}, \\ \mathbf{G}_t &= [\partial f_{1,t}(\mathbf{x}_{1,t}; \xi_{1,t}), \partial f_{2,t}(\mathbf{x}_{2,t}; \xi_{2,t}), \dots, \partial f_{n,t}(\mathbf{x}_{n,t}; \xi_{n,t})] \in \mathbb{R}^{d \times n}.\end{aligned}$$

By letting  $\mathbf{x}_{i,1} = \mathbf{0}$  for any  $i \in [n]$ , the update rule is re-formulated as

$$\mathbf{X}_{t+1} = \mathbf{X}_t \mathbf{W} - \eta \mathbf{G}_t = - \sum_{s=1}^t \eta \mathbf{G}_s \mathbf{W}^{t-s}.$$

Similarly, denote  $\bar{\mathbf{G}}_t = \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$ , and we have

$$\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{x}}_t - \eta \left( \frac{1}{n} \sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) \right) = - \sum_{s=1}^t \eta \bar{\mathbf{G}}_s. \quad (12)$$

Therefore,

$$\begin{aligned}& \sum_{i=1}^n \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \\ & \stackrel{\textcircled{1}}{=} \sum_{i=1}^n \left\| \sum_{s=1}^{t-1} \eta \bar{\mathbf{G}}_s - \eta \mathbf{G}_s \mathbf{W}^{t-s-1} \mathbf{e}_i \right\|^2 \\ & \stackrel{\textcircled{2}}{=} \left\| \sum_{s=1}^{t-1} \eta \mathbf{G}_s \mathbf{v}_1 \mathbf{v}_1^T - \eta \mathbf{G}_s \mathbf{W}^{t-s-1} \right\|_F^2 \\ & \stackrel{\textcircled{3}}{\leq} \left( \eta \rho^{t-s-1} \left\| \sum_{s=1}^{t-1} \mathbf{G}_s \right\|_F \right)^2 \\ & \leq \left( \sum_{s=1}^{t-1} \eta \rho^{t-s-1} \|\mathbf{G}_s\|_F \right)^2.\end{aligned}$$

① holds due to  $\mathbf{e}_i$  is a unit basis vector, whose  $i$ -th element is 1 and other elements are 0s. ② holds due to  $\mathbf{v}_1 = \frac{1}{\sqrt{n}}$ . ③ holds due to Lemma 5.

Thus, we have

$$\begin{aligned}& \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{i=1}^n \sum_{t=1}^T \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2 \\ & \leq \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \left( \sum_{s=1}^{t-1} \eta \rho^{t-s-1} \|\mathbf{G}_s\|_F \right)^2 \\ & \stackrel{\textcircled{1}}{\leq} \frac{\eta^2}{(1-\rho)^2} \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \left( \sum_{t=1}^T \|\mathbf{G}_t\|_F^2 \right) \\ & = \frac{\eta^2}{(1-\rho)^2} \left( \mathbb{E}_{\Xi_{n,T} \sim \mathcal{D}_{n,T}} \sum_{t=1}^T \sum_{i=1}^n \|\partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})\|^2 \right) \\ & \stackrel{\textcircled{2}}{=} \frac{nT\eta^2 G}{(1-\rho)^2}.\end{aligned}$$

① holds due to Lemma 6. ② holds due to Lemma 1.

□

**Lemma 5** (Appeared in Lemma 5 in [Tang et al., 2018]). For any matrix  $\mathbf{X}_t \in \mathbb{R}^{d \times n}$ , decompose the confusion matrix  $\mathbf{W}$  as  $\mathbf{W} = \sum_{i=1}^n \lambda_i \mathbf{v}_i \mathbf{v}_i^T = \mathbf{P} \mathbf{\Lambda} \mathbf{P}^T$ , where  $\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n] \in \mathbb{R}^{n \times n}$ ,  $\mathbf{v}_i$  is the normalized eigenvector of  $\lambda_i$ .  $\mathbf{\Lambda}$  is a diagonal matrix, and  $\lambda_i$  be its  $i$ -th element. We have

$$\|\mathbf{X}_t \mathbf{W}^t - \mathbf{X}_t \mathbf{v}_1 \mathbf{v}_1^T\|_F^2 \leq \|\rho^t \mathbf{X}_t\|_F^2,$$

where  $\rho = \max\{|\lambda_2(\mathbf{W})|, |\lambda_n(\mathbf{W})|\}$ .

**Lemma 6** (Appeared in Lemma 6 in [Tang et al., 2018]). Given two non-negative sequences  $\{a_t\}_{t=1}^\infty$  and  $\{b_t\}_{t=1}^\infty$  that satisfying

$$a_t = \sum_{s=1}^t \rho^{t-s} b_s,$$

with  $\rho \in [0, 1)$ , we have

$$\sum_{t=1}^k a_t^2 \leq \frac{1}{(1-\rho)^2} \sum_{s=1}^k b_s^2.$$