Pricing the Down and Out Call Option (DOC) in Black-Scholes-Merton Model and Heston Model

Name: Yaxin LI

1. Introduction

In this report, the goal is to pricing the down and out call option (DOC), and computing the relevant Greeks in Black-Scholes-Merton model and Heston model.
The down and out call option (DOC) is an exotic option, which will knock out and become worthless if the price of the underlying drops under the barrier level. If the price of the underlying does not drop to the barrier level, the option acts like a vanilla call option, giving the holder the right but not the obligation to buy the underlying asset at strike price K at time to maturity.
The payoff of DOC at time to maturity is:

$$DOC_0(S_0, \mathrm{K}, \mathrm{B}) = \begin{cases} \max(S_T - K, 0) & if \ \min\limits_{0 \le t \le T} S_t \ge B \\ 0 & other\ wise \end{cases} \qquad (1-1)$$

For the Black-Scholes-Merton model, since the analytical solution is available, first, I will compute the theoretical value of DOC. Second, the Monte-Carlo method will be used to price DOC, and sample paths are drawn to visualize the simulation process. In addition, using the method provided by Broadie-Glasserman-Kou (1997), I could change the value of the barrier so as to get more precise price via Monte-Carlo method. What's more, Wiener-Hopf factorization method is also used to price the DOC. Then I will carry on the robust analysis in order to see how the Price of DOC will evolve with the price of the underlying stock, the barrier, the volatility, and the risk free rate. Last by not least, the Greeks (e.g. Delta, Gamma, Vega, Theta, Rho) will be computed, and the robust analysis for Delta in four different time to maturity cases will be carried out as well.
For Heston model, since only the semi-analytical formula is available, I will use Monte-Carlo method to price DOC. Sample paths are drawn to visualize the simulation process. Moreover, the evolution of the stochastic volatility and the underlying stock price will be drawn together so as to show the correlation of the two stochastic process. Then I will carry on the robust analysis in order to see how the Price of DOC will evolve with the price of the underlying stock, the barrier, the correlation coefficient, and the risk free rate. Last by not least, the Greeks (Delta, Gamma, Vega, Theta, Rho) will be computed, and the robust analysis for Delta in four different time to maturities will be carried out as well.
To make all the results comparable, I use the following values of a special case for the following studies:

Table 1-1 The parameters of a special case used in the report

Variable	Values	Variable	Values
S_0	50	r	0.02
K	40	σ	0.2
B	35	T	1

2. Black-Scholes-Merton Model

2.1 Model Presentation

2.1.1 Model Assumptions

The Black-Scholes-Merton Model (Black and Scholes, 1973; Merton, 1973) is valid under the following assumptions:
a) The short-term interest rate, volatility are known and are constant through time.
b) The dynamic of stock price could be described by Geometric Brownian Motion.

c) The stock pays no dividends or other distributions.

d) There are no transaction costs in buying or selling the stock or the option; people are able to short sell and borrow any fraction of the price of a security at risk free rate.

2.1.2 Model Specification

2.1.2.1 Pricing formula for vanilla option

Since the dynamic of the stock price could be described by Geometric Brownian Motion, in the risk neutral world, we have:

$$\frac{dS_t}{S_t} = rdt + \sigma dZ_t \qquad (2-1)$$

With Z_t a standard Q Brownian motion.

By constructing a riskless portfolio, we could get the PDE which is:

$$\frac{\partial \mathrm{C}}{\partial \mathrm{t}} + \frac{\partial \mathrm{C}}{\partial \mathrm{S}} * r + \frac{1}{2}\sigma^2 S^2 * \frac{\partial^2 \mathrm{C}}{\partial \mathrm{S}^2} - r * C = 0 \qquad (2-2)$$

Solving the PDE, we can get the analytical formula for vanilla call option. That is:

$$C_0(S_0, \mathrm{K}) = S_0 * N(d_1) - K * e^{-rT} * N(d2) \qquad (2-3)$$

with $d_1 = \frac{\ln(S_0/(K*e^{-rT}))}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}, d_2 = \frac{\ln(S_0/(K*e^{-rT}))}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$

2.1.2.2 Pricing formula for down and out call option (DOC)

First, according to the payoff of DOC described in (1-1), in the risk neutral world, we have:

$$DOC_0(S_0, \mathrm{K}, \mathrm{B}) = E^Q(\text{payoff(T)} * e^{-rT}) \qquad (2-4)$$

Thus, if $S_t \le B \; for \; t \in [0, T]$, $DOC_0 = 0$

If $S_t \ge B$ at time t, the next timestep, being infinitesimal, will not take us to the barrier. We can therefore apply the usual Black–Scholes hedging analysis and get the PDE:

$$\frac{\partial \mathrm{DOC}}{\partial \mathrm{t}} + \frac{\partial \mathrm{DOC}}{\partial \mathrm{S}} * r + \frac{1}{2}\sigma^2 S^2 * \frac{\partial^2 \mathrm{DOC}}{\partial \mathrm{S}^2} - r * DOC = 0 \qquad (2-5)$$

Reducing to the heat equation by changing variables, and after some computation, we could get the Pricing formula for DOC:

$$DOC_0(S_0, \mathrm{K}, \mathrm{B}) = C_0(S_0, \mathrm{K}) - \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} * C_0\left(\frac{B^2}{S_0}, \mathrm{K}\right) \qquad (2-6)$$

2.2 Pricing DOC

2.2.1 Analytical Method

According to 2.1.1.2, we get the Analytical formula for DOC. That is:

$$DOC_0(S_0, \mathrm{K}, \mathrm{B}) = \begin{cases} C_0(S_0, \mathrm{K}) - \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} * C_0\left(\frac{B^2}{S_0}, \mathrm{K}\right) & if \; \min\limits_{0 \le \mathrm{t} \le \mathrm{T}} S_t \ge B \\ 0 & other\;wise \end{cases} \qquad (2-7)$$

Substituting the values of $S_0, \mathrm{K}, \mathrm{B}, \mathrm{r}, \sigma, T$ into the formula, we could get

$$DOC_0(50{,}40{,}35) = 11.2520$$

2.2.2 Monte-Carlo Method

The Monte-Carlo Method including the following steps:

1. Discretizing the time

Here, we have the time to maturity T=1. Thus, the time-span is [0, T]. Taking n=100, we could get the grid which has 100 uniformly distanced points, and $\Delta \mathrm{t} = \frac{T}{n}$, $t_k^n = \frac{kT}{n}\ for\ k = 1,2, \dots, n$.

2. Get increment for sample path

The dynamic could be rewrite as:

$$dS_t = S_t * (rdt + \sigma dZ_t) \tag{2-8}$$

Thus, we have the discrete piece-wise constant sample:

$$S_{t_{k+1}^n} = S_{t_k^n} * \left(1 + r * \Delta \mathrm{t} + \sigma * \sqrt{\Delta \mathrm{t}} * U_{k+1}\right) \tag{2-9}$$

with U~N(0,1)

3. Simulation

Iterating 100 times, we could get a sample path for the process, and we could using (1) to compute the payoff for this sample path. To summarize, the parameters used for Monte-Carlo simulation is:

Table 2-1 Parameters used for Monte-Carlo simulation

Variable	Values	Variable	Values
nb_sim(N)	10000	nb_points(n)	100

Repeating the iterating process for N=10000 times, we could get 10000 sample paths, and the price of DOC could be computed as:

$$DOC_0(S_0, \mathrm{K}, \mathrm{B}) = e^{-rT} * \sum_{i=1}^{N} \frac{payoff(i)}{N} \tag{2-10}$$

And we could get:

$$DOC_0(50,40,35) = 11.2043$$

4. Visualize the simulation process

To visualize the simulation process, 10 sample paths of the underlying stock are drawn in the following graph as an example. From the graph, we could see that 2 sample paths drops to the barrier level, which would make them worthless; another 1 sample paths ends up to go below the stick price, which will also make it worthless.

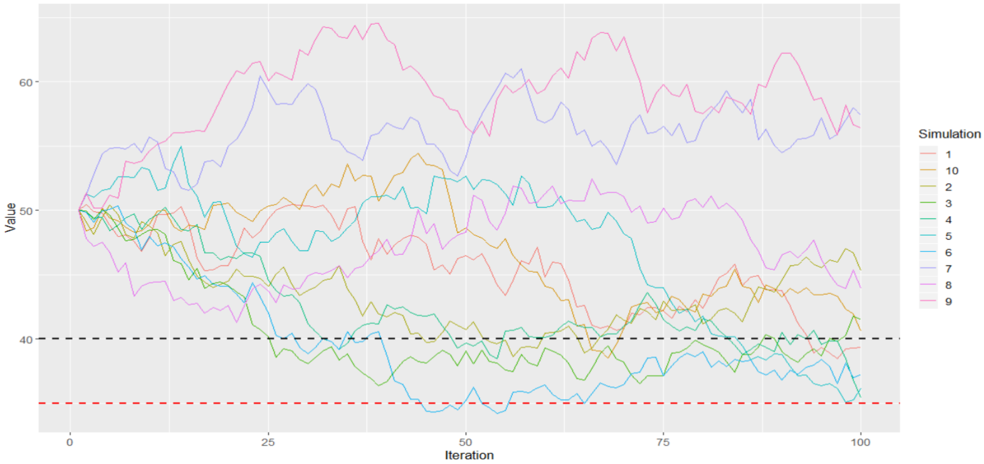

Figure 2-1 10 Sample Paths from Monte-Carlo Simulation

2.2.3 Monte-Carlo Method with Refined Barriers

Using the method provided by Broadie-Glasserman-Kou (1997), I could change the value of the barrier so as to get more precise price via Monte-Carlo method. Now, instead of using B as barrier, in each iteration, I use:

$$\mathrm{B}^* = \mathrm{B} * \mathrm{e}^{\mathrm{sign}(\mathrm{B}-S_0)*\beta\sigma\sqrt{\Delta t}} \qquad (2-11)$$

$$\text{with } \beta = 0.5826.$$

Repeating the Monte-Carlo simulation described in 2.2.2 with B^*, we could get:

$$DOC_0(50,40,35) = 11.2604$$

2.2.4 Wiener-Hopf Factorization method

1. Setting the problem

Let the barrier be $\mathrm{B} = e^b$, strike price be $\mathrm{K} = e^k$, $I_T = \min_{0\le t\le T} S_t$, and we have $X_t = at + \sigma Z_t$, $\emptyset(\mathrm{u}) = \mathrm{aui} - \sigma^2 u^2/2$ for Brownian motion. And for DOC, we have:

$$\mathrm{C}(\mathrm{k},\mathrm{b},\mathrm{T}) = \mathrm{E}\left[(e^{X_T} - e^k)^+ \mathbf{1}_{\min_{0\le t\le T} S_t \ge B}\right] = \int_{R^2} (e^x - e^k)^+ \mathbf{1}_{y>b} * p_t(x,y)dxdy \qquad (2-12)$$

2. Derive the formula

Using double Fourier transform in strike and barrier with the trike provided by Carr & Madan (1999), we get:

$$\int_{R^2} e^{iuk+ivb} * e^{\alpha k-\beta b} * \mathrm{C}(\mathrm{k},\mathrm{b},\mathrm{T})dkdb$$

$$= \int_{R^2} e^{iuk+ivb} * e^{\alpha k-\beta b} * \int_{R^2} (e^x - e^k)^+ \mathbf{1}_{y>b} * p_t(x,y)dxdydkdb$$

$$= \int_{R^2} \left(\int_{-\infty}^{x} e^{iuk} * e^{\alpha k} * (e^x - e^k)dk\right)\left(\int_{-\infty}^{y} e^{ivb} * e^{-\beta b}db\right)p_t(x,y)dxdy$$

$$= \int_{R^2} \frac{e^{(iu+\alpha+1)x} * e^{(iv-\beta)y}}{(iv-\beta)(iu+\alpha+1)(iu+\alpha)} p_t(x,y)dxdy$$

$$= \frac{E\left(e^{(iu+\alpha+1)X_T} * e^{(iv-\beta)I_T}\right)}{(iv-\beta)(iu+\alpha+1)(iu+\alpha)}$$

$$= \frac{E\left(e^{(iu+\alpha+1)(M_T+I_T)} * e^{(iv-\beta)I_T}\right)}{(iv-\beta)(iu+\alpha+1)(iu+\alpha)}$$

$$= \frac{E\left(e^{(iu+\alpha+1)M_T} * e^{(iv+iu+\alpha+1-\beta)I_T}\right)}{(iv-\beta)(iu+\alpha+1)(iu+\alpha)}$$

With:

$$E(e^{iu*M_T} * e^{iv*I_T}) = \emptyset_q^+(u) * \emptyset_q^-(v)\#(2-13)$$

Using Laplace Transform, we could get:

$$q\int_{R^2} e^{iuk+ivb} * e^{\alpha k-\beta b}\int_0^{+\infty} e^{-qt} * \mathrm{C(k,b,T)}dTdkdb = \frac{\emptyset_q^+\big(u-i(\alpha+1)\big) * \emptyset_q^-\big(u+v+i(\beta-\alpha-1)\big)}{(iv-\beta)(iu+\alpha+1)(iu+\alpha)}(2-14)$$

with $\mathrm{lm(u)} < 0$, $\mathrm{lm(v)} < 0$, $\lambda_- = \frac{a}{\sigma^2} + \frac{\sqrt{a^2+2\sigma^2 q}}{\sigma^2}$, $\lambda_+ = -\frac{a}{\sigma^2} + \frac{\sqrt{a^2+2\sigma^2 q}}{\sigma^2}$, $\emptyset_q^+ = \frac{\lambda_-}{\lambda_- + iu}$, $\emptyset_q^- = \frac{\lambda_+}{\lambda_+ - iu}$

3. Slicing time and get the price of DOC

To the time, we use the following parameters.

Table 2-2 Parameters for Wiener-Hopf Factorization

Variable	Values	Variable	Values
α	0.01	β	0.01
u	[-10,10], interval: 0.01	v	[-10,10], interval: 0.01

Using double inverse Fourier Transform and inverse Laplace Transform, finally we get:

$$DOC_0(50{,}40{,}35) = 11.2349$$

2.2.5 Robust Analysis for the price of DOC

2.2.5.1 Summary of the above four pricing methods

Summarizing the results obtained from 2.2.1 to 2.24, we get the following results:

Table 2-3 Summary of the results of the four pricing methods

Methods	Values	Relative Error
Analytical Method	11.2520	
Monte-Carlo Method	11.2043	-0.42%
Monte-Carlo Method with Refined Barriers	11.2604	0.07%
Fourier Transform and Laplace Transform Method	11.2349	-0.15%

We could tell from the results that all the three numerical methods work well in this special case, and the Monte-Carlo Method with Refined Barriers get the closest result compared to the theoretical value (true value) obtained by the Analytical Method.

2.2.5.2 Robust Analysis for the price of DOC

Table 2-4 Range of the parameters

Parameters	Range	interval
S_0	[20, 70]	1
B	[30, 40]	0.2
σ	[0.11, 0.60]	0.01
r	[0.001, 0.050]	0.001

Here, I change the parameters so as to see how the price of DOC evolves with price of the underlying stock, the barrier, the volatility, and the risk free rate respectively. The analytical formula (2-7) for DOC is used here to get the graphs, and the range of the parameters are listed in the above table.

1. The price of DOC and initial stock price S_0

From the graph, we can tell that if the initial stock price is lower than the barrier, the DOC will be worthless from the beginning of the contract and the price of DOC will be 0 all the time. If the initial stock price is higher than the barrier, we can tell from the graph that ceteris paribus, the higher the initial stock price S_0, the higher the price of DOC. Especially when S_0 is bigger than 50, the line is close to a straight line for the reason that the DOC is very likely to end up deeply in the money.

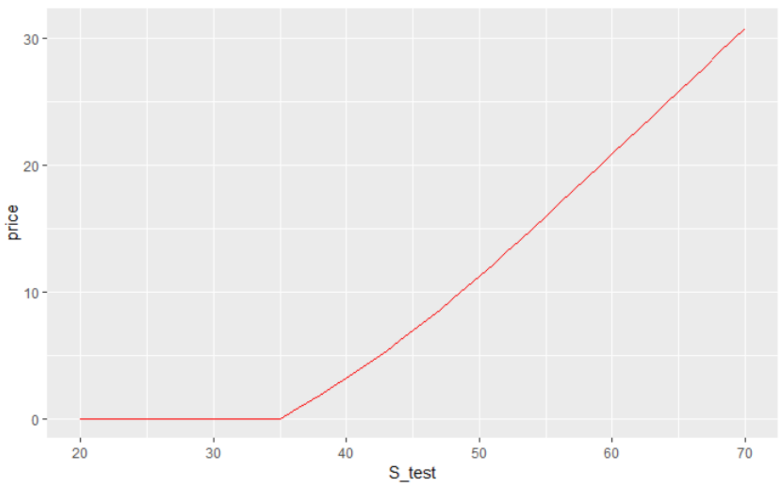

Figure 2-2 The price of DOC and initial stock price

2. The price of DOC and the barrier B

we can tell from the graph that with everything else being equal, the higher the barrier, the lower the price of DOC, because the probability, that the stock price will drop below the barrier and DOC becomes worthless, will increase with the increase of barrier. When B is very low and varies from 30 to 35, the impact of the change of the barrier will be relatively small compared with B varying from 35 to 40, due to the reason that the stock price is not very likely to go below 35.

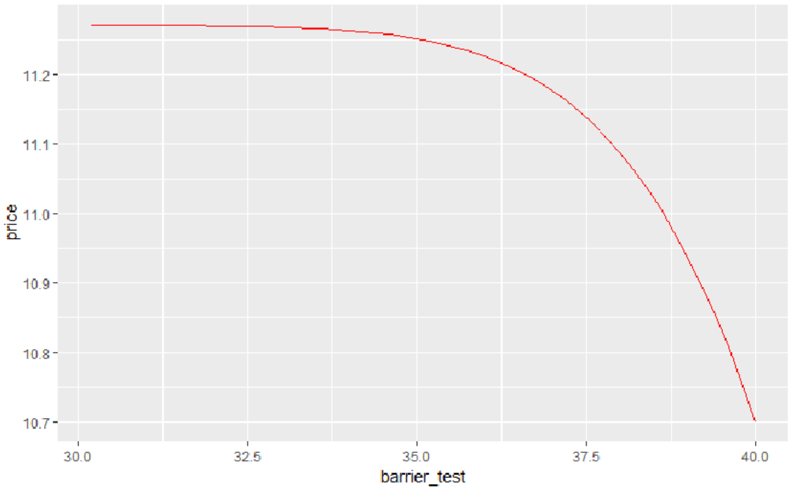

Figure 2-3 The price of DOC and the barrier

3. The price of DOC and the volatility σ

The graph indicates that when sigma is relatively low, e.g. during 0.11 to 0.3, the relationship between DOC and the volatility is slightly convex, which means that the increase of volatility has slightly increasing marginal impact on the price of DOC; while during 0.3 to 0.6, the relationship is slightly concave, indicating slightly decreasing marginal impact.

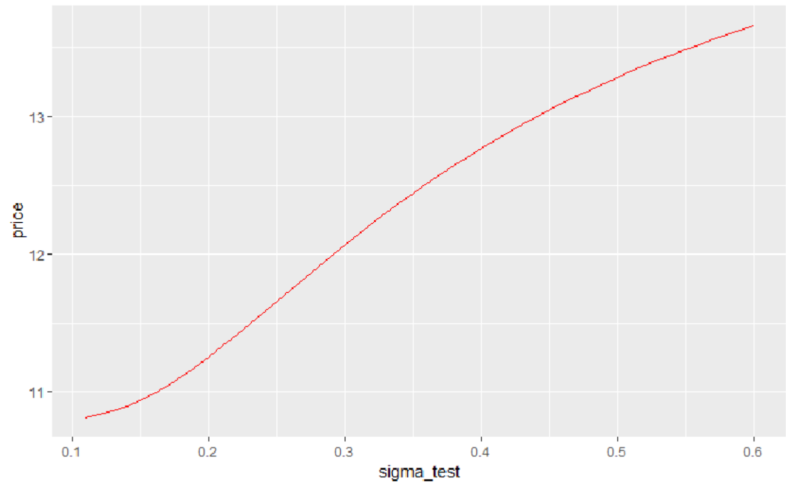

Figure 2-4 The price of DOC and the volatility

4. The price of DOC and the interest rate r

The relationship between price of DOC and risk free rate r is positive, which is similar to the vanilla call option. What's more, here the relationship is relatively linear.

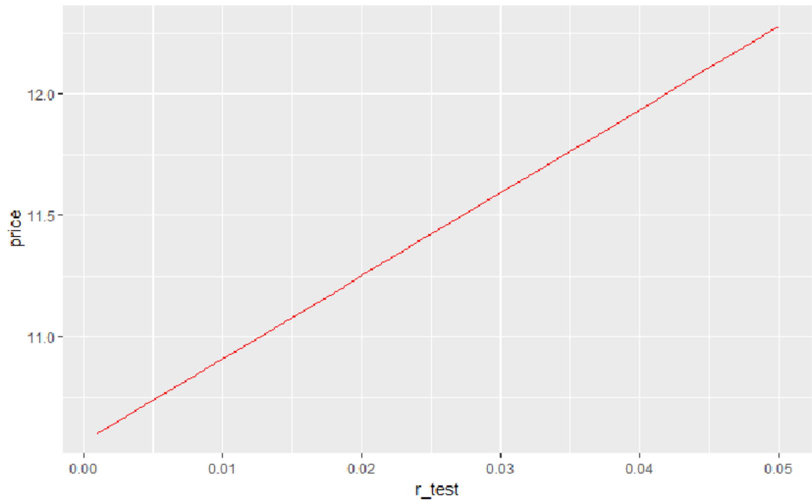

Figure 2-5 The price of DOC and the interest rate

2.3 Greeks

2.3.1 Computing Greeks

Here, I compute all the Greeks: Delta, Gamma, Vega, Theta, Rho for the special case. Since the closed form of Greeks for DOC are very complicated, here, I use numerical approximation method to compute the Greeks.

The main idea of this numerical method is that, assuming smoothness and using the center differentiation,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} \qquad (2-15)$$

And we also have: Delta $= \frac{\partial DOC}{\partial S}$, Vega $= \frac{\partial DOC}{\partial \sigma}$, Rho $= \frac{\partial DOC}{\partial r}$.

Assume h=0.001, we could use the above formula for Delta, Vega, Rho, of which the f(x) will be $DOC_0(x)$, and x will be S, σ, and respectively.

For Gamma, we have:

$$\text{Gamma} = \frac{\partial^2 DOC}{\partial S^2} = \lim_{h \to 0} \frac{DOC_0(S+h) - 2 * DOC_0(S) + DOC_0(S-h)}{2h} \qquad (2-16)$$

For Theta, since time only have one direction, we have:

$$\text{Theta} = \frac{\partial DOC}{\partial T} = \lim_{h \to 0} \frac{DOC_0(T) - DOC_0(T-h)}{2h} \qquad (2-17)$$

Let h=0.0001 and use (2-15) to (2-17), we can get the following results:

Table 2-5 Results of Greeks

Greeks	Values	Greeks	Values
Delta	0.9118	Gamma	0.0075
Vega	7.5489	Theta	1.4367
Rho	34.0894		

From the results, we could tell that the Delta of this special case is 0.9118, which is very close to 1, indicating that the DOC is very likely to be in the money at time to maturity. The Rho is relatively big, for the reason that the interest rate is really small and compared with other variables, the same amount of change of interest rate will lead to a significant change of the DOC price.

2.3.2 Robust Analysis for Delta

To carry on the robust analysis for Delta, here I let the initial stock price vary from 20 to 70. In addition, I use four time to maturities, which is 3 months, 6 months, 9 months and 1 year to draw the following 4 curves respectively, so as to see the impact of time to maturity on Delta as well.

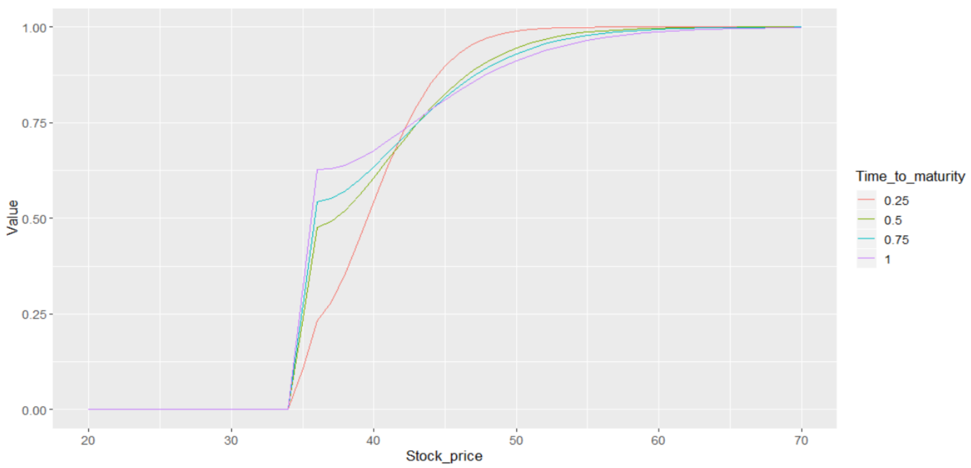

Figure 2-6 Robust analysis for Delta

Table 2-6 Range of the Parameters

Parameters	Range	interval
S_0	[20, 70]	1
T	[0.25, 1]	0.25

From the graph, we can see that for all the four curves, the Delta has a significant jump at barrier level B=35, due to the reason that below the threshold, the price of DOC is 0 all the time, which means we don't need to hedge, while S_0 slightly above the barrier will lead to a price of DOC larger than 0. Then during 36 to 65, the increase of S_0 will lead to the increase of the price of DOC. In addition, when S_0 is bigger than 65, the DOC is very likely to end in the money, leading the Delta be very close to 1.

By comparing the four curves, we can draw the conclusion that with everything else being equal, during the interval [35, 42], the longer the maturity, the higher impact S_0 has on Delta. This probably because the longer the time to maturity, the higher the probability the final price will go above the strike price K=40 and be in the money, thus Delta will increase quickly; while in [42, 65], we have an inverse relationship.

3. Heston Model

3.1 Model Presentation

In 1993, considering the phenomenon of volatility clustering, Heston introduced a model in which the variance is modelled by a Cox-Ingersoll-Ross (CIR) process (Heston, S. ,1993), and the dynamics of stochastic volatility and stock price can be write as follows:

$$\mathrm{d}v_t = (a - bv_t)dt + c * \sqrt{v_t} dZ_t^1 \qquad (3-1)$$

$$\frac{dS_t}{S_t} = mdt + \sqrt{v_t} dZ_t^2 \qquad (3-2)$$

Where Z_t^1 and Z_t^2 are two standard Q Brownian motions with a correlation coefficient ρ; a, b, c and m have to be positive to make sure that the model is well defined.

The CIR process is a mean reverting process, and the c can be interpreted as the volatility of the stochastic volatility. Here, adding $\sqrt{v_t}$ to dZ_t^1 is to make sure that the size of change will keep in step with the size of volatility; once the volatility becomes close to zero, the drift will be significantly larger than the fluctuation, and the volatility will increase immediately towards the mean. In addition, considering the fact that in crisis, the stock prices will decrease while the volatility tends to increase, and in normal case, the prices tends to increase while the volatility tends to decrease, we can reasonably consider Z_t^1 and Z_t^2 are negatively corelated and use $\rho < 0$ for study.

Since the Heston model allows volatility to evolve in a stochastic process instead of being constant all the time, it's a broader framework than BSM model, and it's closer to the reality.

In addition, To study this model, I use the following parameters as a special case:

Table 3-1 Parameters for Heston model

Greeks	Values	Greeks	Values
a	0.2	c	0.2
b	0.2	ρ	-0.7

3.2 Pricing DOC

3.2.1 Monte-Carlo Method

Since only the semi-analytical formula is available, in Heston model, I will use Monte-Carlo method to price DOC. Same as what I did in 2.2.2, I take 4 steps to price the DOC.

Table 3-2 Parameters used for Monte-Carlo simulation

Variable	Values	Variable	Values
nb_sim(N)	5000	nb_points(n)	100

1. Discretizing the time

Similarly, time to maturity T=1. Taking n=100, we get $\Delta t = \frac{T}{n}$, $t_k^n = \frac{kT}{n}$ $for\ k = 1,2,\dots,n.$

2. Get increment for sample path

Here, we discrete both stochastic process and get the discrete piece-wise constant samples:

$$v_{t_{k+1}^n} = v_{t_k^n} + \left(a - b * v_{t_k^n}\right) * \Delta t + c * \sqrt{v_{t_k^n}} * \sqrt{T\Delta t} * U_{k+1}^1)\#(3-3)$$

$$S_{t_{k+1}^n} = S_{t_k^n} * \left(1 + m * \Delta t + \sqrt{v_{t_{k+1}^n}} * \sqrt{T\Delta t} * U_{k+1}^2\right) \quad (3-4)$$

With $U^1 \sim N(0,1)$, $U^2 \sim N(0,1)$, and the correlation coefficient between U^1 and U^2 is $\rho = -0.7$. The initial volatility is 0.2 just as the BSM case.

3. Simulation

Similarly, using:

$$DOC_0(S_0, K, B) = e^{-rT} * \sum_{i=1}^{N} \frac{payoff(i)}{N} \quad (3-5)$$

And we could get

$$DOC_0(50,40,35) = 11.2126$$

4. Visualize the simulation process

Here, 10 sample paths are drawn in the following graph as an example. From the graph, lots of sample paths drops to the barrier level, which would make DOC worthless; only 3 paths end up in the money, which indicates that with stochastic volatility, the stock prices are likely to be more volatile and more likely to be worthless compared with the BSM model.

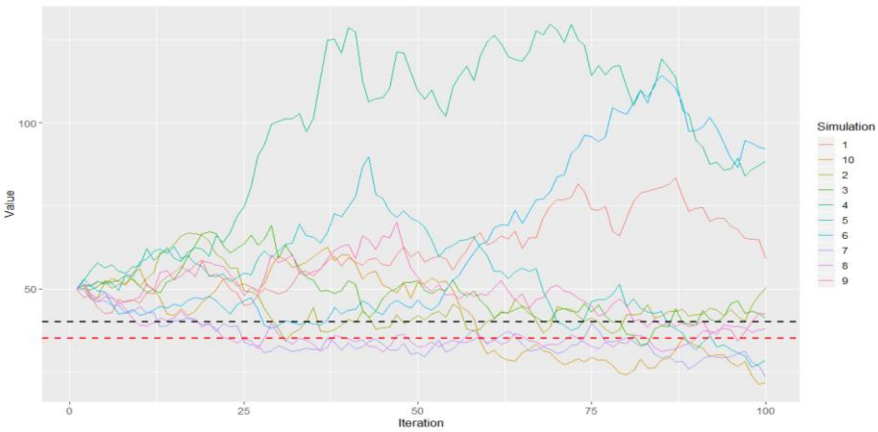

Figure 3-1 10 Sample Paths from Monte-Carlo Simulation

To visualize the relationship between the evolution of the stochastic volatility and the stock price, I draw them together In the following graph. With $\rho = -0.7$, we call tell from the graph that the negative relationship is obvious between the 2 stochastic process, and the volatility clustering can be seen in the latter half.

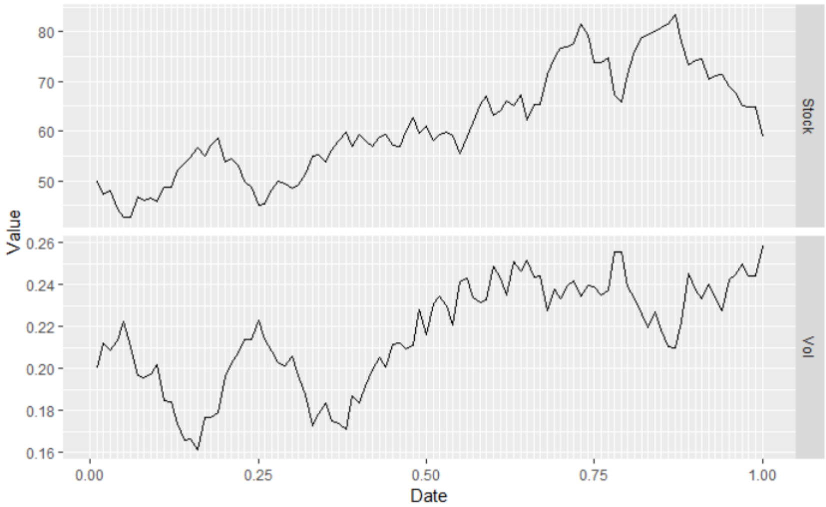

Figure 3-2 A possible evolution of the stochastic volatility and the stock price

3.2.2 Robust Analysis for the price of DOC

Then, I will carry on the robust analysis in order to see how the Price of DOC will evolve with the price of the underlying stock, the barrier, the correlation coefficient, and the risk free rate respectively. The Monte-Carlo method for DOC is used here to get the graphs, and the range of the parameters are listed in the following table.

Table 3-3 Range of the parameters

Parameters	Range	interval
S_0	[20, 70]	1
B	[30, 40]	0.2
ρ	[-0.02, -1]	0.02
r	[0.001, 0.050]	0.001

1. The price of DOC and initial stock price S_0

From the graph, we can tell that similar to what we get from BSM model, if the initial stock price is lower than the barrier, the DOC will be worthless from the beginning of the contract and the price of DOC will be 0 all the time. If the initial stock price is higher than the barrier, we can tell from the graph that ceteris paribus, the relationship between stock price and price of DOC is relatively linear. And the sight fluctuation of the curve is probably drives from the instability of Monte-Carlo method.

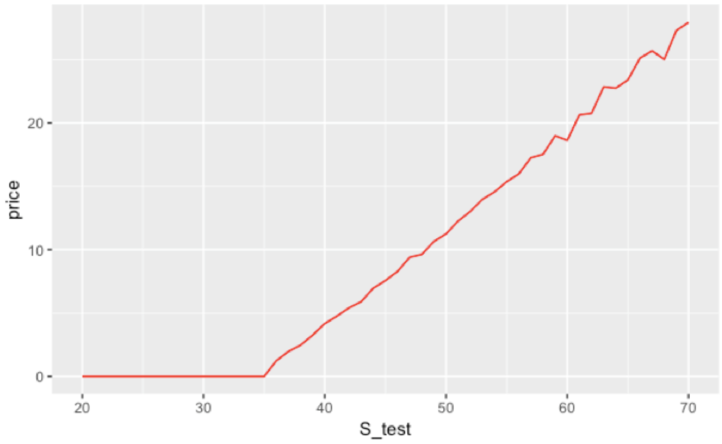

Figure 3-3 The price of DOC and initial stock price

2. The price of DOC and the barrier B

we can tell from the graph that with everything else being equal, in general, the higher the barrier, the lower the price of DOC, because the probability, that the stock price will drop below the barrier and DOC becomes worthless, will increase with the increase of barrier. Similarly, the fluctuation of the curve is probably drives from the instability of Monte-Carlo method.

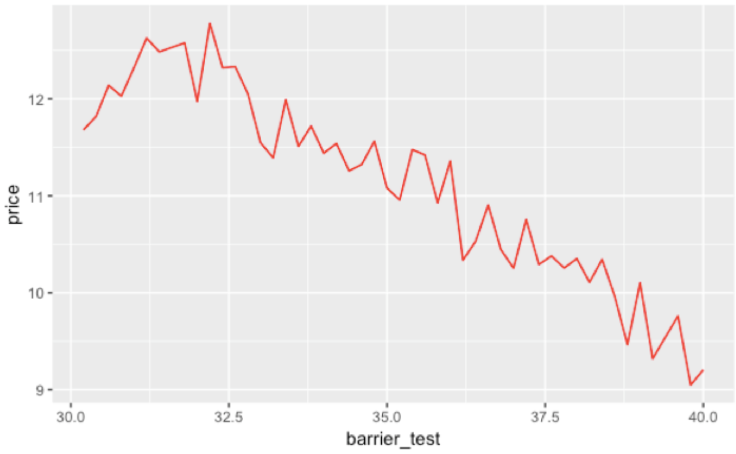

Figure 3-4 The price of DOC and the barrier

3. The price of DOC and the correlation coefficient ρ

From the graph, we can draw the conclusion that in general, the price of the DOC and the correlation coefficient ρ has positive relationship. This phenomenon appears due to the fact that the stronger the negative relationship, when the volatility has a positive ΔZ_t^1 and goes up, the higher probability that the stock price will have a negative ΔZ_t^2 and goes down, and this may lead to higher probability that the stock price will fall below the barrier, and finally lead to lower price of DOC.

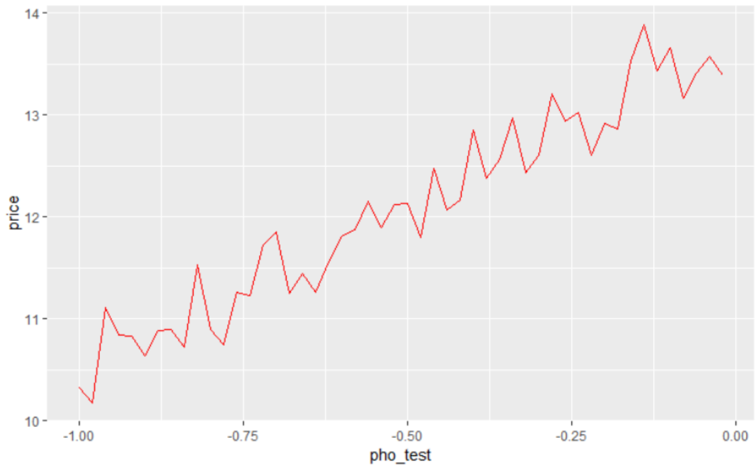

Figure 3-5 The price of DOC and the correlation coefficient

4. The price of DOC and the interest rate r

Roughly speaking, the relationship between price of DOC and risk free rate r is linear and positive, which means that the higher the interest rate, the higher the price of DOC. However, the fluctuation of the curve is large, maybe due to the instability of Monte-Carlo method.

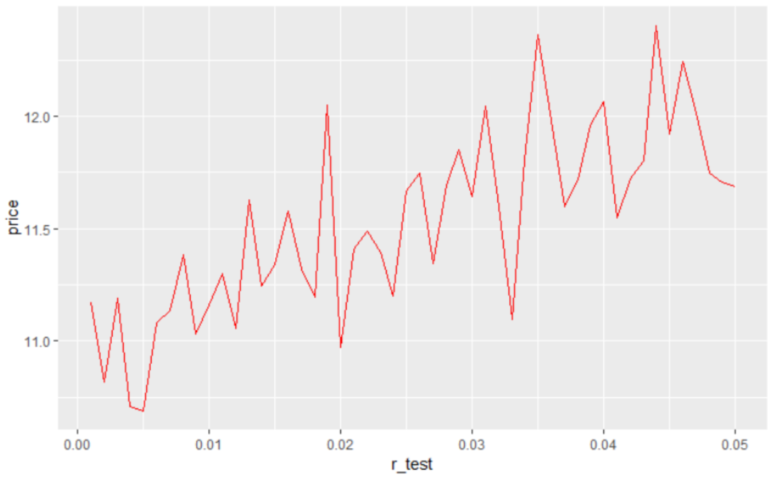

Figure 3-6 The price of DOC and the interest rate

3.3 Greeks

3.3.1 Computing Greeks

The Greeks: Delta, Gamma, Vega, Theta, Rho) are also computed using the method mentioned in 2.3.1 and formulas (2-15) to (2-17), and the results are as follows. Monte-Carlo method with 500 simulations are used here for computing the Greeks. And to save computation time, here for the computation of the price of DOC, I use 100 simulations with 10 iterations. In addition, considering the instability of Monte-Carlo method, I choose different h for different parameters, so as to get relatively stable and reasonable results.

Table 3-4 Results of Greeks

Greeks	h	Values	Greeks	h	Values
Delta	0.1	0.9737	Gamma	0.1	9.8827
Vega	0.001	40.1443	Theta	0.001	69.4542
Rho	0.0001	235.8730			

From the results, we could tell that some Greeks of DOC in Heston model could be extremely big, and Greeks can have significant changes from time to time, which indicates that even after 500 simulations, using the Monte-Carlo pricing method to calculate Greeks of DOC are still relatively unstable. Thus, simply using Monte-Carlo pricing method to simulate Greeks may lead to hedging problems. Further study on Delta will be carried out in the 3.3.2.

3.3.2 Robust Analysis for Delta

Here comes the robust analysis for Delta in four different time to maturity with h=0.1. Same as before, if stock price is in the interval [20,35], the price of DOC will be 0 all the time, leading to Delta also being 0. The significant jump at $S_0 = 35$ also can be seen in the graph. However, we can see from graph that the Delta is extremely unstable and fluctuates with the increase of stock price. Thus, if we use Monte-Carlo pricing method to compute Greeks, there will be plenty of issues when hedging DOC.

Comparing the four curves, we can see that generally speaking, the longer the time to maturity, the less volatile the Delta will be. This phenomenon appears probably due to the fact that the longer the time to maturity, the higher the probability the final price will go above the strike price K=40 and be in the money.

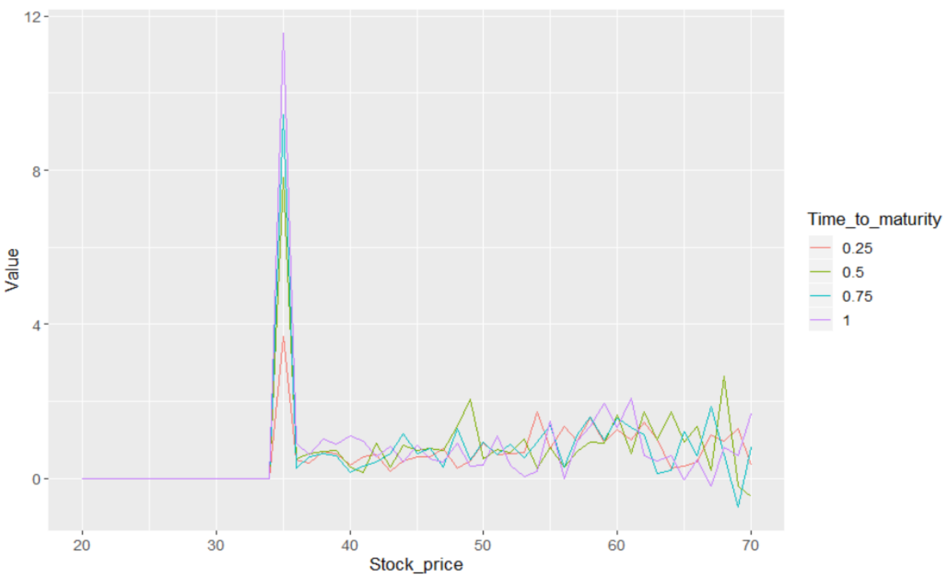

Figure 3-7 Robust analysis for Delta with h=0.1 for 4 time to maturities

Table 3-5 Range of the parameters

Parameters	Range	interval
S_0	[20, 70]	1
T	[0.25, 1]	0.25

If we take h=1 and T=1, we could get the following graph. In this graph, the curve is obviously less volatile than the previous one, which is easier to handle if we want to hedge DOC. However, since the h is not very small, the Delta will just be a roughly approximation and may also lead to problems due to its inaccuracy.

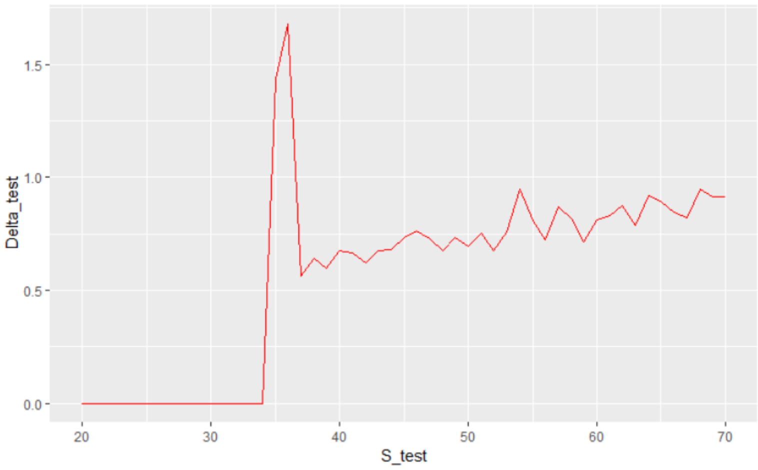

Figure 3-8 Robust analysis for Delta with h=1

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