Decentralized Online Learning: Exchanging Local Models to Track Dynamics

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Abstract

In this paper, we consider online learning in the decentralized setting up, which is motivated by the scenario where users want to take benefits from the data from other users, but do not want to share their private data to a third party or other users. Instead, they can only share their private prediction model, e.g., recommendation model. We study the decentralized online gradient method in which each user maintains a private model and share its private model with its neighbors (or users he/she trusts) periodically. In addition, to consider more practical scenario we allow users' interest changing over time, unlike most online work which assumes that the optimal prediction model is constant. We prove that decentralized online gradient can efficiently and effectively propagate the values in all private data without leaking them to track the dynamics, by admitting a tight dynamic regret $\mathcal{O}\left(n\sqrt{TM}+\sqrt{nTM}\sigma\right)$ where n is the number of users, T is the number of time steps, M measures the dynamics (this is, how much the users' interest changes over time), and σ measures the randomness of the private data. Empirical studies are also conducted to validate our analysis.

1. Introduction

For any online algorithm $A \in \mathcal{A}$, the previous dynamic regret $\widetilde{\mathcal{R}}_T^A$ is defined by

$$\widetilde{\mathcal{R}}_T^A = \sum_{i=1}^{n}\sum_{t=1}^{T}\left(g_{i,t}(\mathbf{x}_{i,t}) - g_{i,t}(\mathbf{x}_t^*)\right), \tag{1}$$

Notations and definitions In the paper, we make the following notations.

- For any $i \in [n]$ and $t \in [T]$, the random variable $\xi_{i,t}$ is subject to a distribution D_t, that is, $\xi_{i,t} \sim D_t$. Besides, a set of random variables $\Xi_{n,T}$ and the corresponding set of distributions are defined by

$$\Xi_{n,T} = \{\xi_{i,t}\}_{1\le i\le n, 1\le t\le T}, \text{ and } \mathcal{D}_T = \{D_t\}_{1\le t\le T},$$

  respectively. For math brevity, we use the notation $\Xi_{n,T} \sim \mathcal{D}_T$ to represent that $\xi_{i,t} \sim D_t$ holds for any $i \in [n]$ and $t \in [T]$. $\mathbb{E}$ represents mathematical expectation.

- For a decentralized network, we use $\mathbf{W} \in \mathbb{R}^{n\times n}$ to represent its confusion matrix. It is a symmetric doublely stochastic matrix, which implies that every element of $\mathbf{W}$ is non-negative, $\mathbf{W}\mathbf{1} = \mathbf{1}$, and $\mathbf{1}^{\mathrm{T}}\mathbf{W} = \mathbf{1}^{\mathrm{T}}$. We use $\{\lambda_i\}_{i=1}^n$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ to represent its eigenvalues.

- ∂ and ∇ represent sub-gradient and gradient operators, respectively. $\|\cdot\|$ represents the ℓ_2 norm in default.

- $\lesssim$ represents "less than equal up to a constant factor".

2. Related work

Online learning has been studied for decades of years. The static regret of a sequential online convex optimization method can achieve $\mathcal{O}\left(\sqrt{T}\right)$ and $\mathcal{O}\left(\log T\right)$ bounds for convex and strongly convex loss functions, respectively (Hazan, 2016; Shalev-Shwartz, 2012). Recently, both the decentralized online learing and the dynamic regret have drawn much attention due to their wide existence in the practical big data scenarios.

2.1. Decentralized online learning

Online learning in a decentralized network has been studied in (Shahrampour and Jadbabaie, 2018; Kamp et al., 2014; Koppel et al., 2018; Zhang et al., 2018a; 2017b; Xu et al., 2015; Akbari et al., 2017; Lee et al., 2016; Nedi et al., 2015; Lee et al., 2018; Benczúr et al., 2018; Yan et al., 2013). Shahrampour and Jadbabaie (2018) studies decentralized online mirror descent, and provides $\mathcal{O}\left(n\sqrt{nTM}\right)$ dynamic regret. Here, n, T, and M represent the number of nodes in the newtork, the number of iterations, and the budget of

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dynamics (defined in (3)), respectively. When the Bregman divergence in the decentralized online mirror descent is chosen appropriately, the decentralized online mirror descent becomes identical to the decentralized online gradient descent. Using the same definition of dynamic regret (defined in (1)), our method obtains $\mathcal{O}\left(n\sqrt{TM}\right)$ dynamic regret for a decentralized online gradient descent, which is better than $\mathcal{O}\left(n\sqrt{nTM}\right)$ in Shahrampour and Jadbabaie (2018). The improvement of our bound benefits from a better bound of network error (see Lemma 1). Kamp et al. (2014) studies decentralized online prediction, and presents $\mathcal{O}\left(\sqrt{nT}\right)$ static regret. It assumes that all data, used to yielded the loss, is generated from an unknown distribution. The strong assumption is not practical in the dynamic environment, and thus limits its novelity for a general online learning task. Additionally, many decentralized online optimization methods are proposed, for example, decentralized online multi-task learning (Zhang et al., 2018a), decentralized online ADMM (Xu et al., 2015), decentralized online sub-gradient descent (Akbari et al., 2017), decentralized continuous-time online saddle-point method (Lee et al., 2016), decentralized online Nesterov's primal-dual method (Nedi et al., 2015; Lee et al., 2018). Those previous methods are proved to yield $\mathcal{O}\left(\sqrt{T}\right)$ static regret, which do not have theoretical guarantee of regret in the dynamic environment. Besides, Yan et al. (2013) provides necessary and sufficient conditions to preserve privacy for decentralized online learning methods, which is interesting to extend our method to be privacy-preserving in the future work.

2.2. Regret in dynamic environment

Dynamic regret has been widely studied for decades of years (Zinkevich, 2003; Hall and Willett, 2015; 2013; Jadbabaie et al., 2015; Yang et al., 2016; Bedi et al., 2018; Zhang et al., 2017a; Mokhtari et al., 2016; Zhang et al., 2018b; György and Szepesvári, 2016; Wei et al., 2016; Zhao et al., 2018). Zinkevich (2003) first defines the dynamic regret by (1), and then proposes an online gradient descent method. The method yields $\mathcal{O}\left(\sqrt{TM}\right)$ by choosing an appropriate learning rate. The following researches achieve the sublinear dynamic regret, but extend the analysis of regret by using different reference points. For example, Hall and Willett (2015; 2013) choose the reference points $\{\mathbf{x}_t^*\}_{t=1}^T$ satisfying $\sum_{t=1}^{T-1}\left\|\mathbf{x}_{t+1}^*-\Phi(\mathbf{x}_t^*)\right\|\leq M$, where $\Phi(\mathbf{x}_t^*)$ is the predictive optimal decision variable. When the function Φ predicts accurately, a small M is enough to bound the dynamics. The dynamic regret is thus effectively decreased. Jadbabaie et al. (2015); Yang et al. (2016); Bedi et al. (2018); Zhang et al. (2017a); Mokhtari et al. (2016); Zhang et al. (2018b) chooses the reference points $\{\mathbf{y}_t^*\}_{t=1}^T$ with $\mathbf{y}_t^*=\operatorname{argmin}_{\mathbf{z}\in\mathcal{X}} f_t(\mathbf{z})$, where f_t is the loss function at the t-th iteration. György and Szepesvári (2016) provides a new analysis framework, which achieves $\mathcal{O}\left(\sqrt{TM}\right)$ dynamic regret[1] for any given reference points. Besides, Zhao et al. (2018) presents that the lower bound of the dynamic regret defined by 1 is $\Omega\left(\sqrt{TM}\right)$. The previous definition of the regret, i.e., (1), is a special case of our new definition. When setting $\beta=1$, we achieve the state-of-the-art regret, that is, $\mathcal{O}\left(\sqrt{TM}\right)$.

In some literatures, the regret in a dynamic environment is measured by the number of changes of a reference point over time. It is usually denoted by shifting regret or tracking regret (Herbster and Warmuth, 1998; György et al., 2005; Gyorgy et al., 2012; György and Szepesvári, 2016; Mourtada and Maillard, 2017; Adamskiy et al., 2016; Wei et al., 2016; Cesa-Bianchi et al., 2012; Mohri and Yang, 2018; Jun et al., 2017). Both the shifting regret and the tracking regret can be considered as a variation of the dynamic regret, and is usually studied in the setting of "learning with expert advice". But, the dynamic regret is usually studied in a general setting of online learning.

3. Problem formulation

For any online algorithm $A\in\mathcal{A}$, we define its dynamic regret $\mathcal{R}_T^A$ by

$$\mathcal{R}_T^A := \mathop{\mathbb{E}}_{\Xi_{n,T}\sim\mathcal{D}_T}\left(\sum_{i=1}^n\sum_{t=1}^T f_{i,t}(\mathbf{x}_{i,t};\zeta_{i,t},\xi_{i,t})\right) - \mathop{\mathbb{E}}_{\Xi_{n,T}\sim\mathcal{D}_T}\left(\sum_{i=1}^n\sum_{t=1}^T f_{i,t}(\mathbf{x}_t^*;\zeta_{i,t},\xi_{i,t})\right), \quad (2)$$

where n is the number of nodes in the decentralized network. $\{\mathbf{x}_t^*\}_{t=1}^T$ is the sequence of reference points. $\mathbf{x}_{i,t}$ is the decision variable played by an online algorithm A at the t-th round. The local loss function $f_{i,t}(\mathbf{x};\zeta_{i,t},\xi_{i,t})$ is defined by

$$f_{i,t}(\mathbf{x};\zeta_{i,t},\xi_{i,t}) := \beta g_{i,t}(\mathbf{x};\zeta_{i,t}) + (1-\beta)h_t(\mathbf{x};\xi_{i,t})$$

with $0<\beta<1$. $\zeta_{i,t}$ represents the adversary part of data. $\xi_{i,t}$ represents the stochastic part of data, which is drawn from the distribution D_t. Note that $g_{i,t}$ is an adversary loss function, which is caused by the adversary data. $h_t(\cdot;\xi_{i,t})$ is a stochastic loss function, which depends on the stochastic data $\xi_{i,t}$. The expectation is taken with respect to $\{\xi_{i,t}\}_{1\leq i\leq n, 1\leq t\leq T}$.

The sequence of reference points $\{\mathbf{x}_t^*\}_{t=1}^T$ satisfies

$$\{\mathbf{x}_t^*\}_{t=1}^T \in \left\{\{\mathbf{z}_t\}_{t=1}^T : \sum_{t=1}^{T-1}\|\mathbf{z}_t-\mathbf{z}_{t+1}\|\leq M\right\}.$$

[1] György and Szepesvári (2016) uses the notation of "shifting regret" instead of "dynamic regret". In the paper, we keep using "dynamic regret" as used in most previous literatures.

Here, M is the budget of the dynamics, that is,

$$\sum_{t=1}^{T-1} \left\| \mathbf{x}_{t+1}^* - \mathbf{x}_t^* \right\| \leq M. \tag{3}$$

When $M = 0$, all $\mathbf{x}_t^*$s are same, and it degenerates to the static online learning problem. When the dynamic environment changes significantly, M becomes large to model the dynamics. Let us take an example to explain the dynamics. Suppose we want to conduct online music recommendation task by using users' browsing records in Youtube. Every user has his/her own favorite music, and users' preference changes over time due to time-varying trends of hot topics in Internet. It leads to the dynamics of the optimal recommendation model.

Recall that the previous definition of the dynamic regret is (1). Using (1), the classic online learning in a decentralized network only considers the loss function, i.e., $g_{i,t}$, incurred by the adversary part of data on every node, ignoring the potential relation of data among different nodes. Comparing with it, our definition of the dynamic regret, i.e., (2), still considers the stochastic loss function, i.e., $h_t(\cdot; \xi_{i,t})$. It is incurred by the stochastic part of data. Since every node shares its private model to neighbours, the regret due to stochastic part of data would be decreased effectively, which is varified by the theoretical results in Section 4.2.

4. Decentralized online gradient method

In the section, we first present the decentralized online gradient method, and then prove that it leads to $\mathcal{O}\left(n\sqrt{TM} + \sqrt{nTM}\sigma\right)$ dynamic regret.

4.1. Algorithm

Algorithm 1 DOG: Decentralized Online Gradient method.

Require: The learning rate η, number of iterations T, and the confusion matrix $\mathbf{W}$. $\mathbf{x}_{i,1} = \mathbf{0}$ for any $i \in [n]$.

1: **for** $t = 1, 2, ..., T$ **do**
2:     **For the i-th node with** $i \in [n]$**:**
3:     Predict $\mathbf{x}_{i,t}$.
4:     Observe the loss function $f_{i,t}$, and suffer loss $f_{i,t}(\mathbf{x}_{i,t}; \zeta_{i,t}, \xi_{i,t})$.
5:     **Update:**
6:     Query a sub-gradient $\partial f_{i,t}(\mathbf{x}_{i,t}; \zeta_{i,t}, \xi_{i,t})$.
7:     $\mathbf{x}_{i,t+1} = \sum_{j=1}^n \mathbf{W}_{i,j}\mathbf{x}_{j,t} - \eta\partial f_{i,t}(\mathbf{x}_{i,t}; \zeta_{i,t}, \xi_{i,t})$.
8: **end for**

The Decentralized Online Gradient method, namely DOG, is presented in Algorithm 1. This algorithm works iteration by iteration. At each iteration, every node needs to collect the decision variable, e.g., $\mathbf{x}_{i,t}$, from its neighbours, and compute a weighted sum as $\sum_{j=1}^n \mathbf{W}_{i,j}\mathbf{x}_{j,t}$. Then, the weighte sum is updated by an online gradient descent step. In addition, we denote $\bar{\mathbf{x}}_t = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_{i,t}$ to facilitate the theoretical analysis. We can verify that $\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{x}}_t - \frac{\eta}{n}\sum_{i=1}^n \partial f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t})$ (see Lemma **??**).

4.2. Theoretical analysis

Denote

$$H_t(\cdot) := \mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} h_t(\cdot; \xi_{i,t}) \qquad \text{for } \forall i \in [n],$$

and

$$F_{i,t}(\cdot) := \mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} f_{i,t}(\cdot; \zeta_{i,t}, \xi_{i,t}).$$

Assumption 1. *We make following assumptions to analyze the dynamic regret theoretically.*

- *For any $i \in [n]$, $t \in [T]$, and $\mathbf{x}$, there exists a constant G such that*

  $$\max\left\{ \mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} \left\|\nabla h_t(\mathbf{x}; \xi_{i,t})\right\|^2, \left\|\partial g_{i,t}(\mathbf{x}; \zeta_{i,t})\right\|^2 \right\} \leq G,$$

  and

  $$\mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} \left\|\nabla h_t(\mathbf{x}; \xi_{i,t}) - \nabla H_t(\mathbf{x})\right\|^2 \leq \sigma^2.$$

- *For given vectors $\mathbf{x}$ and $\mathbf{y}$, we assume $\left\|\mathbf{x} - \mathbf{y}\right\|^2 \leq R$.*
- *For any $i \in [n]$ and $t \in [T]$, we assume the function $f_{i,t}$ is convex, but may be non-smooth.*
- *Given a symmetric doublely stochastic matrix $\mathbf{W}$, and a constant ρ with $\rho := \max\{|\lambda_2(\mathbf{W})|, |\lambda_n(\mathbf{W})|\}$, we assume $\rho < 1$.*

The bound of dynamic regret yielded by Algorithm 1 is presented in the following theorem.

Theorem 1. *Denote constants C_0, and C_1 by*

$$\begin{aligned} C_0 :=& 1 + \frac{1}{2(1-\rho)^2} + 16\beta; \\ C_1 :=& \frac{L + 2\eta L^2 + 4(1-\beta)^2 L^2 \eta}{(1-\rho)^2} + 2L. \end{aligned}$$

Using Assumption 1, and choosing $\eta > 0$ in Algorithm 1, we have

$$\begin{aligned} &\mathop{\mathbb{E}}_{\Xi_{n,T}\sim\mathcal{D}_T} \sum_{t=1}^T \sum_{i=1}^n f_{i,t}(\mathbf{x}_{i,t}; \xi_{i,t}) - f_{i,t}(\mathbf{x}_t^*; \xi_{i,t}) \\ \leq& C_0\eta T n\beta G + (1-\beta)\eta T\sigma^2 + 4n(1-\beta)T\eta G \\ &+ C_1 n T\eta^2 G + \frac{n}{2\eta}\left(4\sqrt{R}M + R\right). \end{aligned}$$

By choosing an approximate learning rate η, we obtain sublinear regret as follows.

Corollary 1. *Using Assumption 1, and choosing*

$$\eta = \sqrt{\frac{nM\sqrt{R} + nR}{nTG + (1-\beta)T\sigma^2}}$$

in Algorithm 1, we have

$$\begin{aligned}
&\mathcal{R}_T^{\mathrm{DOG}} \\
\lesssim &\frac{n\sqrt{T\left(M+\sqrt{R}\right)G}}{(1-\rho)^2} + \sqrt{n(1-\beta)T\left(M+\sqrt{R}\right)\sigma^2} \\
&+ \frac{n\left(M+\sqrt{R}\right)}{(1-\rho)^2} + \sqrt{TM\left(n^2G + n(1-\beta)\sigma^2\right)} \\
&+ \sqrt{T\left(n^2G + (1-\beta)n\sigma^2\right)}.
\end{aligned} \tag{4}$$

First, corollary 1 shows that the dynamic regret of DOG is sublinear. Second, we would like make some comments on the effects of different parameters on the dynamic regret. The regret becomes large with the increase of the budget of dynamics M. When $n = 1$ and $\rho = 0$, the dynamic regret is $\mathcal{O}\left(\sqrt{TM} + \sqrt{T}\right)$, which is tight in the case of $n = 1$ (Zhao et al., 2018). When $\beta < 1$, the regret $\mathcal{R}_T^{\mathrm{DOG}}$ has $\sqrt{nTM\sigma^2}$ dependence on σ^2, instead of $\sqrt{n^2TM\sigma^2}$. It benefits from the communication among nodes in the decentralized setting. Since every node shares its decision variable with its neighbours, the variance of the average of stochastic gradients $\frac{1}{n}\sum_{i=1}^n \nabla h_t(\mathbf{x}_{i,t};\xi_{i,t})$ is decreased to be $\frac{\sigma^2}{n}$, thus eventually reducing the regret caused by the stochastic data. Additionally, the regret is affected by the topology of the network, which is measured by ρ with $0 \le \rho < 1$. For a fully connected network[2], $\rho = 0$, then the regret is better than those for other topologies.

4.3. Connections with previous work

Theorem 2 (Appeared in Theorem 3 in Shahrampour and Jadbabaie (2018))**.** *Use Assumption 1, and choose $\eta > 0$ in Algorithm 1. The dynamic regret $\widetilde{\mathcal{R}}_T^{\mathrm{DOG}}$ is bounded by*

$$d$$

Using the classic dynamic regret defined in (1), Shahrampour and Jadbabaie (2018) has provided a $\mathcal{O}\left(n\sqrt{nTM}\right)$ regret for DOG, which is a special case of our dynamic regret when $\beta = 1$. In addition, compared with the result in Shahrampour and Jadbabaie (2018), our regret enjoys the state-of-the-art dependence on T and M, and meanwhile improves the dependence on n. This improvement is achieved by a better bound on the difference between $\mathbf{x}_{i,t}$ and $\bar{\mathbf{x}}_t$[3].

[2]When a network is fully connected, a decentralized method de-generates to a centralized method.

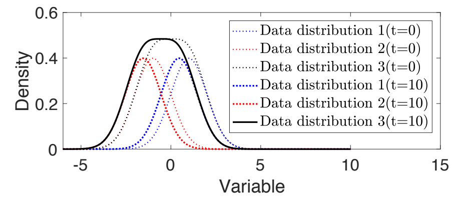


Figure 1. An illustration of the dynmaics caused by the time-varying distributions of data. Data distributions 1 and 2 satisfy $N(1+\sin(t), 1)$ and $N(-1+\sin(t), 1)$, respectively. Data distribution 3 is the sum of them, which changes over time.

Lemma 1. *Using Assumption 1, and setting $\eta > 0$ in Algorithm 1, we have*

$$\mathop{\mathbb{E}}_{\Xi_{n,T}\sim\mathcal{D}_T} \sum_{i=1}^n \sum_{t=1}^T \left\|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\|^2 \le \frac{nT\eta^2 G}{(1-\rho)^2}.$$

Additionally, previous work usually assumes $\partial f_{i,t}$ is bounded, e.g., $\left\|\partial f_{i,t}(\mathbf{x};\zeta_{i,t},\xi_{i,t})\right\|^2 \le G$, which implies $\left\|\nabla h_t(\mathbf{x};\xi_{i,t})\right\|^2 \le G$, and $\mathbb{E}_{\xi_{i,t}\sim D_t}\left\|\nabla h_t(\mathbf{x};\xi_{i,t})\right\|^2 \le \sigma^2 + G$ according to Lemma 2.

Lemma 2. *Assume $\left\|\nabla h_t(\mathbf{x};\xi_{i,t})\right\|^2 \le G$. It implies*

$$\mathop{\mathbb{E}}_{\xi_{i,t}\sim D_t} \left\|\nabla h_t(\mathbf{x};\xi_{i,t})\right\|^2 \le \sigma^2 + G.$$

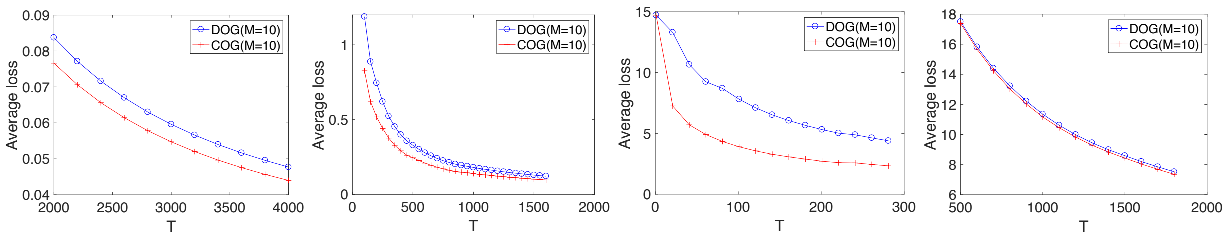
5. Empirical studies

For simplicity, in the experiments we only consider online logistic regression with squared ℓ_2 norm regularization, i.e., $f_{i,t}(\mathbf{x};\xi_{i,t}) = \log\left(1+\exp(-\mathbf{y}_{i,t}\mathbf{A}_{i,t}^{\mathrm{T}}\mathbf{x})\right) + \frac{\gamma}{2}\left\|\mathbf{x}\right\|^2$, where $\gamma = 10^{-3}$ is a given hyper-parameter. Under this setting, we compare the proposed Decentralized Online Gradient method (DOG) and the Centralized Online Gradient method (COG). The learning rate η is set to be $C\sqrt{\frac{M}{T}}$ with $C \in [10^{-2}, 20]$. M is fixed as 10 to determine the space of reference points, while C is tuned for each data sperately. We evaluate the learning performance by measuring the average loss $\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T f_{i,t}(\mathbf{x}_{i,t};\xi_{i,t})$, instead of the dynamic regret $\mathbb{E}_{\Xi_{n,T}\sim\mathcal{D}_{n,T}}\sum_{i=1}^n\sum_{t=1}^T\left(f_{i,t}(\mathbf{x}_{i,t};\xi_{i,t}) - f_{i,t}(\mathbf{x}_t^*)\right)$, since the optimal reference point $\{\mathbf{x}_t^*\}_{t=1}^T$ is the same for DOG and COG .

5.1. Datasets

To test the proposed algorithm, we utilized a toy dataset and three real-world datasets, whose details are as follows.

[3]Shahrampour and Jadbabaie (2018) denotes $\left\|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\right\|$ by "network error".



(a) *synthetic data*, 100 nodes (b) *room-occupancy*, 5 nodes (c) *usenet2*, 5 nodes (d) *spam*, 5 nodes

Figure 2. The average loss yielded by DOG is comparable to that yielded by COG.

Synthetic Data For the i-th node, a data matrix $\mathbf{A}_i \in R^{10\times T}$ is generated, s.t. $\mathbf{A}_i = 0.1\tilde{\mathbf{A}}_i + 0.9\hat{\mathbf{A}}_i$, where $\tilde{\mathbf{A}}_i$ represents the adversary part of data, and $\hat{\mathbf{A}}_i$ represents the stochastic part of data. Specifically, elements of $\tilde{\mathbf{A}}_i$ is uniformly sampled from the interval $[-0.5+\sin(i), 0.5+\sin(i)]$. Note that $\tilde{\mathbf{A}}_i$ and $\tilde{\mathbf{A}}_j$ with $i \neq j$ are drawn from different distributions. $\hat{\mathbf{A}}_{i,t}$ is generated according to $\mathbf{y}_{i,t} \in \{1, -1\}$ which is generated uniformly. When $\mathbf{y}_{i,t} = 1$, $\hat{\mathbf{A}}_{i,t}$ is generated by sampling from a time-varying distribution $N((1+0.5\sin(t))\cdot\mathbf{1}, \mathbf{I})$. When $\mathbf{y}_{i,t} = -1$, $\hat{\mathbf{A}}_{i,t}$ is generated by sampling from another time-varying distribution $N((-1+0.5\sin(t))\cdot\mathbf{1}, \mathbf{I})$. Due to this correlation, $y_{i,t}$ can be considered as the label of the instance $\hat{\mathbf{A}}_{i,t}$. The above dynamics of time-varying distributions are illustrated in Figure 1, which shows the change of the optimal learning model over time and the importance of studying the dynamic regret.

Real Data Three real public datasets are *room-occupancy*[4], *usenet2*[5], and *spam*[6]. *room-occupancy* is a time-series dataset, which is from a natural dynamic enviroment. Both *usenet2* and *spam* are "concept drift" (Katakis et al., 2010) datasets, for which the optimal model changes over time.

5.2. Results

First, figure 2 summarizes the performance of DOG compared with COG on all the datasets. For the synthetic dataset, we simulated a decentralized network consisting of 100 nodes; For the three real datasets, we simulated a network consisting of 5 nodes. In these networks, the nodes are connected by a ring topology. Under these settings, we can observe that both DOG and COG are effective for the online learning tasks on all the datasets, while DOG achieves slightly worse performance.

[4]https://archive.ics.uci.edu/ml/datasets/Occupancy+Detection+

[5]http://mlkd.csd.auth.gr/concept_drift.html

[6]http://mlkd.csd.auth.gr/concept_drift.html

Second, figure 3 summarizes the effect of the network size on the performance of DOG. It shows that the performance of DOG is not sensitive to the network size, which confirms our theoretical result, that is, the average regret does not increase with the number of nodes.

Third, figure 4 shows the effect of the topology of the network on the performance of DOG, for which four different topologies are used. Besides the ring topology, the *Fully connected* means all nodes are connected, where DOG degenerates to be COG. The topology *WattsStrogatz* represents a Watts-Strogatz small-world graph, for which we can use a parameter to control the number of random edges (set as 0.5 and 1 in this paper). The result shows *Fully connected* enjoyes the best performance, because that $\rho = 0$ for it while $\rho > 0$ for other topologies.

6. Conclusion

We investigate a new online learning problem in a decentralized network, where the loss incurs by both adversary and stochastic data. We provide a new analysis framework, which achieves sublinear regret. Extensive empirical studies varify the theoretical result.

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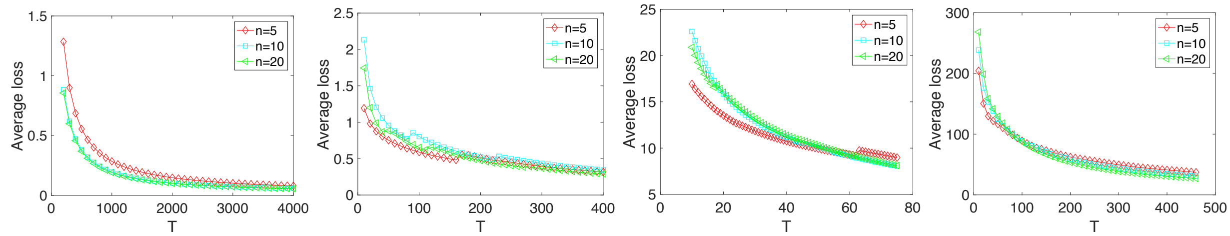
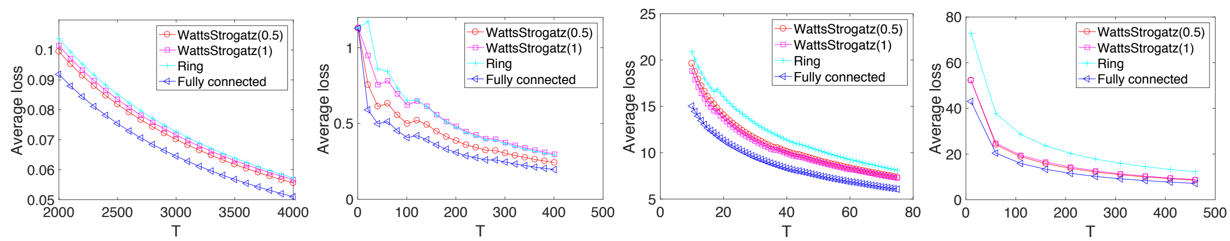


(a) *synthetic data*, ring topology (b) *room-occupancy*, ring topology (c) *usenet2*, ring topology (d) *spam*, ring topology

Figure 3. The average loss yielded by DOG is insensitive to the network size.



(a) *synthetic data*, 100 nodes (b) *room-occupancy*, 20 nodes (c) *usenet2*, 20 nodes (d) *spam*, 20 nodes

Figure 4. The average loss yielded by DOG is insensitive to the topology of the network.

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