Elizabethtown College

JOINT MATHEMATICS MEETINGS
JAN 6–9, 2021
AMS • MAA

Go Big or Go Home?

Yawen Zhang, Advisor: Timothy McDevitt
Elizabethtown College

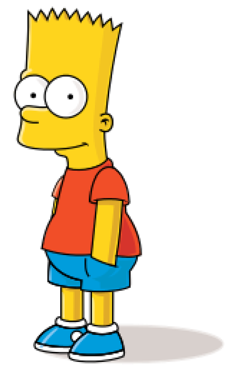
Background

We often hear that "Winning isn't everything; it's the only thing" and we usually regard ties with disappointment. However, we show that accepting ties can be advantageous, at least in some situations.

Problem Formulation

Inspired by Penney's game, we designed a game involving two players, Lisa and Bart. They toss a weighted coin that lands on heads with probability $p \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, which is known to both players. Each player simultaneously chooses a sequence of heads and tails of length n. Then a coin is tossed repeatedly and the player whose sequence appears first wins. If they select the same sequence, then they tie. Lisa selects strategies to maximize her probability of winning (aggressive), and Bart minimizes his probability of losing (conservative).

Lisa: I want to win!
Go big or go home!



Bart: I also want to win,
but I can accept ties.

Our game	Penney's game
Two players choose simultaneously	One player chooses first
Biased coin	Fair coin

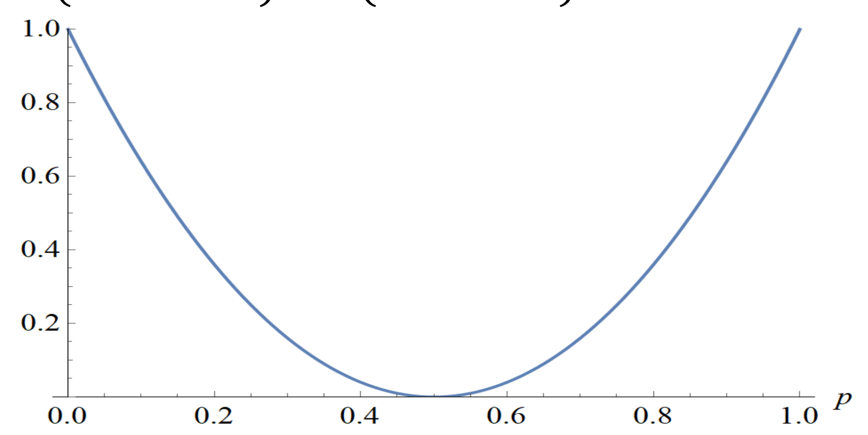
Basic Cases

$n = 1$

Bart's strategy

Lisa's strategy		H	T
	H	0	p
	T	$1-p$	0

P(Bart wins) − P(Lisa wins)

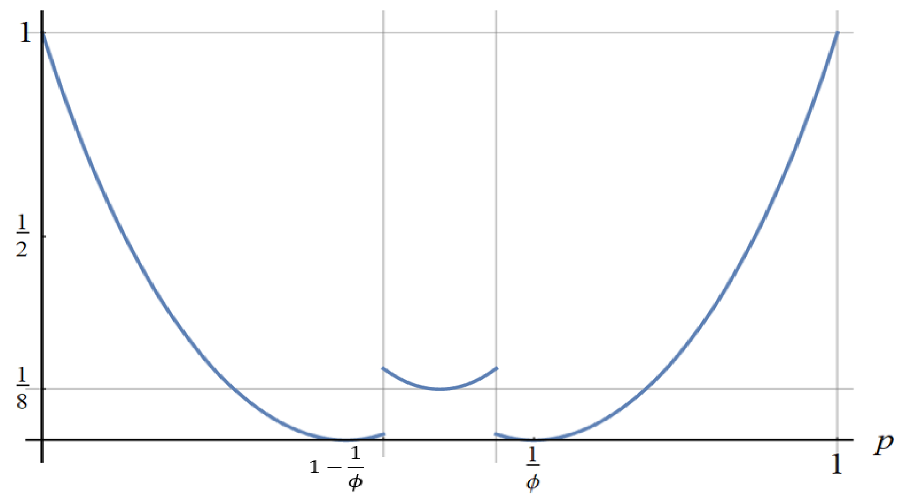


$n = 2$

Bart's strategy

Lisa's strategy		HH	HT	TH	TT
	HH	0	p	p^2	$\frac{p^2(2-p)}{1-p(1-p)}$
	HT	$1-p$	0	p	$p(2-p)$
	TH	$1-p^2$	$1-p$	0	p
	TT	$1-\frac{p^2(2-p)}{1-p(1-p)}$	$1-p(2-p)$	$1-p$	0

P(Bart wins) − P(Lisa wins)

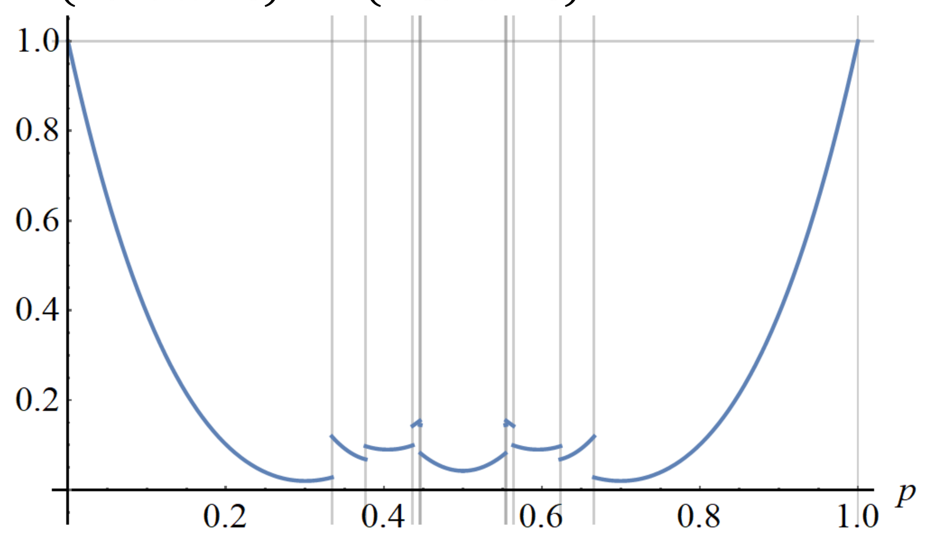


$n = 3$

Bart's strategy

Lisa's strategy		HHH	HHT	HTH	...
	HHH	0	p	$\frac{p}{1+p-p^2}$	...
	HHT	$1-p$	0	$\frac{1}{2-p}$	...
	HTH	$\frac{-1+p^2}{-1-p+p^2}$	$-\frac{1+p}{-2+p}$	0	...
	...	...	...	...	

P(Bart wins) − P(Lisa wins)



(Matrices indicate $P(Lisa\ wins)$)

Conclusion: Conservative strategy is almost always better!

$$P(Lisa\ wins) < P(Bart\ wins)$$

Exceptions: n=2, p=1/2 and n=3, p=1/ϕ, where ϕ is the golden ratio!

Penney's Game

In Penney's game, the second player (Bart) chooses his sequence after the first player (Lisa) has chosen hers. Bart always has an advantage over Lisa because the payoffs are nontransitive. Here are two examples to show the odds in favor of Bart:

Lisa's choice	Bart's choice	Odds in favor of Bart
HHH	THH	7 to 1
THH	TTH	2 to 1

By having the players choose their sequences simultaneously, our version of the game removes Bart's advantage of choosing second, but he still has a *significant* advantage because of his openness to ties.

Future Work

We consider tempering Lisa's aggressiveness with the probability ϵ of accepting ties. We are also looking for real-life applications where being conservative pays off.

$$\min_y \max_x P(Lisa\ wins) + \epsilon P(tie) = \min_y \max_x x^T(L + \epsilon I)y$$