Introduction to Universal Q Compiler

Raban Iten,^{1,*} Oliver Reardon-Smith,² Emanuel Malvetti,³ Luca Mondada,¹ Gabrielle Pauvert,² Ethan Redmond,² Ravjot Singh Kohli,² and Roger Colbeck^{2,†}

¹Institute for Theoretical Physics, ETH Zürich, 8093 Zürich, Switzerland ²Department of Mathematics, University of York, YO10 5DD, UK ³Department of Chemistry, TUM, Lichtenbergstraβe 4, 85747 Garching, Germany (Dated: 29th March 2021)

We introduce an open source software package UniversalQCompiler written in Mathematica that allows the decomposition of arbitrary quantum operations into a sequence of single-qubit rotations (with arbitrary rotation angles) and controlled-NOT (C-NOT) gates. Together with the existing package QI, this allows quantum information protocols to be analysed and then compiled to quantum circuits. Our decompositions are based on Phys. Rev. A 93, 032318 (2016), and hence, for generic operations, they are near optimal in terms of the number of gates required. UniversalQ-Compiler allows the compilation of any isometry (in particular, it can be used for unitaries and state preparation), quantum channel, positive-operator valued measure (POVM) or quantum instrument, although the run time becomes prohibitive for large numbers of qubits. The resulting circuits can be displayed graphically within Mathematica or exported to LATEX. We also provide functionality to translate the circuits to OpenQASM, the quantum assembly language used, for instance, by the IBM Q Experience.

I. INTRODUCTION

A universal quantum computer should be able to perform arbitrary computations on a quantum system. It is common to break down a given computation into a sequence of elementary gates, each of which can be implemented with low cost on an experimental architecture. However, given an abstract representation of the desired computation, such as a unitary matrix, it is in general difficult and time consuming to find a low-cost circuit implementing it. Here, we introduce an open source Mathematica package, *UniversalQCompiler*¹ that allows for automation of the compiling process on a small number of qubits. The package requires an existing Mathematica package QI^2 , which can easily handle common computations in quantum information theory, such as partial traces over various qubits or the Schmidt decomposition. Since the code is provided for Mathematica, our packages are well adapted for analytic calculations and can be used alongside the library of mathematical tools provided by Mathematica. Together, these constitute a powerful set of tools for analysing protocols in quantum information theory and then compiling the computations into circuits that can finally be run on a experimental architecture, such as IBM Q Experience (see Figure 1 for an overview). Universal Q Compiler focuses on the compilation process, and performs a few basic simplifications on the resulting quantum circuit. Hence, one might want to put the gate sequences obtained from Universal QCompiler into either a source-to-source compiler or a transpiler (see for example [1–3]) in order to optimize the gate count of the circuits further or to map them to a different hardware, which may have restrictions on the qubit-connectivity [4–8].

The package *UniversalQCompiler* provides code for all the decompositions described in [11], which are near optimal in the required number of gates for generic computations in the quantum circuit model (in fact, the achieved C-NOT counts differ by a constant factor of about two from a theoretical lower bound given in [11]). Note that our decompositions may not lead to optimal gate counts for computations of a special form lying in a set of measure zero (see [11] for the details), as for example for a unitary that corresponds to the circuit performed for Shor's algorithm [12]. Hence, to optimize the gate counts when decomposing operations of certain special forms, such as diagonal gates, multi-controlled singlequbit gates and uniformly-controlled gates, we provide separate commands. In addition, we provide methods for analyzing, simplifying and manipulating gate sequences. Outputs are given in a bespoke gate list format, and can be exported as graphics, or to LATEX using the format of Q-circuit [13].

Universal QCompiler is intended to be an academic software library that focuses on simplicity and adaptability of the code and it was not our focus to optimize the (classical) run time of the decomposition methods (the theoretical decompositions mainly focused on minimizing the C-NOT count). A detailed documentation as well as an example notebook are published together with our code and should help the user to get started quickly. The aim of this paper is to give an overview over the package Universal QCompiler and to provide some theoretical background about the decomposition methods that it uses. A separate manual is provided with the package that provides more details.

^{*} itenr@ethz.ch

 $^{^{\}dagger}$ roger.colbeck@york.ac.uk

See our webpage for a reference to the github repository and the documentation: http://www-users.york.ac.uk/~rc973/ UniversalQCompiler.html.

 $^{^2 \; \}texttt{https://github.com/rogercolbeck/QI}$

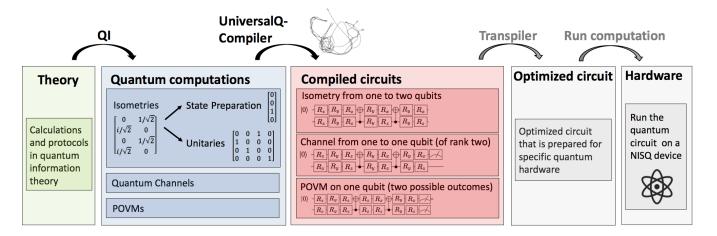


FIG. 1. Overview over the use of UniversalQCompiler. The Mathematica package QI can be used to do common computations in quantum information and manipulate quantum operations, such as unitary matrices. Given an abstract representation of a quantum computation, UniversalQCompiler takes it as an input and outputs a quantum circuit implementing the computation. Thereby, we distinguish the following classes of operations: isometries (including unitaries and state preparation as special cases), quantum channels and POVMs. The picture depicts example-circuits for each class of operations on two qubits. The circuit could then be further optimized and prepared for a specific quantum hardware architecture by an (external) transpiler. To simplify interfacing with a transpiler or a circuit optimizer (for instance the PyZX quantum circuit optimizer [3]), we provide a python script (based on ProjectQ [9, 10]) to translate the Mathematica outputs to the quantum assembly language OpenQASM.

We work with the universal gate library consisting of arbitrary single-qubit rotations and C-not gates (we also explain how to convert gate sequences from this universal set to another that comprises single-qubit rotations and Mølmer-Sørensen gates (see Appendix A), which are common on experimental architectures with trapped ions). *UniversalQCompiler* decomposes different classes of quantum operations into sequences of these elementary gates keeping the required number of gates as small as possible.

In Section II, we define the elementary gates we are working with, i.e., the single-qubit rotations and the C-NOT gate.

In Section III, we describe how to use UniversalQCompiler to decompose arbitrary isometries from m to $n \geq m$ qubits describing the most general evolution that a closed quantum system can undergo. Mathematically, an isometry from m to n qubits is an inner-product preserving transformation that maps from a Hilbert space of dimension 2^m to one of dimension 2^n . Physically, such an isometry can be thought of as the introduction of n-m ancilla qubits in a fixed state (conventionally $|0\rangle$) followed by a general n-qubit unitary on the m input qubits and ancilla qubits. Unitaries and state preparation on n qubits are two important special cases of isometries from m to n qubits, where m=n and m=0, repsectively.

In Section IV, we consider the decomposition of quantum channels from m to n qubits (no longer restricting to $m \leq n$). A quantum channel describes the most general evolution an open quantum system (i.e., a quantum system that may interact with its environment) can undergo. Mathematically, a quantum channel is a com-

pletely positive trace-preserving map from the space of density operators on m qubits to the space of density operators on n qubits. Universal QCompiler takes a mathematical description of such a quantum channel (which can be supplied in Kraus representation or as a Choi state) and returns a gate sequence that implements the channel (in general after tracing out some qubits at the end of the circuit). The decomposition is nearly optimal for generic channels working in the quantum circuit model [11]. However, working in more general models would allow further reducions in the number of gates [14]. We plan to implement code for the decompositions described in [14] in the future. For an overview of possible applications of implementing channels, see [15].

In Section V, we describe how to implement arbitrary POVMs on m qubits describing the most general measurements that can be performed on a quantum system. Similarly to the case of channels, working in generalized models can reduce the gate count further [14], and we plan to implement these in a future version. See also [16] for an application of UniversalQCompiler for synthesis of POVMs.

In Section VI, we extend from POVMs to quantum instruments. These can be thought of as the most general type of quantum measurement where we care about the post-measurement state (in contrast to a POVM where we only care about the distribution over the classical outcomes). Our decompositions for these are based on those used for channels, and again could be improved using additional methods from [14].

In Section VII, we describe some simple rules that can be used to simplify circuits and that are implemented within *UniversalQCompiler*.

Finally, in Section VIII, we explain how to automatically translate our circuits to the open quantum assembly language (OpenQASM) [17], which allows our package to interface with other quantum software packages.

II. UNIVERSAL GATE LIBRARY

Our gate library consists of arbitrary single-qubit rotations and C-NOT gates. This set of gates is known to be universal [21], i.e., any quantum computation can be decomposed into a sequence of gates in this set. We use the following convention for rotation gates

$$R_x(\theta) = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{1}$$

$$R_{x}(\theta) = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \qquad (1)$$

$$R_{y}(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \qquad (2)$$

$$R_{z}(\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}. \qquad (3)$$

$$R_z(\theta) = \begin{pmatrix} e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2} \end{pmatrix}. \tag{3}$$

Note that in [11], we used the convention $R'_x(\theta) =$ $R_x(-\theta)$, $R'_y(\theta) = R_y(-\theta)$ and $R'_z(\theta) = R_z(-\theta)$. In addition, we use the following two-qubit gate

$$C-NOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4}$$

In Appendix A, we explain how to convert gate sequences from this universal set to the one that comprises single-qubit rotations and Mølmer-Sørensen gates without increasing the number of two-qubit gates.

Note that, when displaying circuits, we use - to represent measurement in the computational basis where the classical outcome is retained, and - to represent the qubit being traced out (equivalent to measuring and forgetting the outcome).

COMPILATION OF ISOMETRIES III.

Universal QCompiler provides three different decompositions for generic isometries from m to n qubits given as a $2^n \times 2^m$ complex matrix V satisfying $V^{\dagger}V = 1$ (and an additional decomposition that only works for m = 0, i.e., for state preparation). For an overview of the gate counts and the running time complexity of the different methods, see Table I. For a comparison with a theoretical lower bound on the number of C-NOT gates, see Table I in [11]. In the following, first we give some information about the different decomposition schemes. Then we explain the methods Declsometry and DeclsometryGeneric, which choose the optimal decomposition scheme automatically. An example of an

output circuit created by Declsometry is given in Figure 2.

Column-by-column decomposition

column-by-column decomposition The (method: ColumnByColumnDec) was introduced in achieves near optimal C-NOT counts for generic isometries from m to n qubits. As the name suggests, the isometry is decomposed in a column-wise fashion (see [11] for the details). This decomposition achieves the lowest known C-NOT counts for generic isometries with $1 \le m \le n-2$. For isometries of a special form, it may also achieve lower C-NOT counts for m = 0, n - 1, n(after running the simplifications described in Section VII and removing gates that implement the identity during the decomposition). In particular, it usually performs well for isometries with many zeros, since the number of gates to decompose the columns is reduced in such cases.

The column-by-column decomposition requires 2^{m+n} C-NOTS to leading order for the decomposition of an isometry from m to n qubits. Its classical time complexity is $\mathcal{O}(n2^{2m+n})$ (see Appendix B), which scales significantly better in m than the other decomposition methods for isometries. Note also that it is straightforward to parallelize parts of the column-by-column decomposition, which may help to lower the run time significantly for practical implementations (but we have not done so in the version 0.1 of our package).

Quantum Shannon Decomposition (QSD)

The Quantum Shannon Decomposition (method: QSD) was introduced for unitaries in [18] and adapted to isometries in [11]. It achieves lower C-NOT counts than the column-by-column decomposition for generic isometries from m to n qubits with m = n - 1 or m = n, whereas the QSD is not well adapted to the case $m \ll n$. Its classical time complexity is independent of m and given by $\mathcal{O}(2^{3n})$ (see Appendix B 2).

Knill's decomposition

Knill's decomposition scheme (method: KnillDec) described in [11] and based on [19, 20] expands an isometry V to a unitary U maximizing the number of eigenvalues of U that are equal to one. The unitary U can then be decomposed into a circuit (described in [11, 19]) that requires $c \cdot (k+1)2^n + k\mathcal{O}(n^2)$ C-NOT gates for a unitary on n qubits with k eigenvalues that are not equal to one, where c = 23/24 if n is even and c = 115/96if n is odd. For a generic isometry V from m to nqubits, the unitary extension, U, can be chosen to have

TABLE I. Overview of the asymptotic number of C-NOT gates and the classical run time required to decompose m to n isometries using different decomposition schemes. Abbreviations used: ^aColumn-by-column decomposition of an isometry; ^bDecomposition of an isometry using the Quantum Shannon Decomposition; ^cState preparation.

Method	C-not count for a generic m to n isometry	Classical run time	References
CCD^a	$2^{m+n} - \frac{1}{24}2^n + \mathcal{O}\left(n^2\right)2^m$	$\mathcal{O}(n2^{2m+n})$	[11]
QSD^b	$\frac{23}{144}\left(4^m+2\cdot 4^n\right)+\mathcal{O}\left(m\right)$	$\mathcal{O}(2^{3n})$	[18],[11]
Knill	$\frac{23}{24}(2^{m+n}+2^n)+\mathcal{O}(n^2)2^m$ if <i>n</i> is even	$\mathcal{O}(2^{3n})$	[19],[11]
	$\frac{115}{96}(2^{m+n}+2^n)+\mathcal{O}(n^2)2^m$ if n is odd	$\mathcal{O}(2^{3n})$	[19],[11]
SP^c	$\frac{23}{24}2^n$ [here $m = 0$]	$\mathcal{O}(2^{rac{3n}{2}})$	[20],[11]

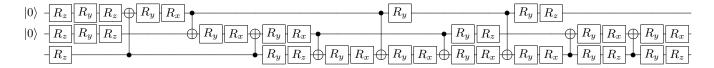


FIG. 2. Circuit for a randomly chosen isometry from one to three qubits. The two ancilla qubits are always initialized in the state $|0\rangle$, where an arbitrary state $|\psi\rangle$ is provided as an input on the least significant qubit. The output of the computation is read out from all three qubits at the end of the circuit. The circuit was produced by running st=Declsometry[PickRandomlsometry[2, 8]] in Mathematica and then calling LatexQCircuit[st] to export the circuit to LaTeX. To save space we do not depict the angles here, but these are found by our code and can be output if desired.

 2^m eigenvalues that are not equal to one and hence requires $c \cdot (2^{n+m} + 2^n) + \mathcal{O}(2^{m+n/2})$ C-not gates to leading order. However, for isometries of a special form for which the unitary extensions has fewer than 2^m eigenvalues that are not equal to one, Knill's decomposition may achieve lower C-not counts than the others (for an example, see the notebook Examples.nb that is provided together with the package). The classical time complexity of the decomposition is independent of m to leading order and given by $\mathcal{O}(2^{3n})$ (see Appendix B 3).

D. Householder decomposition

We also include a method for decomposing isometries using Householder reflections [22] (method: DenseHouseholder Dec). A Householder reflection with respect to $|v\rangle$ is a unitary of the form $1 - 2|v\rangle\langle v|$. Generalizations of these can be used to construct unitaries that map any computational basis state to a particular state. Such unitaries can then be applied in sequence to construct an isometry, each mapping one basis state to the corresponding column. Householder reflections are particularly useful for this because their successive action does not mess up previously created columns. For a generic isometry from m to n qubits, the number of C-NOTs required for this decomposition scales as $c \cdot (2^{n+m} + 2^n) + \mathcal{O}(2^{m+n/2})$, where c = 23/24 if n is even and c = 115/96 if n is odd [22]. Note that this scaling is identical to that for Knill's decomposition; the advantage of using the Householder decomposition over Knill's is only for small m and n. Our implementation of the Householder decomposition uses one ancilla qubit (which can start in any state and is returned to its initial state after the computation). The classical complexity of this decomposition is $\mathcal{O}(2^{m+n}(2^m+2^{n/2}))$ [22].

E. State preparation

For the special case of state preparation on n qubits (i.e., for an isometry from 0 to n qubits), the best known decomposition scheme is based on the Schmidt decomposition of the quantum state [20] (method: StatePreparation). The scheme was slightly improved for state preparation for an odd number of qubits in [11] leading to a C-NOT count of $23/24 \cdot 2^n$ for state preparation on n qubits. This is lower than the number of C-NOTs required to decompose an n-qubit state with uniformly controlled gates [23], which is 2^n to leading order³. However, the classical time complexity is $\mathcal{O}(2^{\frac{3n}{2}})$ (see Appendix B 4 for the details), which is worse than the complexity $\mathcal{O}(n2^n)$ for state preparation using uniformly controlled gates.

Remark (States with low Schmidt rank). The Schmidt rank of a bipartite quantum state ψ_{AB} is given by the

³ By default, the decomposition based on uniformly controlled gates [23] is used for the decomposition of the first column in the method ColumnByColumnDec. The option FirstColumn → StatePreparation allows use of the scheme based on the Schmidt decomposition [20] for its decomposition.

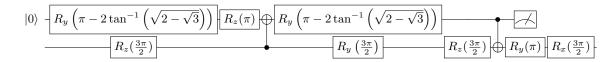


FIG. 3. Circuit for the amplitude damping channel given by Kraus operators $K_0(\gamma) = \{\{1,0\},\{0,\sqrt{1-\gamma}\}\}$ and $K_1(\gamma) = \{\{0,\sqrt{\gamma}\},\{0,0\}\}$ for $\gamma=1/3$. The ancilla qubit is always initialized in the state $|0\rangle$, where an arbitrary state $|\psi\rangle$ is provided as an input on the second qubit. The output of the computation is read out from the second qubit after tracing out the first one (i.e., after measuring the first qubit and forgetting about the classical outcome). The circuit was produced by running $st=\text{DecChannelInQCM}[\{K_0(1/3),K_1(1/3)\}]$ in Mathematica and then $\text{LatexQCircuit}[st,AnglePrecision \to 1]$ to export the circuit to LATeX.

minimal number of Schmidt coefficients required for its Schmidt decomposition. States with a small Schmidt rank correspond to weakly entangled quantum systems, and occur naturally in the study of the grounds states of certain types of Hamiltonian (see, e.g., [24]). In the future, we plan to adapt the decomposition for state preparation introduced in [20] to states with low Schmidt ranks (where the splitting of the sub-systems has to be specified). We expect this adaptation to lower the C-NOT count significantly for such states. Moreover, in the related task of approximate state preparation, one could lower the gate count by setting small Schmidt coefficients equal to zero.

F. Sparse isometries

We also have methods for implementing the decompositions of [22] that are designed for sparse isometries. These work using Householder reflections and can be seen as an adaptation of the method of Section IIID that takes advantage of sparse structure. For sparse states of n qubits with nnz non-zero entries, the decompositions result in $\mathcal{O}(n \cdot \text{nnz})$ C-nots with a classical runtime of $\mathcal{O}(\binom{n}{s} + n2^{2s})$ with $s \in \mathbb{N}$ such that $nnz \leq 2^s$ (method: SparseStatePreparation). The method is described in Section 2 and Section 5.2 in [22] and uses one ancilla. For sparse isometries from $m \geq 1$ to n qubits (method: SparseHouseholderDec) it is more difficult to be precise about the number of required C-Nots, since this number is affected by the structure of the non-zero elements in the isometry and not just the number of them. The decomposition is based on Algorithm 3 in [22] (with minor modifications) and requires one ancilla.

G. Choosing the optimal decomposition

The method Declsometry [V] decomposes the isometry V into a sequence of single-qubit rotations and C-NOT gates by running all four decompositions (and in the case m=0 also the one for state preparation) and in addition the decomposition for sparse V if the number of zeros in V is larger than $2^{m+\frac{n}{3}}$, simplifying the gate sequences using the methods described in

Section VII, and choosing the output gate sequence that achieves the lowest C-not count. To decompose a random isometry V (of high dimensions), we recommend using DeclsometryGeneric[V], which chooses the decomposition method that achieves the lowest C-not count for a generic isometry with the same dimensions as V and hence has a shorter running time compared to Declsometry[V] (since it runs only one decomposition).

As sub-routines, we use optimal decompositions of twoqubit gates [25] (method: DecUnitary2Qubits) and optimal state preparation on three qubits [26] (method: StatePrep3Qubits).

IV. COMPILATION OF QUANTUM CHANNELS

In the following, we consider the implementation of quantum channels in the quantum circuit model (method: DecChannellnQCM). For the decomposition of channels, it is most convenient to work with the Kraus representation of the channel. Every quantum channel \mathcal{E} from m to n qubits with Kraus rank K can be represented by Kraus operators A_i , which are complex matrices of dimension $2^n \times 2^m$ such that $\sum_{i=1}^K A_i^{\dagger} A_i = \mathbb{1}$ and $\mathcal{E}(\rho) = \sum_{i=1}^K A_i \rho A_i^{\dagger}$ for all density operators ρ of dimension 2^m [27]. To change the Kraus representation to a Choi state [27] or vice versa, we provide the methods KrausToChoi and ChoiToKraus, respectively.

An arbitrary channel from m to n qubits (given as a list of Kraus operators) can be provided as an input to DecChannellnQCM, which returns a gate sequence (with some trace-out operations at the end) that implements the channel (see Figure 3 for an example). The decomposition uses Stinespring's theorem [28] stating that a channel of Kraus rank $K=2^k$ can be represented by an isometry from m to n+k qubits. Then, the channel can be implemented by using one of the decomposition schemes for isometries and tracing out the ancillas at the end of the circuit. The C-NOT count for a channel from m to n qubits of Kraus rank 2^k is therefore 2^{m+n+k} to leading order. This is nearly optimal for the decomposition of generic channels in the quantum circuit model [11].

Note also that all channels from m to n qubits have a Kraus representation with at most 2^{m+n} elements. The command MinimizeKrausRank is provided to do the reduction to the minimal number of Kraus operators (which may be lower than 2^{m+n} for channels of a special form).

V. COMPILATION OF POVMs

Positive-operator valued measures (POVMs) describe the most general measurements that can be performed on quantum systems. In the following, we consider the implementation of POVMs on m qubits in the quantum circuit model (method: DecPOVMInQCM). Every POVM \mathcal{M} on m qubits with L possible measurement outcomes can be represented by L operators $0 \le E_i \le 1$ satisfying $\sum_{i=0}^{L-1} E_i = 1$. The probability of getting the i^{th} outcome when performing the POVM on an m qubit state ρ is then given by $\operatorname{tr}[E_i\rho]$.

To demonstrate the use of DecPOVMInQCM, we consider state discrimination (see for example [29] for a review). Suppose a state is chosen from a known set of (not necessarily pure) density operators $\{\phi_i\}_i$, where ϕ_i is chosen with probability p_i . The goal is to correctly guess which state was chosen, by performing a measurement on the given state. In general, the optimal strategy for such a task is to perform a (non-projective) POVM. Since it is (in general) difficult to find the optimal POVM, a "pretty good" choice was introduced in [30, 31]. Using DecPOVMInQCM we can find a quantum circuit to implement these POVM elements. Running this circuit on some quantum hardware would then give us a (classical) output that corresponds to a pretty good guess of which state was given to us. As a concrete example, we take $\phi_1 = |0\rangle\langle 0|$, $\phi_2 = |1\rangle\langle 1|$ and $\phi_3 = |+\rangle\langle+|$, where $|+\rangle := 1/\sqrt{2}(|0\rangle + |1\rangle)$, assuming that each state is chosen with the same probability $p_i=1/3$. The pretty good measurement has POVM elements $M_i=p_i\phi^{-1/2}\phi_i\phi^{-1/2}$, where $\phi=\sum_i p_i\phi_i$. Using DecPOVMInQCM gives a quantum circuit with two ancilla qubits (see Figure 4). The outcomes can be interpreted as follows: (x,y) = (0,0) corresponds to a guess that the chosen state was ϕ_1 , (x,y) = (0,1) corresponds to a guess of ϕ_2 and (x,y) = (1,0) corresponds to a guess of ϕ_3 . Note that (x,y)=(1,1) has probability zero.

VI. COMPILATION OF INSTRUMENTS

Instruments are generalizations of both channels and POVMs. Again we consider an implementation in the quantum circuit model (method: DecInstrumentInQCM). To specify an instrument, we give the Kraus operators of the subnormalized channels comprising the instrument. Consider an instrument from m to n qubits with L outcomes. This can be specified using Kraus

operators of dimension $2^n \times 2^m$. If the j^{th} subnormalized channel has Kraus representation $\{A_i^j\}_{i=1}^{K_j}$, then to be a valid instrument requires $\sum_{j=1}^L \sum_{i=1}^{K_j} (A_i^j)^\dagger A_i^j = 1$. Starting from state ρ , the outcome j occurs with probability $\operatorname{tr}(\mathcal{E}^j(\rho))$ and the post-measurement state is $\mathcal{E}^j(\rho)/\operatorname{tr}(\mathcal{E}^j(\rho))$, where $\mathcal{E}^j(\rho) = \sum_{i=1}^{K_j} A_i^j \rho(A_i^j)^\dagger$. In general, the circuit output by DecInstrumentInQCM will involve some final measurements and trace out operations, as well as some output qubits.

A circuit for an instrument can be converted to one for the corresponding POVM by tracing out all of the output qubits, although the circuit formed by this procedure may be longer than that formed by first calculating the POVM and decomposing it using DecPOVMinQCM. Furthermore, if one takes the circuit output by DecPOVMinQCM for the POVM $\{M_1, M_2, \ldots\}$ and removes all trace out operations at the end, then one gets a circuit for the instrument $\{\{\sqrt{M_1}\}, \{\sqrt{M_2}\}, \ldots\}$, i.e., where each subnormalized channel has just one Kraus operator equal to the square root of the corresponding POVM element.

VII. SIMPLIFICATIONS OF GATE SEQUENCES

Given a sequence of single-qubit rotations and C-NoT gates, it may be possible to find a shorter gate sequence that implements the same operation. We provide the method SimplifyGateList, which uses some simple rules to reduce the gate count. The number of single-qubit gates is reduced by merging single-qubit rotations in cases where more than two occur consecutively on the same qubit. The merged single-qubit unitary can then be decomposed using the following well-known Lemma [32].

Lemma 1 (ZYZ decomposition) For every unitary operation U acting on a single qubit, there exist real numbers α, β, γ and δ such that

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta). \tag{5}$$

By symmetry, Lemma 1 holds for any two orthogonal rotation axes. We decompose the merged unitary using the ZYZ decomposition if the previous C-not gate controls on the considered qubit, and we decompose it using the XYX decomposition otherwise. Since R_z gates commute with the control of C-not gates and R_x gates with the target, one of the rotation gates can be commuted to the left of the C-not, as summarized in the following circuit equivalence.

We do this procedure starting from the end of the circuit and we also cancel C-NOT gates where we have two in a row (or with commuting C-NOT gates in-between

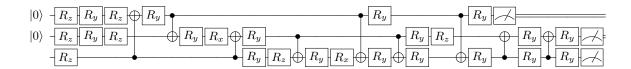
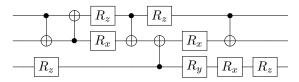


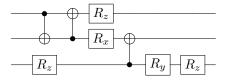
FIG. 4. Circuit for a POVM on one qubit implementing the pretty good measurement for distinguishing the states $\phi_1 = |0\rangle\langle 0|$, $\phi_2 = |1\rangle\langle 1|$ and $\phi_3 = |+\rangle\langle +|$ (the rotation angles are not depicted for simplicity). The POVM elements are given by $M_1 = 1/4 \cdot \{\{1,1\},\{1,1\}\}, M_2 = 1/8 \cdot \{\{3+2\sqrt{2},-1\},\{-1,3-2\sqrt{2}\}\}$ and $M_3 = 1/8 \cdot \{\{3-2\sqrt{2},-1\},\{-1,3+2\sqrt{2}\}\}$. The probability to measure $|i\rangle$ (with $i \in \{1,2,3\}$) on the two ancilla qubits for a given state ρ on the third qubit is given by $\operatorname{tr}(M_i\rho)$. The circuit was produced by running $st=\mathsf{DecPOVMInQCM}[\{M_1,M_2,M_3\}]$ in Mathematica and then $\mathsf{LatexQCircuit}[st]$ to export the circuit to $\mathsf{LFT}_{\mathsf{FX}}$. Again, the rotation angles are not depicted for simplicity.

them) with the same control and target. The resulting circuit contains at most four single-qubit rotations after each C-not gate. Note that to do the simplifications we have to traverse only once through the circuit, hence the classical run time of this procedure is linear in the number of gates of the circuit.

For example, the following circuit (for arbitrary rotation angles)



gets simplified to the following (by only traversing the circuit once).



VIII. EXPORTING CIRCUITS TO OpenQASM

We also provide a python script to translate the gate sequences produced by Mathematica to the QASM language [17]. The gate sequence can then be imported into for example the IBM library Qiskit and further optimized or also directly sent to quantum hardware for evaluation. The script is based on ProjectQ [9, 10] and is simple to use (see our documentation for more details).

IX. FUTURE WORK

In a future version we plan to implement code for the decompositions of quantum channels and POVMs in more general models than the quantum circuit model that allow for measurements in between the gate sequence and to classical control on the measurement results [14]. This will significantly reduce the C-NOT count for channels and POVMs. A remaining open question is how to use these decomposition as a starting point for circuit optimization. As a straightforward application, one could do peephole optimization, by taking a large circuit and extracting parts of it that act on, e.g., three qubits, and resynthesize the unitary corresponding to the circuit. If this leads to a shorter circuit, this could then replace the original. Alternatively, one could build up sets of increasingly complicated templates [1], i.e., circuits that implement the identity operator, using our universal decomposition schemes. Indeed, choosing a (parametrized) circuit, writing it as a unitary and synthesizing a new circuit for it, directly leads to a (parametrized) template. These templates could then be used to simplify parts of larger circuits.

X. ACKNOWLEDGEMENTS

We thank Thomas Häner, Dmitri Maslov and Joseph M. Renes for helpful discussions. We are grateful to the Department of Mathematics, University of York for partfunding some summer projects that enabled this work. RI acknowledges support from the Swiss National Science Foundation through SNSF project No. 200020-165843 and through the National Centre of Competence in Research Quantum Science and Technology (QSIT).

Appendix A: Transforming our gate library to one that is well adapted for trapped ions

The common universal gate set used for trapped ions consists of single-qubit gates $R(\theta, \phi)$ and the Mølmer-Sørensen gate $XX(\phi)$ (see for example [33]) defined as follows.

$$\mathbf{R}(\theta,\phi) = \begin{bmatrix} \cos(\theta/2) & -i\exp(-i\phi)\sin(\theta/2) \\ -i\exp(i\phi)\sin(\theta/2) & \cos(\theta/2) \end{bmatrix},$$

$$\mathbf{X}\mathbf{X}(\phi) = \begin{bmatrix} \cos(\phi) & 0 & 0 & -i\sin(\phi) \\ 0 & \cos(\phi) & -i\sin(\phi) & 0 \\ 0 & -i\sin(\phi) & \cos(\phi) & 0 \\ -i\sin(\phi) & 0 & 0 & \cos(\phi) \end{bmatrix}.$$

In particular, we have $R_x(\theta) = R(-\theta, 0)$ and $R_y(\theta) = R(-\theta, \pi/2)$. Having a circuit consisting of C-NOT gates, one can use the following identity to replace each C-NOT gates with a single XX gate and single qubit gates [34]. Note that this transformation does not increase the two-qubit gate count.

$$= - \begin{bmatrix} R_y(-\frac{\pi}{2}) \\ XX(\frac{\pi}{4}) \end{bmatrix} - \begin{bmatrix} R_x(\frac{\pi}{2}) \\ R_x(\frac{\pi}{2}) \end{bmatrix} - \begin{bmatrix} R_y(\frac{\pi}{2}) \\ R_y(\frac{\pi}{2}) \end{bmatrix} -$$

In the following, we show how to merge the single-qubit gates in a circuit containing XX and single-qubit rotations, such that the resulting circuit contains at most one $R(\theta, \phi)$ gate after each XX gate, and additionally a possible R_x gate on each of the qubits at the beginning of the circuit. To do so, we use that the XX gate commutes with the R_x gate (on both qubits) [34] together with the following decomposition.

Lemma 2 (R- R_x decomposition) Given a 2 × 2 unitary matrix, U, there exist reals α, θ, ϕ and δ such that

$$U = e^{i\alpha} R(\theta, \phi) R_x(\delta). \tag{A1}$$

Proof. From (the generalized) Lemma 1, it follows that there exist reals $\alpha, \beta, \gamma, \tilde{\delta}$, such that

$$U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_x(\tilde{\delta}). \tag{A2}$$

The circuit equivalence (12) in [34] implies that there exists reals θ , ϕ such that $R(\theta, \phi)R_x(-\beta)R_y(-\gamma)R_x(-\beta) = 1$, or, equivalently, $R_x(\beta)R_y(\gamma) = R(\theta, \phi)R_x(-\beta)$. It follows that

$$U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_x(\tilde{\delta}) \tag{A3}$$

$$= e^{i\alpha} R(\theta, \phi) R_x(-\beta) R_x(\tilde{\delta})$$
 (A4)

$$= e^{i\alpha} R(\theta, \phi) R_x(\delta) , \qquad (A5)$$

where $\delta := \tilde{\delta} - \beta$.

This leads to the circuit equivalence

which we can apply recursively starting at the end of the circuit as follows. We first merge all the single-qubit rotations into single-qubit unitaries before applying the above circuit equivalence at the last XX gate appearing in the circuit and merge the R_x gates into the proceeding single-qubit unitary. We then apply the circuit equivalence to the second to last XX gate in the circuit, and so on. The single-qubit unitaries that remain at the end can be written in terms of R-gates.

We provide the commands CNOTRotationsToXXR-Gates and XXRGatesToCNOTRotations to convert between circuits that use single qubit rotations and C-NOTS and those using XX and R-gates.

Appendix B: Classical time complexity for the decomposition of isometries

In this section we give some details about how to find the classical time complexity for the different decomposition schemes for isometries. Note that these complexities refer to numerical cases (not analytic calculations).

1. Classical complexity for the column-by-column decomposition

To leading order, we only have to consider the decomposition and simulation (i.e., the application to quantum states) of the uniformly controlled gates. The decomposition scheme for a uniformly controlled single-qubit gate with k controls introduced in [23] has time complexity $\mathcal{O}(k2^k)$, and computing the updated state after its application to an n-qubit state has time complexity $\mathcal{O}(2^n)$ (one has to update all the entries of the state vector in general). Hence, to update all 2^m columns of an isometry from m to n qubits has time complexity $\mathcal{O}(2^{m+n})$. Note that it is straightforward to parallelize the application of a uniformly controlled gate to the different columns of an isometry (and also the application to a single-column), which can speed up practical implementations significantly (but we do not take this into account here for the complexity measure). The complexity of decomposing one column is

$$\sum_{k=0}^{n-1} \mathcal{O}(k2^k + 2^{m+n}) = \mathcal{O}(n2^{m+n})$$

Hence, the complexity to decompose all of the 2^m columns is $\mathcal{O}(n2^{2m+n})$.

2. Classical complexity for the Quantum Shannon Decomposition

The Quantum Shannon Decomposition of an isometry from m to n qubits is based on the Cosine-Sine-Decomposition of an unitary expansion of the isometry [11, 18]. Since the unitary expansion is a matrix of dimension $2^n \times 2^n$, the time complexity to perform the Cosine-Sine-Decomposition of it is $\mathcal{O}(2^{3n})$ [35].

⁴ Decomposing the l^{th} column with the column-by-colum decomposition, the columns $1,\ldots,l-1$ are in the states $e^{i\phi_0} |0\rangle,\ldots,e^{i\phi_{l-2}}|l-2\rangle$ for some real phases ϕ_0,\ldots,ϕ_{l-2} , and hence the application of uniformly controlled gates on these columns has constant time complexity. We ignore this here, since it does not change the overall time complexity of the column-by-column decomposition.

3. Classical Complexity for Knill's decomposition

Knill's decomposition of an isometry from m to n qubits requires running several subroutines from linear algebra to find the unitary U from Lemma 5 and its eigenvalue decomposition, which is required for the decomposition (see [11, 19] and Appendix C for the details). The most time consuming operations are:

- Finding an orthonormal basis of the null space of $V^{\dagger} I_{2^m,2^n}$ of dimension $2^m \times 2^n$ in the proof of Lemma 5,
- Multiplying the matrices W and W'' of dimension $(2^n 2^m) \times (2^n 2^m)$ in the proof of Lemma 4,
- Finding the eigenvalues and eigenvectors of a $2^n \times 2^n$ unitary U in Lemma 3.

All of these operations can be implemented with time complexity $\mathcal{O}(2^{3n})$.

The time complexity of the remaining part of Knill's decomposition is dominated by the decomposition of the state preparation operations, which are denoted by V_i in Lemma 3. By Appendix B 4, state preparation (using the decomposition scheme introduced in [20]) has time complexity $\mathcal{O}(2^{3n/2})$. For Knill's decomposition, we have to perform this decomposition 2^{m+1} times, and hence the "decomposition-phase" of Knill's scheme has time complexity $\mathcal{O}(2^{m+3n/2})$.

We conclude that the whole decomposition has time complexity $\mathcal{O}(2^{3n} + 2^{m+3n/2}) = \mathcal{O}(2^{3n})$ (since $m \leq n$).

4. Classical complexity for state preparation

The method introduce in [20] requires calculating the Schmidt decomposition of the given state on n qubits as well as decomposing two unitaries on each half of the qubits. To calculate the Schmidt decomposition, one performs the singular value decomposition on a matrix of dimension $2^{\lfloor n/2 \rfloor} \times 2^{\lceil n/2 \rceil}$, which has complexity $\mathcal{O}(2^{3n/2})$ [36]. Decomposing the unitaries can also be done with complexity $\mathcal{O}(2^{3n/2})$ (see Appendix B 2). We conclude that state preparation as done in [20] has classical time complexity $\mathcal{O}(2^{3n/2})$.

Appendix C: Theoretical details required for the implementation of Knill's decomposition

In this appendix, we give some details about Knill's decomposition introduced in [19] that are required for its implementation. To help keep track of dimensions, throughout this section we will use $\{|i\rangle_D\}_{i=0}^{D-1}$ to denote an orthonormal basis for \mathbb{C}^D for any $D\in\mathbb{N}$.

Lemma 3 Let U be an $N \times N$ unitary matrix with eigendecomposition $U = \mathbb{1} + \sum_{i=0}^{t-1} (e^{i\theta_i} - 1)|v_i\rangle\langle v_i|$, with $\{|v_i\rangle\}$ orthonormal, $e^{i\theta_i} \neq 1$ and $t \geq 1$ (i.e., U has t eigenvalues that differ from 1). Then $U = \prod_{i=0}^{t-1} V_i P_i V_i^{\dagger}$, where V_i is any unitary that takes $|0\rangle$ to $|v_i\rangle$ and $P_i = \mathbb{1} + (e^{i\theta_i} - 1)|0\rangle\langle 0|$.

Proof. Write $V_i = |v_i\rangle\langle 0| + R_i$, where $R_i = \sum_{j=1}^{t-1} |r_j^i\rangle\langle j|$ so that $R_i |0\rangle = 0$. In order that V_i is unitary, we require $R_i R_i^{\dagger} = \mathbb{1} - |v_i\rangle\langle v_i|$. Then,

$$V_i P_i V_i^{\dagger} = (|v_i\rangle\langle 0| + R_i)(\mathbb{1} + (e^{i\theta_i} - 1)|0\rangle\langle 0|)(|0\rangle\langle v_i| + R_i^{\dagger})$$

$$= e^{i\theta_i}|v_i\rangle\langle v_i| + R_i R_i^{\dagger}$$

$$= \mathbb{1} + (e^{i\theta_i} - 1)|v_i\rangle\langle v_i|.$$

The product is hence U.

We now show (along the lines of [19]) that any $N \times M$ isometry can be extended to a unitary with at most M eigenvalues that differ from 1.

Lemma 4 Let X and Y be $N \times M$ matrices such that $X^{\dagger}X = Y^{\dagger}Y$. Then there exists an $N \times N$ unitary U such that UX = Y.

Proof. Let $M \leq N$, and $Y = W\Sigma V$ be the SVD of Y, with $\Sigma = \sum_{i=1}^{M} \sigma_i |i\rangle_N \langle i|_M$ where $\{\sigma_i\}$ are non-negative real numbers. Note that W is $N \times N$, V is $M \times M$ and Σ is $N \times M$.

We have $Y^{\dagger}Y = V^{\dagger}\Sigma^{\dagger}\Sigma V$. Thus $VX^{\dagger}XV^{\dagger} = \Sigma^{\dagger}\Sigma = \sum_{i}\sigma_{i}^{2}|i\rangle\langle i|_{M}$. Let $XV^{\dagger} = \sum_{i}|v_{i}\rangle\langle i|_{M}$, for some (non-normalized) vectors $\{v_{i}\}_{i=1}^{M}$; $v_{i}\in\mathbb{C}^{N}$. Then $VX^{\dagger}XV^{\dagger} = \sum_{i,j}\langle v_{i}|v_{j}\rangle|i\rangle_{M}\langle j|_{M}$, hence $\langle v_{i}|v_{j}\rangle = \sigma_{i}^{2}\delta_{ij}$. We hence define the $N\times N$ matrix $W' = \sum_{i:\sigma_{i}\neq 0}\sigma_{i}^{-1}|i\rangle_{N}\langle v_{i}|$, so that $W'XV^{\dagger} = \Sigma$. Note that $W'(W')^{\dagger} = \sum_{i:\sigma_{i}\neq 0}|i\rangle_{N}\langle i|_{N}$, and that W' can be extended to a unitary W'' without affecting its action on XV^{\dagger} . Then, if we take U = WW'', we have $UX = WW''X = W\Sigma V = Y$. The case M > N can be treated similarly. \blacksquare

Lemma 5 Let N and $M \leq N$ be positive integers and V be an $N \times M$ matrix satisfying $V^{\dagger}V = \mathbb{1}_{M}$, i.e., V is an isometry. There exists an $N \times N$ unitary matrix U such that $U | i \rangle_{N} = V | i \rangle_{M}$ for $i \in \{0, \ldots, M-1\}$, and which has at least N-M eigenvalues equal to 1.

Proof. First note that V can be written in terms of its columns $|v_i\rangle \in \mathbb{C}^N$ via $V = \sum_{i=0}^{M-1} |v_i\rangle \langle i|_M$. We can then find N-M further vectors $|v_i\rangle \in \mathbb{C}^N$ numbered from i=M to N-1 such that $\{|v_i\}\}_{i=0}^{N-1}$ is an orthonormal basis for \mathbb{C}^N . The matrix $\sum_{i=0}^{N-1} |v_i\rangle \langle i|_N =: \tilde{U}$ is then a unitary satisfying $\tilde{U}|i\rangle_N = V|i\rangle_M$ for $i \in \{0,\ldots,M-1\}$. We need to show that it is always possible to choose the set $\{|v_i\rangle\}_{i=M}^{N-1}$ such that U has at least N-M eigenvalues equal to 1. Note that this is equivalent to U^{\dagger} having at least N-M eigenvalues equal to 1.

Let us write $U^\dagger = \begin{pmatrix} V^\dagger \\ W \end{pmatrix}$, where W is $(N-M) \times N$ so that for any $N \times K$ matrix X for some positive integer $K,\ U^\dagger X = \begin{pmatrix} V^\dagger X \\ W X \end{pmatrix}$. Note that, by unitarity, $VV^\dagger + W^\dagger W = \mathbbm{1}_N$.

 $V^{\dagger}-I_{M,N}$ has dimension $M\times N$ and hence its null space dimension q is at least N-M (here $I_{M,N}$ denotes the $M\times N$ matrix $I_{M,N}=\sum_{i=0}^{M-1}|i\rangle_M\langle i|_N$). Let us take $|f_i\rangle\in\mathbb{C}^N$ to be an orthonormal basis for this null space for $i\in\{0,\ldots,q-1\}$ so that $V^{\dagger}|f_i\rangle=|f_i\rangle$ for $i\in\{0,\ldots,q-1\}$. Consider now the $N\times(N-M)$ matrix $X=\sum_{i=0}^{q-1}|f_i\rangle\langle i|_{N-M}$. We can rewrite X in terms of its rows as $X = \sum_{i=1}^N |i\rangle_N \langle x_i|$, where $|x_i\rangle \in \mathbb{C}^{N-M}$ and divide this into X_1 and X_2 , where X_1 comprises the first M rows, and X_2 the remaining N-M rows (e.g., $X_1 = \sum_{i=0}^{M-1} |i\rangle_N \langle x_i|$). By construction, $V^\dagger X = I_{M,N} X = X_1$ and hence $X_1^\dagger X_1 = X^\dagger V V^\dagger X$. Furthermore, $\mathbbm{1}_{N-M} = X^\dagger X = X_1^\dagger X_1 + X_2^\dagger X_2$ and hence $X^\dagger W^\dagger W X = X^\dagger (\mathbbm{1} - V V^\dagger) X = X^\dagger X - X_1^\dagger X_1 = X_2^\dagger X_2$.

Since there is unitary freedom in W, it follows from Lemma 4 that it can be chosen such that $WX = X_2$. With this choice, $U^{\dagger}X = X$, and hence the N-M columns of X are eigenvectors of U^{\dagger} with eigenvalue 1.

- D. Maslov, G. W. Dueck, D. M. Miller, and C. Negrevergne, "Quantum circuit simplification and level compaction," IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems 27, 436–444 (2008).
- [2] Y. Nam, N. J. Ross, Y. Su, A. M. Childs, and D. Maslov, "Automated optimization of large quantum circuits with continuous parameters," npj:Quantum Information 4, 23 (2018).
- [3] R. Duncan, A. Kissinger, S. Perdrix, and J. van de Wetering, "Graph-theoretic simplification of quantum circuits with the ZX-calculus," Quantum 4, 279 (2020).
- [4] M. Siraichi, V. F. D. Santos, C. Collange, and F. M. Q. Pereira, "Qubit allocation," in CGO 2018 - International Symposium on Code Generation and Optimization (2018) pp. 113–25.
- [5] A. Zulehner, A. Paler, and R. Wille, "An efficient methodology for mapping quantum circuits to the IBM QX architectures," IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems 38, 1226–1236 (2019).
- [6] W. Hattori and S. Yamashita, "Quantum circuit optimization by changing the gate order for 2D nearest neighbor architectures," in *Reversible Computation*, Lecture Notes in Computer Science, Vol. 11106, edited by J. Kari and I. Ulidowski (Springer, 2018) pp. 228–243.
- [7] G. Li, Y. Ding, and Y. Xie, "The laws of physics and cryptographic security," e-print arXiv:1809.02573 (2018).
- [8] K. Smith, M. Soeken, B. Schmitt, G. D. Micheli, and M. Thornton, "Using ZDDs in the mapping of quantum circuits," in *Proceedings 16th International Conference* on Quantum Physics and Logic (2019) pp. 106–118.
- [9] D. S. Steiger, T. Häner, , and M. Troyer, "ProjectQ: An open source software framework for quantum computing," Quantum 2, 49 (2018).
- [10] T. Häner, D. S. Steiger, K. M. Svore, and M. Troyer, "A software methodology for compiling quantum programs," Quantum Science and Technology, 020501 (2018).
- [11] R. Iten, R. Colbeck, I. Kukuljan, J. Home, and M. Christandl, "Quantum circuits for isometries," Phys. Rev. A 93, 032318 (2016).
- [12] P. W. Shor, "Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer," SIAM Journal on Computing 26, 1484–1509 (1997).

- [13] B. Eastin and S. T. Flammia, "Q-circuit tutorial," e-print quant-ph/0406003 (2004).
- [14] R. Iten, R. Colbeck, and M. Christandl, "Quantum circuits for quantum channels," Physical Review A 95, 052316 (2016).
- [15] F. Ticozzi and L. Viola, "Quantum and classical resources for unitary design of open-system evolutions," Quantum Science and Technology 2, 034001 (2017).
- [16] H. Chen, L. Wossnig, S. Severini, H. Neven, and M. Mohseni, "Universal discriminative quantum neural networks," Quantum Machine Intelligence 3, 1 (2021).
- [17] A. W. Cross, L. S. Bishop, J. A. Smolin, and J. M. Gambetta, "Open quantum assembly language," e-print arXiv:1707.03429 (2017).
- [18] V. V. Shende, S. S. Bullock, and I. L. Markov, "Synthesis of quantum-logic circuits," IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems 25, 1000–1010 (2006).
- [19] E. Knill, "Approximation by quantum circuits," e-print arXiv:quant-ph/9508006 (1995).
- [20] M. Plesch and Č. Brukner, "Quantum-state preparation with universal gate decompositions," Phys. Rev. A 83, 032302 (2011).
- [21] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, "Elementary gates for quantum computation," Phys. Rev. A 52, 3457–3467 (1995).
- [22] E. Malvetti, R. Iten, and R. Colbeck, "Quantum circuits for sparse isometries," Quantum 5, 412 (2021).
- [23] V. Bergholm, J. J. Vartiainen, M. Möttönen, and M. M. Salomaa, "Quantum circuits with uniformly controlled one-qubit gates," Phys. Rev. A 71, 052330 (2005).
- [24] M. B. Hastings, "An area law for one-dimensional quantum systems," Journal of Statistical Mechanics: Theory and Experiment 2007, P08024–P08024 (2007).
- [25] V. V. Shende, S. S. Bullock, and I. L. Markov, "Recognizing small-circuit structure in two-qubit operators," Phys. Rev. A 70, 012310 (2004).
- [26] O. Giraud, M. Žnidarič, and B. Georgeot, "Quantum circuit for three-qubit random states," Phys. Rev. A 80, 042309 (2009).
- [27] M.-D. Choi, "Completely positive linear maps on complex matrices," Linear Algebra and its Applications 10, 285–290 (1975).
- [28] W. F. Stinespring, "Positive functions on C^* -algebras,"

- Proceedings of the American Mathematical Society 6, 211-216 (1955).
- [29] S. M. Barnett and S. Croke, "Quantum state discrimination," Advances in Optics and Photonics 1, 238–278 (2009).
- [30] V. P. Belavkin, "Optimal multiple quantum statistical hypothesis testing," Stochastics 1, 315–345 (1975).
- [31] P. Hausladen and W. K. Wootters, "A 'pretty good' measurement for distinguishing quantum states," Journal of Modern Optics 41, 2385–2390 (1994).
- [32] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press,

- 2000).
- [33] D. Hanneke, J. P. Home, J. D. Jost, J. M. Amini, D. Leibfried, and D. J. Wineland, "Realization of a programmable two-qubit quantum processor," Nature Physics 6, 13–16 (2010).
- [34] D. Maslov, "Basic circuit compilation techniques for an ion-trap quantum machine," New Journal of Physics 19, 023035 (2017).
- [35] B. D. Sutton, "Computing the complete CS decomposition," Numerical Algorithms 50 (2009), 10.1007/s11075-008-9215-6.
- [36] G. H. Golub and C. F. Van Loan, Matrix Computations, 4th ed. (John Hopkins University Press, 2013).