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New look on quantum representation of images: Fourier transform representation

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Abstract
Image representation plays an essential role in quantum image processing and quantum computer vision in numerous, computationally expensive applications. Quantum image processing is a new discipline with tremendous potential. However, there is a critical problem in applying quantum computing to process data: how to represent data (signal, image, and video data) using quantum states without losing information. In this paper, we describe briefly a few known models for image representation and propose a new approach for representing discrete signals and images in quantum computing, by mapping the input data into the unit circle, or only part of the circle. Such a representation allows for introducing the concept of the Fourier transform qubit representation. For grayscale images, we consider the similar concept of the Fourier representation of images and, for color images, we introduce models with the concept of the 3-point DFT of color qubits. The circuits for proposed signal and image representations are described.

Keywords Quantum image representation · Quantum image processing · Quantum computing · Fourier transform qubit representation

1 Introduction

Many quantum image processing algorithms are based on the classical theory and methods of image processing that are well developed today. Therefore, much attention is paid to the issue of representing images in quantum calculations. This is exactly the bridge that needs to be transferred from classical theory to quantum theory of image processing. Such representations should be developed, analyzed, and united in order

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to select a unified format in the future or several such ones in quantum imaging, such as the well-known formats in the RGB, CMYK, XYZ color models. We believe that such unified formats will facilitate research work in the field of quantum visualization. It is also expected that algorithms specific to quantum computing will be developed in the future, and this will open a new page in image processing. Thus, the image representation in quantum space is the first step in processing images in quantum algorithms. In last decades, different approaches were proposed for quantum image representation [1–3], which include the qubit lattice model (QLM) [4], the real ket model (RKM) [5], the flexible representation for quantum images (FRQI) [6], the novel enhanced quantum representation (NEQR) [7], the normal arbitrary superposition state (NASS) and its version with three components (NASSTC) [8, 9], and the generalized quantum image representation (GQIR) [10, 11]. In our opinion, the above-mentioned and many other known methods of image representation in quantum computing can be divided into the following two classes. The first class includes representations of the image in which the basis states and single qubit or a few qubits that are assigned to each pixel to carry color information are interconnected by the tensor product in superposition. The amplitudes of states of a multi-qubit superposition of color images can be determined in various ways. The QLM, FRQI, NEQR, and GQIR are methods of the first class. In the second class of image representation, the information of the image at each pixel (n, m) is stored in amplitudes of the basis states $|n, m\rangle$ in the standard quantum representation. The NASS and RKM are examples in this class. In NASSTC, such quantum representation is used for each color component and then is united with a 2-qubit state.

In this paper, we focus only on the quantum image representation, not quantum image processing. We consider the concept of the quantum pixel, which is used in many models in quantum imaging, and then describe briefly a few known models for image representation. In addition, we present a new approach for representing discrete signals and images in quantum computing, by mapping the input data into the unit circle. In the above classification, the presented method belongs to the second class of algorithms. For discrete signals, such representation allows us to describe the Fourier transform qubit representation. For grayscale images, we consider the similar concept of the Fourier representation of images and, for color images in the RGB model [12], we introduce the concept of the 3-point DFT of color qubits. The properties of such Fourier representation are described. This representation differs from the concept of the quantum Fourier transform (DFT) that is applied to amplitudes of the states of superposition [13, 14].

2 Image representation: quantum pixel

In this section, we consider the concept of quantum pixel [15] for a discrete image $f = \{f_{n,m}\}$ of size $N \times M$, $N = 2^r$, $M = 2^s$, $r, s > 1$, with the range of intensities in

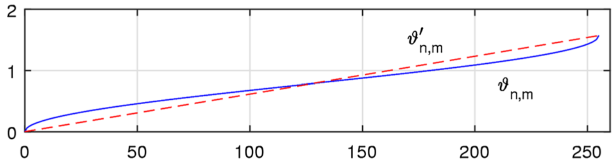

Fig. 1 The graphs of two mappings of values of intensities into the angles

the interval of integers [0, 255]. The quantum pixel (n, m) is defined by the following transform to the state of superposition of a single qubit:

$$f_{n,m} \to \left|f_{n,m}\right\rangle = \frac{1}{\sqrt{255}}\left[\sqrt{255 - f_{n,m}}|0\rangle + \sqrt{f_{n,m}}|1\rangle\right]. \tag{1}$$

Considering the basis states $|0\rangle$ and $|1\rangle$ as the black and white colors, respectively, the color black will be obtained with the probability $\left(1 - f_{n,m}\right)$ and the white color with the probability $f_{n,m}$, when measuring this qubit. Defining the angle

$$\vartheta_{n,m} = \cos^{-1}\sqrt{1 - \frac{f_{n,m}}{255}}, \tag{2}$$

the quantum pixel state can be written as the qubit

$$\left|f_{n,m}\right\rangle = \cos\vartheta_{n,m}|0\rangle + \sin\vartheta_{n,m}|1\rangle. \tag{3}$$

Some redundancy takes place in this qubit in the sense that the same information, the angle $\vartheta_{n,m}$, is recorded in the amplitudes of both states $|0\rangle$ and $|1\rangle$ in the pixel. The values of the image are mapped into the interval $[0, \pi/2]$ angles. Such mapping is shown in Fig. 1 together with the simple linear transform

$$\vartheta'_{n,m} = \frac{\pi}{2}\frac{f_{n,m}}{255}, \quad f_{n,m} \in \{0, 1, 2, \ldots, 255\},$$

which can also be considered for angles in qubit representation (3).

Thus, one qubit can be assigned to each pixel. The superposition state of NM quantum pixels for the entire image can be presented by the following state of $(r + s + 1)$ qubits:

$$\begin{aligned}\left|\check{f}\right\rangle &= \frac{1}{\sqrt{NM}}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\left|f_{n,m}\right\rangle \otimes |n, m\rangle \\ &= \frac{1}{\sqrt{NM}}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\left[\cos\vartheta_{n,m}|0\rangle + \sin\vartheta_{n,m}|0\rangle\right] \otimes |n, m\rangle.\end{aligned} \tag{4}$$

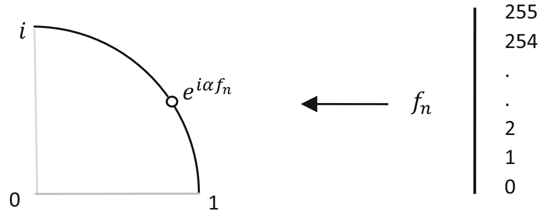

Fig. 2 Transformation of the integer interval [0,255] into the quarter circle

Here, the operation $\otimes$ denotes the tensor product. For integers $n = 0, 1, \ldots, N-1$ and $m = 0, 1, \ldots, M-1$, the states $|n\rangle$ and $|m\rangle$ are the quantum computational basis states which are written in their binary forms, and

$$|n\rangle|m\rangle = |n, m\rangle = |n\rangle \otimes |m\rangle = |n_{r-1}n_{r-2}\ldots n_1 n_0\rangle \otimes |m_{s-1}m_{s-2}\ldots m_1 m_0.\rangle \quad (5)$$

If it were possible to measure all states of this superposition $\left|\check{f}\right\rangle$, we could compose a binary image with intensities 0 or 1 in each pixel, when measuring $|0\rangle$ or $|1\rangle$, respectively. The image representation in Eq. 4 requires $(r+s+1)$ qubits; $(r+s)$ qubits for all pixel coordinates paired with one additional qubit, which carries information of the image intensity in the form of angles (Fig. 2).

The state of pixel (n, m) can also be defined as

$$q_{n,m} = a_{n,m}|n\rangle|m\rangle, \quad (6)$$

where the coefficient $a_{n,m}$ is the value $f_{n,m}$ of the image after the normalization,

$$a_{n,m} = \frac{f_{n,m}}{E}, \quad E = \sqrt{\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f_{n,m}^2}. \quad (7)$$

Therefore, the image can be presented by the NM-quantum superposition state

$$\left|\check{f}\right\rangle = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} q_{n,m} = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} a_{n,m}|n, m\rangle. \quad (8)$$

The representation requires $(r+s)$ qubits, while the discrete image with 8-bit intensities uses $8NM$ bits. This representation is similar to the image representation in the classical computer when the image value $f_{n,m}$ or $a_{n,m}$ is written to a table with the location (n, m), which is indicated by the state $|n, m\rangle$ in the quantum representation.

To describe this quantum image representation in more detail, we consider for simplifying the calculation of the similar concept in the one-dimensional (1-D) case. Let $f = (f_0, f_1, f_2, \ldots, f_{N-1})$ be the signal of length $N = 2^r$, $r > 1$. This signal can be associated in the r-qubit state space as the N-dimensional state qubit in the standard basis $\{|n, n\rangle = 0 : (N-1)\}$, i.e.,

$$\left|\check{f}\right\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + \cdots + a_{N-1}|N-1\rangle, \quad (9)$$

with the condition that $|a_0|^2+|a_1|^2+|a_2|^2+\cdots+|a_{N-1}|^2=1$. In such a probabilistic representation, the coefficients can be calculated as

$$a_n = \frac{f_n}{E}, \quad n = 0:(N-1), \quad \text{where } E = E(f) = \sqrt{\sum_{k=0}^{N-1} |f_k|^2}. \tag{10}$$

Let us analyze this representation in the example with a real-valued signal of length $N = 8$.

Example 1 We consider the real signal or vector $f = (f_0, f_1, f_2, \ldots, f_7) = (2, 1, 0, 1, 2, 4, 3, 1)$ and its quantum superposition state

$$\left|\check{f}\right\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + \cdots + a_7|7\rangle. \tag{11}$$

The energy of the signal $E^2(f) = 4+1+0+1+4+16+9+1 = 36$, and $E(f) = 6$. Therefore,

$$\left|\check{f}\right\rangle = \frac{2}{6}|0\rangle + \frac{1}{6}|1\rangle + 0|2\rangle + \frac{1}{6}|3\rangle + \frac{2}{6}|4\rangle + \frac{4}{6}|5\rangle + \frac{3}{6}|6\rangle + \frac{1}{6}|7\rangle. \tag{12}$$

Thus, the probability amplitudes of this state $\left|\check{f}\right\rangle$ are $a_0 = a_4 = 1/3$, $a_1 = a_3 = a_7 = 1/6$, $a_2 = 0$, $a_5 = 2/3$, and $a_6 = 1/2$. One measurement of $\left|\check{f}\right\rangle$ gives only one state $|k\rangle$, $k \in \{0, 1, 2, \ldots, 7\}$, with probability a_k^2.

The following question arises. The above superposition of states is considered as a probabilistic representation of the signal f. Each coefficient a_n in (12) determines the probability that the 3-qubit state vector after measurement may be in one of the basis states $|n\rangle$. For instance, the measurement may be in $|0\rangle$ or $|4\rangle$ with the probability 1/9. Also, it may be in states $|1\rangle$, $|3\rangle$, or $|7\rangle$ with the same probability 1/36, or be in state $|2\rangle$ with zero probability or never measured. All these coefficients describe the values of the signal f, and in the 3-qubit state $\left|\check{f}\right\rangle$ these values determine the probability of the measurement of $\left|\check{f}\right\rangle$, up to the constant 1/6. It is clear that, by representing the real signal in quantum state space in such a way, we create the vector that is not a correct reflection of the real signal. When a signal is processed in the classical computer, all values of the signal are treated in the same way. Why to values of the signal at points 0 and 4 should be given twice as likely to see (or measure) them at $\left|\check{f}\right\rangle$, than the values at points 1, 3 and 7? The zero amplitude a_2 means that the basis state $|2\rangle$ cannot be measured or measured with zero probability. In general, zero values may be in many points of the signal f and, as the state $|2\rangle$ in the above example, they may be lost in measurements of $\left|\check{f}\right\rangle$.

Now, let us consider the new signal $g = f + 1 = (f_0 + 1, f_1 + 1, f_2 + 1, \ldots, f_7 + 1) = (3,2,1,2,3,5,4,2)$ and its 3-qubit state

$$\left|\check{g}\right\rangle = b_0|0\rangle + b_1|1\rangle + b_2|2\rangle + \cdots + b_7|7\rangle. \tag{13}$$

Table 1 Probabilities of states representing the signals f, (f + 1), and (f − 1)

	$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$	$\|5\rangle$	$\|6\rangle$	$\|7\rangle$
$\check{f}$	0.1111	0.0278	0	0.0278	0.1111	0.4444	0.2500	0.0278
$\check{f}+1$	0.1246	0.0554	0.0138	0.0554	0.1246	0.3460	0.2215	0.0554
$\check{f}-1$	0.0625	0	0.0625	0	0.0625	0.5625	0.2500	0

The energy of the signals $E(g) = 6\sqrt{2}$ and $E(f) = 6$. Therefore, the representation of the new signal in quantum state space is

$$\left|\check{g}\right\rangle = \frac{3}{E}|0\rangle + \frac{2}{E}|1\rangle + \frac{1}{E}|2\rangle + \frac{2}{E}|3\rangle + \frac{3}{E}|4\rangle + \frac{5}{E}|5\rangle + \frac{4}{E}|6\rangle + \frac{2}{E}|7\rangle. \tag{14}$$

Thus, the new amplitudes b_n are greater or smaller than amplitudes of $\left|\check{f}\right\rangle$; $b_n = (a_n + 1/6)/\sqrt{2}$, $n = 0:7$.

For the signal $g = f - 1$ =(1,0,-1,0,1,3,2,0), the energy equals $E(g) = 4$. Therefore, the representation of the new signal in quantum state space is

$$\left|\check{g}\right\rangle = \frac{1}{4}|0\rangle + 0|1\rangle - \frac{1}{4}|2\rangle + 0|3\rangle + \frac{1}{4}|4\rangle + \frac{3}{4}|5\rangle + \frac{2}{4}|6\rangle + 0|7\rangle. \tag{15}$$

Three amplitudes of the new 3-qubit vector increase, three states have amplitudes 0, and one amplitude has not been changed. The measurement of the 3-qubit vector with the probability 0 may be in the states $|1\rangle$, $|3\rangle$, and $|7\rangle$, i.e., they may never occur in measurements of this qubit vector. The probabilities (i.e., the squares of amplitudes) of the 3-qubit vectors representing the above three signals f, $(f+1)$, and $(f-1)$ are given in Table 1.

The operations $f \pm 1$ are point-wise. Because of equation $|a_0|^2+|a_1|^2+\cdots+|a_7|^2=1$, the increase or decrease of any coefficient $|a_k|$ in the basis state $|k\rangle$ of superposition results in decrease or increase of one or a few amplitudes of other states. In addition, it should be noted that since the amplitudes of states are normalized, the same quantum representation holds for signals $f, 2f, 3f, 2/3f, \ldots$. In other words, the superposition $\left|\check{f}\right\rangle = \left|\left(2\check{f}\right)\right\rangle = \left|(3\check{f})\right\rangle = \left|\left(2/3\check{f}\right)\right\rangle$. Thus, simple arithmetic operations are not identified by similar operations in the above quantum representations of signals (and images).

3 New multi-qubit representation

Many effective quantum signal and image representations were developed [1, 2, 4], and a few such representations are given in Sect. 4. Our vision on this problem can be described as follows. We may think first converting the signal into another one and then representing it as a multi-qubit state. It is assumed that all signal/image values are valuable and counted in the same way, and in measurements they will have the same probability of representing the image. As an example, we consider the following transformation of the real signal in the range of [0, 255]; 8-bit intensity is the standard format for many grayscale images. Let f_n be the signal of length $N = 2^r$, $r > 1$, that is transformed as

$$f_n \rightarrow T[f_n] = e^{i2\pi f_n/4\times 256}, n = 0 : (N-1). \tag{16}$$

Values of the signal $T[f_n]$ are different. The additional factor of 4 is used to have all values of the transform in the first quarter circle (see Fig. 2). Defining the constant $\alpha = 2\pi/1024$, the transform can be written as $T[f_n] = e^{i\alpha f_n}$.

Now, we consider the representation of the signal in the form of the following r-qubit state:

$$\left|\check{f}\right\rangle = \frac{1}{\sqrt{N}}\Big[e^{i\alpha f_0}|0\rangle + e^{i\alpha f_1}|1\rangle + e^{i\alpha f_2}|2\rangle + \cdots + e^{i\alpha f_{N-1}}|N-1\rangle\Big]. \tag{17}$$

In this representation, which we call *the Fourier transform qubit representation (FTQR)*. The N-point DFT of the signal f_n at frequency-point $p = 1$ is defined as

$$F_p = \sum_{n=0}^{N-1} \phi_p(n) f_n = \sum_{n=0}^{N-1} \left(e^{-i2\pi/Npn}\right) f_n.$$

The base functions are $\phi_p(n) = e^{-i2\pi/Nnp}$, $p = 0, 1, \ldots, N-1$. In quantum representation, the states $|n\rangle$ are the basis states. Thus, in the sum defining the component F_p, we need to swap the time point n with f_n, as $(e^{-i2\pi/Npf_n})|n\rangle$. In addition, it is enough to consider the sum (superposition) of such states only for the case $p = 1$, and we can use other constants instead of $\alpha = 2\pi/N$ in the basis exponential function. Therefore, the representation in Eq. 17 is called the FTQR of the signal.

All states of the image in FTQR have the same probability

$$\left|\frac{1}{\sqrt{N}} e^{i\alpha f_k}\right|^2 = \frac{1}{N}, \quad k = 0 : (N-1).$$

The representation of the signal f requires r qubits, as in the representation written in (9),

$$\left|\check{f}\right\rangle = \frac{1}{E}(f_0|0\rangle + f_1|1\rangle + f_2|2\rangle + \cdots + f_{N-1}|N-1\rangle),$$

where the coefficients of the basis states are the normalized values of the signal. The amplitudes of states are normalized by the square root of the signal energy, $E^2 = f_0^2 + f_1^2 + \cdots + f_{N-1}^2$.

Example 2 The signal $f = (f_0, f_1, f_2, \ldots, f_7) = (2,1,0,1,2,4,3,1)$ with 3-bit values can be written in the Fourier transform qubit representation as

$$\left|\check{f}\right\rangle = \frac{1}{\sqrt{8}}\Big[e^{i\alpha 2}|0\rangle + e^{i\alpha}|1\rangle + |2\rangle + e^{i\alpha}|3\rangle + e^{i\alpha 2}|4\rangle + e^{i\alpha 4}|5\rangle + e^{i\alpha 3}|6\rangle + e^{i\alpha}|7\rangle\Big]$$
$$= \frac{1}{\sqrt{8}}\Big[|2\rangle + e^{i\alpha}(|1\rangle + |3\rangle + |7\rangle) + e^{i\alpha 2}(|0\rangle + |4\rangle) + e^{i\alpha 3}|6\rangle + e^{i\alpha 4}|5\rangle\Big].$$

Here, the constant $a = 2\pi/(4 \times 8) = \pi/16$.

Properties:

1. (*Inverse transform*) The value f_k of the signal can be reconstructed from the amplitude of state of measurement $|k\rangle$ as

$$\frac{1}{\sqrt{N}}e^{i\alpha f_k} \rightarrow C_k = \cos(\alpha f_k) \rightarrow f_k = \frac{1}{\alpha}\cos^{-1}(C_k). \tag{18}$$

2. (*Constant Adding*) When adding a constant to the signal, for instance $f_n \rightarrow f_n+1$, while preserving the range of the signal, the qubit superposition states are changed as

$$\left|\check{f}+1\right\rangle = \frac{1}{\sqrt{N}}\Big[e^{i\alpha f_0}|0\rangle + e^{i\alpha f_1}|1\rangle + e^{i\alpha f_2}|2\rangle + \cdots + e^{i\alpha f_{N-1}}|N-1\rangle\Big]e^{i\alpha}$$
$$= e^{i\alpha}\left|\check{f}\right\rangle,$$

and when considering the signal $f_n - 1$, its r-qubit state is

$$\left|\check{f}-1\right\rangle = \frac{1}{\sqrt{N}}\Big[e^{i\alpha f_0}|0\rangle + e^{i\alpha f_1}|1\rangle + e^{i\alpha f_2}|1\rangle + \cdots + e^{i\alpha f_{N-1}}|N-1\rangle\Big]e^{-i\alpha}$$
$$= e^{-i\alpha}\left|\check{f}\right\rangle.$$

In general, given a constant A, the qubit representation of the new signal $g_n = f_n + A$ equals

$$\left|\check{g}\right\rangle = e^{i\alpha A}\left|\check{f}\right\rangle. \tag{19}$$

3 (*Amplification of the signal*) Given a constant B, the representation of the signal $g_n = Bf_n$ is

$$\left|\check{g}\right\rangle = \frac{1}{\sqrt{N}}\Big[e^{i\alpha Bf_0}|0\rangle + e^{i\alpha Bf_1}|1\rangle + e^{i\alpha Bf_2}|2\rangle + \cdots + e^{i\alpha Bf_{N-1}}|N-1\rangle\Big].$$

The coefficients of this new r-qubit state are

$$e^{i\alpha Bf_k} = \left(e^{i\alpha f_k}\right)^B, k = 0, 1, \ldots, (N-1). \tag{20}$$

4. (*Sum of signals*) The qubit representation of the sum of two signals $(f_n + g_n)$ equals

$$\left|\widetilde{f+g}\right\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{i\alpha(f_k+g_k)}|k\rangle$$

and the coefficients of states of the sum equal the products of coefficients of qubits $\left|\check{f}\right\rangle$ and $\left|\check{g}\right\rangle$

$$e^{i\alpha(f_k+g_k)} = e^{i\alpha f_k} e^{i\alpha g_k},\ k = 0, 1, \ldots, (N-1). \tag{21}$$

5. (*Shift of the signal*) The circular shift of the signal $f_n \to f_{n-1 mod N}$ changes the qubit representation of the signal as follows:

$$\begin{aligned} &e^{i\alpha f_n}|n\rangle \to e^{i\alpha f_{n-1}}|n\rangle, \quad n = 1, 2, \ldots, (N-1), \\ &e^{i\alpha f_0}|0\rangle \to e^{i\alpha f_{N-1}}|0\rangle. \end{aligned}$$

Preparation for FTQR. The algorithm of the FTQR of the signal f_n can be described as follows.

1. Apply r Hadamard gates H, i.e., the Hadamard transform $H^{\otimes r}$ on r qubits, all initialized to $|0\rangle$, to obtain the equal superposition of N basis states,

$$|\psi\rangle = H^{\otimes r}|0\rangle^{\otimes r} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle. \tag{22}$$

2. Apply the phase shift gates ($\phi = \alpha f_k$) to the basis states $|k\rangle$, $k = 0 : N-1$),

$$|k\rangle \quad \overset{\phi}{\text{———}\bullet\text{———}} \quad e^{i\phi}|k\rangle \tag{23}$$

$$|\psi\rangle \to \left|\check{f}\right\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{i\alpha f_k}|k\rangle, \quad \text{where } \alpha = 2\pi/1024.$$

The operation with phase gates can be described by the corresponding diagonal $N \times N$ matrices

$$S_k = \mathrm{diag}\{s_{i,i}\}_{i=0:(N-1)}, \quad s_{i,i} = 1, \quad \text{if } i \neq k, \text{ and } s_{k,k} = e^{i\alpha f_k}. \tag{24}$$

Thus,

$$\left(\prod_{k=0}^{N-1} S_k\right)|\psi\rangle = \left|\check{f}\right\rangle. \tag{25}$$

The circuit for implementation of the FTQR is shown in Fig. 3.

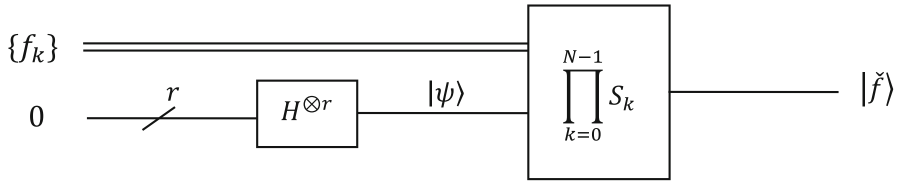

Fig. 3 The circuit for the FTQR of the signal

4 Quantum representations of images

In this section, we describe a few known qubit representations of images in Cartesian system of coordinates and, then, present the Fourier transform qubit representation of grayscale and color images. Different representations of images in qubits have been proposed in the past decade, and many of them are similar in terms of what they represent as a qubit in each pixel. It is common to describe a single qubit

$$|\varphi\rangle = c_0|0\rangle + c_1|1\rangle = \cos\vartheta|0\rangle + e^{i\gamma}\sin\vartheta|1\rangle \tag{26}$$

as the state, or a point in the unit sphere [16]. Here, ϑ and γ are angles from the interval $[0, \pi/2]$. The coefficients c_0 and c_1 can be selected in different ways, but with the required condition that $|c_0|^2+|c_1|^2= 1$.

A. In the qubit lattice model (QLM) of images, a representation that is similar to Eq. 26 is used for pixels [4]. The grayscale or color image $f = f_{n,m}$ of size $N \times M$ pixels is presented by the matrix with states

$$\boldsymbol{Q} = \left\{\cos\vartheta_{n,m}|0\rangle + e^{i\gamma}\sin\vartheta_{n,m}|1\rangle;\quad n = 0 : (N-1), m = 0 : (M-1)\right\}. \tag{27}$$

Here, the values of grays or colors are written/encoded in angles ϑ_{nm}, and γ is a constant. The quantum state of such data can be written as

$$\left|\check{f}\right\rangle = \frac{1}{\sqrt{NM}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(\cos\vartheta_{n,m}|0 + e^{i\gamma}\sin\vartheta_{n,m}|1\right)\otimes|n,m\rangle. \tag{28}$$

The phase factor $e^{i\gamma}$ is added to the qubit state for each pixel. This representation requires $(r + s + 1)$ qubits, as in the model described in Eq. 4 with quantum pixels.

B. We mention the flexible representation for quantum images (FRQI) [6]. In the FRQI model, the discrete image of size $2^r \times 2^r$ pixel is written into a 4^r-dimensional vector and then presented as the following 4^r-qubit state:

$$\left|\check{f}\right\rangle = \frac{1}{2^r}\sum_{n=0}^{4^r-1}(\cos\vartheta_n|0\rangle + \sin\vartheta_n|1\rangle)\otimes|n\rangle, \tag{29}$$

where $|n\rangle$ are computational basis quantum states and the information of image colors is written into angles ϑ_n. This qubit representation is similar to the representation in Eq. 28 when $\gamma = 0$. The FRQI is similar to the model with the quantum pixels, which is given in Eq. 4. The same redundancy occurs in a single qubit carrying color information.

C. The concept of the quantum pixel in a color image in the RGB model can also be used as proposed in [17]. At each pixel number n, the two components, for instance, the red and green components, are normalized and then moved into the interval $[-1, 1]$, as $r \to 2r - 1$ and $g \to 2g - 1$. The corresponding angles are calculated by $\varphi_r = \sin^{-1}(2r - 1)$ and $\varphi_g = \sin^{-1}(2g - 1)$. Thus, the data of the red and green components are moved to angles in the interval $[-\pi/2, \pi/2]$. The qubit pixel is defined as

$$q = q_n = z_0|0\rangle + z_1|1\rangle, \text{ where } z_0 = z_{n,0} = \sqrt{1-b^2}e^{i\varphi_r}, \ z_1 = z_{n,1} = be^{i\varphi_g}. \tag{30}$$

Thus, the probability of measuring the basis states in this qubit is defined by the blue color; $|z_0|^2 = 1 - b^2$ and $|z_1|^2 = b^2$. The transformation of colors $(r, g, b) \to (z_0, z_1)$ is invertible:

$$b = |z_1| \neq 0, 1, \quad \sin\varphi_g = \text{Imag}\frac{z_1}{b}, \quad g = \frac{\sin\varphi_g + 1}{2}, \quad \text{and } \sin\varphi_r = \text{Imag}\frac{z_0}{\sqrt{1-b^2}},$$
$$r = \frac{\sin\varphi_r + 1}{2}.$$

The cases when $b = 0$ and 1 should be considered separately. The 4^r-dimensional vector presents the color image by the following 4^r-quantum state:

$$\left|\check{f}\right\rangle = \frac{1}{2^r}\sum_{n=0}^{4^r-1}\left(z_{n,0}|0\rangle + z_{n,1}|1\rangle\right) \otimes |n\rangle. \tag{31}$$

This representation of the image requires $(2r + 1)$ qubits, as in the model QLM.

D. Image representation can also be described by transferring data of colors into angles in other ways, for instance, as proposed in the normal arbitrary superposition state with three components (NASS) for the RGB color images [8, 9]. Each color of the image $f_{n,m} = \left(r_{n,m}, g_{n,m}, b_{n,m}\right)$ is considered as the number in the 256th representation,

$$c = c(n, m) = 256^2 r_{n,m} + 256 g_{n,m} + b_{n,m}. \tag{32}$$

The base of this presentation, 256, is chosen for images in the standard range [0, 255] of grays. This representation corresponds to the system with a 24-bit, or 3-byte memory word. In other words, the RGB image is presented as the grayscale image with intensities in the range $\left[0, 256^3 - 1\right]$. The number of grays is large, $256^3 = 16777216$. One can note that many similar colors are far apart in such an image. For example, the colors (100,20,10) and (101,21,10) are located at a distance of $256^2 + 256 = 65792$ from each other. In the method by phase (NASSMP), the coefficients $c(n, m)$ are scaled in the interval of angles $[0, \pi/2]$ as

$$c \to \vartheta_c = \lambda c, \quad \lambda = \frac{\pi}{2} \frac{1}{\left(256^3 - 1\right)} = 9.3627 \times 10^{-8}. \tag{33}$$

Then, the grayscale image is $c(n, m)$ of size $2^r \times 2^r$ pixels, as the 4^r-dimensional vector with angle components $c(i) = c(n, m)$, where $i = n2^r + m$, is represented by the following superposition of $2r$ qubits:

$$\left|\check{f}_A\right\rangle = \sum_{i=0}^{4^r-1} \vartheta_{c(i)} |i\rangle. \tag{34}$$

Here, the coefficients

$$\vartheta_{c(i)} = c(i) \Big/ \sqrt{\sum_{i=0}^{4^r-1} c^2(i)}, \quad c(i) \in \left\{0, 1, 2, \ldots, 256^3 - 1\right\}. \tag{35}$$

It can be noted that the scaling factor λ is very small, and in fact scaling does not change the normalized amplitudes in the quantum representation of the image. In other words, there is no need for such a factor λ in (33).

The amplitudes of states determine a probabilistic representation of the quantum image in (34). It is not difficult to see from the 256th representation of colors, which is given in (32), that colors with red components, even of low intensity, have a very high probability of measurement compared to green and blue. For instance, the pixels with a pure red value even small one, for instance (7,0,0), have probability that is proportional to the number $\left(7 \times 256^2\right)^2$, whereas a strong pure green, for instance (0,200,0), has the number $(200 \times 256)^2$, i.e., 80 times smaller. For a pure blue color, for instance (0,0,220), such a number is $(220)^2$. This may mean that after measuring states of the superposition of $2r$ qubits in (34), the color image is most likely to be a red image with a few pixels or without it with green and blue colors. In addition, the starting of Eq. (28) in NASS is written for 8-bit images, and many images in medical imaging and other applications use 10, 12, and 16-bit image formats. For 16-bit images, the numbers $2^8 = 256$ in (32) will be changed by $2^{16} = 65536$ and the range of the grayscale image in 65536 th representation will be very large, $\left[0, 2^{16\times 3} - 1\right]$.

In NASSTC, the standard basis states of pixels $|i\rangle$ are united with an incomplete 2-qubit state of colors as follows:

$$\left|\check{f}_C\right\rangle = \sum_{i=0}^{4^r-1} [c_R(i)|10\rangle + c_G(i)|01\rangle + c_B(i)|11\rangle]|i\rangle. \tag{36}$$

Here, the coefficients $c_R(i)$ for the red component of the image are calculated by

$$c_R(i) = r(i) \Big/ \sqrt{\sum_{i=0}^{4^r-1} \left[r^2(i) + g^2(i) + b^2(i)\right]}, \quad r(i), g(i), b(i) \in \{0, 1, 2, \ldots, 255\}. \tag{37}$$

For the green and blue components, the corresponding coefficients $c_G(i)$ and $c_B(i)$ are calculated similarly. This representation of the RGB color image requires $(2r+2)$ qubits. For comparison, we mention the model NCQI [18], i.e., the novel quantum representation of RGB color images

$$\left|\check{f}_C\right\rangle = \frac{1}{2^r} \sum_{i=0}^{4^r-1} |r(i)\rangle|g(i)\rangle|b(i)\rangle|i\rangle, \tag{38}$$

which uses $(2r+24)$ qubits. Each color component at pixel number i is in the range [0, 255], i.e., with 8 bits, or 8-qubit state in the above superposition.

As mentioned in Eq. 26, the one-qubit state in the standard form $|\varphi\rangle = \cos\vartheta\, |0\rangle + \mathrm{e}^{i\gamma} \sin\vartheta |1\rangle$ carries information of two values, the phase $e^{i\gamma}$ and angle ϑ. In the extension of the NASS with relative phases that is called NASSRP [8], such a qubit state $|\varphi(i)\rangle$ is considered to present some additional information, $\varphi(i)$, in the pixel i. In this case, the following superposition state is used:

$$\left|\check{f}_{AP}\right\rangle = \sum_{i=0}^{4^r-1} \vartheta_{c(i)} |i\rangle|\varphi(i)\rangle. \tag{39}$$

This quantum representation of the color image with additional information requires $(2r+1)$ qubits.

E. In the real ket model (RKM), the discrete image of square size $N \times N = 2^r \times 2^r$ is divided consequently down into four equal parts, and therefore, the image can be presented as

$$\left|\check{f}\right\rangle = \sum_{n_1, n_2, \ldots, n_r = 1,2,3,4} c_{n_1, n_2, \ldots, n_r} |n_1, n_2, \ldots, n_r\rangle. \tag{40}$$

The intensities $f_{n,m}$ of the image are stored in the coefficients $c_{n_1,n_2,\ldots,n_r}$ [5]. Such numbering of pixels allows for using only r qubits for the image, and each such qubit is a superposition of four values. The numbering of the coefficients for the 8×8 image is shown in Fig. 4. In this representation, for example, the coefficient in the pixel (3,2) is numbered as (1,4,3), and the coefficient in the pixel (6,5) is numbered as (4,3,2). Therefore, the values $f_{3,2}$ and $f_{6,5}$ are stored as $c_{1,4,3}|143\rangle$ and $c_{4,3,2}|432\rangle$, respectively. Here, $c_{1,4,3}$ and $c_{4,3,2}$ are amplitudes that are defined from the values $f_{3,2}$ and $f_{6,5}$, respectively. For instance, these coefficients can be considered $c_{1,4,3} = a_{3,2}$ and $c_{4,3,2} = a_{6,5}$ when using the amplitudes calculated by (7).

1	2
3	4

1	2	1	2
3	4	3	4
1	2	1	2
3	4	3	4

1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4

Fig. 4 The scheme of block structuring numbering for the 8×8 image

F. It should be mentioned that existing models extend the dimension NM of states in the quantum representation of images. The novel enhanced quantum representation (NEQR) was proposed for the discrete image of size $N \times M = 2^r \times 2^s$, with integers r and $s > 1$ [7]. This method extended for other size of images by the generalized quantum image representation model (GQIR) [10, 11]. The qubit states of pixels are nested into the higher dimension basis states, by using the operation of tensor product with the intensity (or brightness) of the image, which is written in the binary form as a multi-qubit state. For example, if the range of the image is [0, 255], i.e., with 8 bits, the value of the image at pixel (n, m) can be written in binary form

$$f_{n,m} \rightarrow \left[\left(f_{n,m}\right)_7, \left(f_{n,m}\right)_6, \ldots, \left(f_{n,m}\right)_1, \left(f_{n,m}\right)_0\right]$$

and the state in this pixel is defined as

$$\begin{aligned} \left|q_{n,m}\right\rangle &= \left|\left(f_{n,m}\right)_7, \left(f_{n,m}\right)_6, \ldots, \left(f_{n,m}\right)_1, \left(f_{n,m}\right)_0\right\rangle |n, m\rangle \\ &= \left|\left(f_{n,m}\right)_7, \left(f_{n,m}\right)_6, \ldots, \left(f_{n,m}\right)_1, \left(f_{n,m}\right)_0, n, m\right\rangle. \end{aligned} \tag{41}$$

For instance, the value $f_{1,2} = 36$ of the 4×4 image can be written as the basis state

$$\left|q_{1,2}\right\rangle = |00100100\rangle |1, 2\rangle = |00100100\rangle |0110\rangle = |001001000110\rangle.$$

Thus, the image with 8-bit intensity can be presented as

$$\left|\check{f}\right\rangle = \frac{1}{\sqrt{2^{r+s}}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left|q_{n,m}\right\rangle \tag{42}$$

and that requires $(r + s + 8)$ qubits.

5 Multi-qubit Fourier transform representation of images

In this section, we apply the concept of the FTQR for grayscale and color images and then introduce the 3-point DFT on color qubits in two models for color image representation. The grayscale image $f = f_{n,m}$ of size $N \times M = 2^r \times 2^s$ in the range

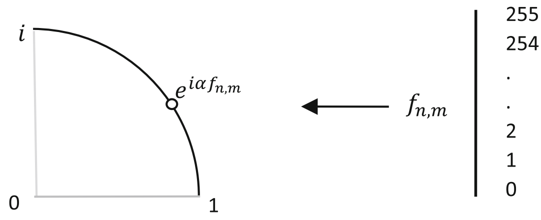

Fig. 5 Transformation of the interval [0,255] into the quarter circle, for the grayscale image

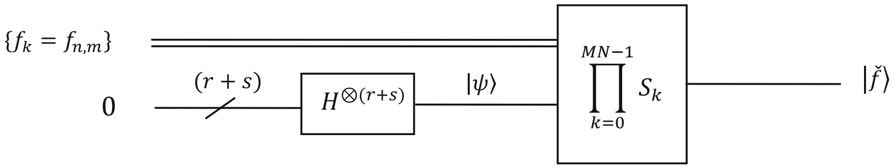

Fig. 6 The circuit for the FTQR of the grayscale image

of integers [0, 255] can be represented by the exponential coefficients in a way that is similar to the 1-D signals (see Fig. 5). The image is transforming as

$$f_{n,m} \to T\left[f_{n,m}\right] = e^{i\alpha f_{n,m}},$$

and then, the $(r+s)$-qubit Fourier transform representation is defined by the following superposition:

$$\left|\hat{f}\right\rangle = \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} T\left[r_{n,m}\right]|n,m\rangle = \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{i\alpha f_{n,m}}|n,m\rangle. \qquad (43)$$

Note that this representation is not the classical discrete Fourier transform (DFT) and the image reconstruction from this representation relates to the measurement of the superposition state. The inverse classical DFT requires knowledge of all components of the transform. When measuring $\left|\check{f}\right\rangle$ in (43) and getting the state $|n,m\rangle$, its amplitude $e^{i\alpha f_{n,m}} / \sqrt{NM}$ allows for calculating the value $f_{n,m}$ of the image.

The circuit for implementation of the FTQR is shown in Fig. 6. Here, for simplicity of notations, the image $f_{n,m}$ is written into the 1-D vector f_k, $k = nN + m$, where $n = 0 : (N-1)$ and $m = 0 : (M-1)$. Therefore, this circuit is described similarly to the one in Fig. 3.

5.1 Models for RGB color images

In many applications of color image processing by the traditional computer, the color components of the image are processed separately [12, 19]. Therefore, the above-proposed models of quantum image representation can be applied for each image component, including the Fourier transform representation. For example, if the image is in the RGB format, then the red, green, and blue components can be represented by the $(r+s)$-qubit Fourier transform representation each, as in Eq. 43. Thus, the

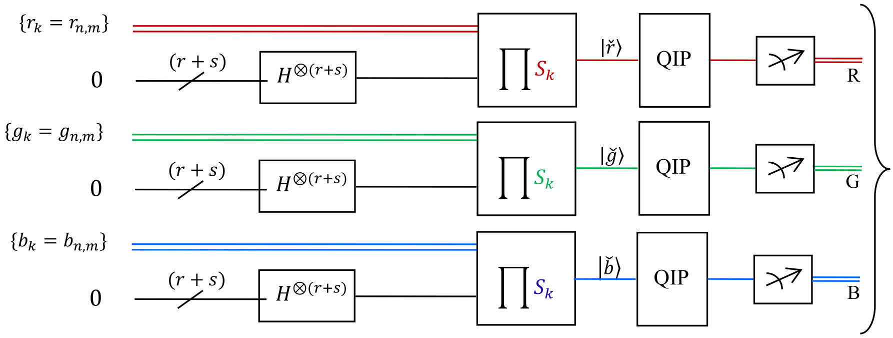

Fig. 7 The circuit for processing separately color components of the RGB image

following quantum representation is considered for three components of the RGB image:

$$\left|\check{r}\right\rangle = \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} T\left[r_{n,m}\right] |n, m\rangle, \quad \left|\check{g}\right\rangle = \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} T\left[g_{n,m}\right] |n, m\rangle,$$
$$\left|\check{b}\right\rangle = \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} T\left[b_{n,m}\right] |n, m\rangle. \tag{44}$$

Here, the transform $T\left[f_{n,m}\right] = e^{i\alpha f_{n,m}}$, $\alpha = 2\pi/1024$. Then, many known methods of color image processing can be applied for processing color image components in quantum computers, followed by measurement of each color image superposition. The circuit for implementation of the FTQR for three color components of the RGB image followed by their quantum image processing (QIP) and measurements is shown in Fig. 7. The image components are written as 1-D vectors r_k, g_k, and b_k, for the red, green, and blue colors, respectively.

However, the concept of the quantum superposition allows for uniting the color components without additional qubits and develops new models of quantum representation, when color components of the image are processed and measured in one model. We consider two such models and introduce the concept of the 3-point discrete Fourier transform of the qubits.

Model 1. We represent the color image $f_{n,m} = \left(r_{n,m}, g_{n,m}, b_{n,m}\right)$ in the RGB model as a multi-qubit state in the following way. Because of three primary colors, the unit circle is divides by three parts, each of 120 °, and each color is map to one circular arc, as shown in Fig. 8. The range of each color component is considered to be in the integer interval [0, 255]. Thus, to the red, green, and blue colors three separate places are highlighted on the circle. Each pixel is defined with three points on the unit circle, and each point is in its color circular arc. For a grayscale image, each such triplet of points composes an equilateral triangle.

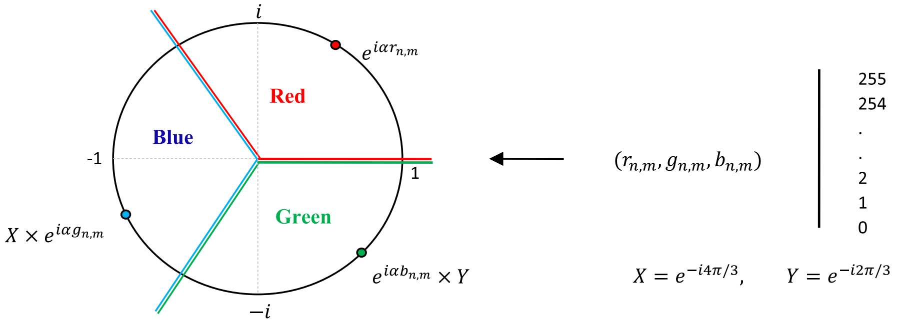

Fig. 8 Transformation of colors (for the RGB model)

The red color component of the image can be presented as follows:

$$\left|\check{r}\right\rangle = \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} T\left[r_{n,m}\right]|n, m\rangle = \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{i\alpha r_{n,m}}|n, m\rangle, \tag{45}$$

where $\alpha = 2\pi/(3 \times 256)$. The green and blue components of the color image can be mapped similarly to other two circular arcs of 120 degree each,

$$\left|\check{g}\right\rangle = e^{-i2\pi/3} \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} T\left[g_{n,m}\right]|n, m\rangle = \frac{1}{\sqrt{NM}} e^{-i2\pi/3} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{i\alpha g_{n,m}}|n, m\rangle, \tag{46}$$

$$\left|\check{b}\right\rangle = e^{-i4\pi/3} \frac{1}{\sqrt{NM}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} T\left[b_{n,m}\right]|n, m\rangle = \frac{1}{\sqrt{NM}} e^{-i4\pi/3} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{i\alpha b_{n,m}}|n, m\rangle. \tag{47}$$

We can unite these states into the following superposition of states of colors:

$$\left|(\check{r}, \check{g}, \check{b})_1\right\rangle = \frac{1}{A} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(T\left[r_{n,m}\right] + T\left[g_{n,m}\right]e^{-i2\pi/3} + T\left[b_{n,m}\right]e^{-i4\pi/3}\right)|n, m\rangle, \tag{48}$$

where A is the normalization coefficient calculated by

$$A = \sqrt{\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left|T\left[r_{n,m}\right] + T\left[g_{n,m}\right]e^{-i2\pi/3} + T\left[b_{n,m}\right]e^{-i4\pi/3}\right|^2}.$$

Preparation for Model 1. The algorithm of the color image superposition in Eq. 48 can be described as follows.

1. Apply $(r+s)$ Hadamard gates H, i.e., the Hadamard transform $H^{\otimes(r+s)}$ on $(r+s)$ qubits all initialized to $|0\rangle$, to obtain the equal superposition of NM basis states,

$$|\psi\rangle = H^{\otimes r}|0\rangle^{\otimes(r+s)} = \frac{1}{\sqrt{NM}}\sum_{k=0}^{NM-1}|k\rangle. \tag{49}$$

2. For each pixel number k, calculate the coefficient

$$F_1(k) = T[r_k] + T[g_k]e^{-i2\pi/3} + T[b_k]e^{-i4\pi/3}. \tag{50}$$

   It is the component of the 3-point DFT of the vector $(T\left[r_{n,m}\right], T\left[g_{n,m}\right], T\left[b_{n,m}\right])$, which is calculated at the frequency-point 1.
3. Calculate the amplitudes $|F_1(k)|$ and phases ϕ_k of obtained complex coefficients $F_1(k) = |F_1(k)|e^{i\phi_k}$.
4. Add amplitudes $|F_1(k)|$ to the basis states $|k\rangle$, to compose the new superposition of $(r+s)$ qubits,

$$\left(|\psi\rangle = \frac{1}{\sqrt{NM}}\sum_{k=0}^{NM-1}|k\rangle\right) \rightarrow \left(|\psi\rangle_a = \frac{1}{A}\sum_{k=0}^{NM-1}|F_1(k)||k\rangle\right). \tag{51}$$

5. Apply the phase shift gates $|k\rangle \rightarrow e^{i\phi_k}|k\rangle$ to the basis states, to obtain the superposition in Eq. 48,

$$|\psi\rangle_a \rightarrow \frac{1}{A}\sum_{k=0}^{NM-1} e^{i\phi_k}|F_1(k)||k\rangle = \frac{1}{A}\sum_{k=0}^{NM-1} F_1(k)|k\rangle = \left|(\check{r}, \check{g}, \check{b})_1\right\rangle. \tag{52}$$

The operation in Step 5 with phase gates can be described by the corresponding diagonal $NM \times NM$ matrices

$$S_k = \mathrm{diag}\left\{s_{i,i}\right\}_{i=0:(N-1)}, \quad s_{i,i} = 1, \text{ if } i \neq k, \text{and } s_{k,k} = e^{i\phi_k}. \tag{53}$$

Thus,

$$\left(\prod_{k=0}^{NM-1} S_k\right)|\psi\rangle_a = \left|(\check{r}, \check{g}, \check{b})_1\right\rangle. \tag{54}$$

The circuit for implementation of superposition (48) is given in Fig. 9. The T-block is used to transform $T[x] = e^{i\alpha x}$, given number x. This circuit uses the incomplete 3-point DFT.

The above image superposition can also be written as the following operation on qubits:

$$\left|(\check{r}, \check{g}, \check{b})_1\right\rangle = \frac{1}{B_1}\left(\left|\check{r}\right\rangle + \left|\check{g}\right\rangle + \left|\check{b}\right\rangle\right), \tag{55}$$

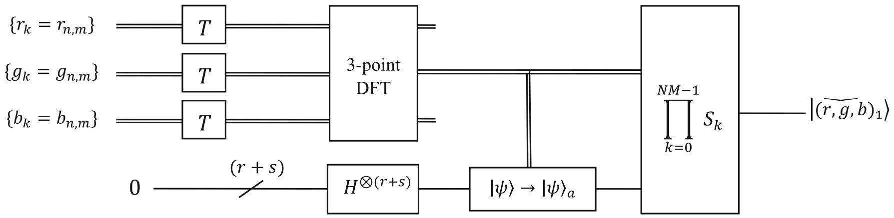

Fig. 9 The circuit for the color image representation in Model 1

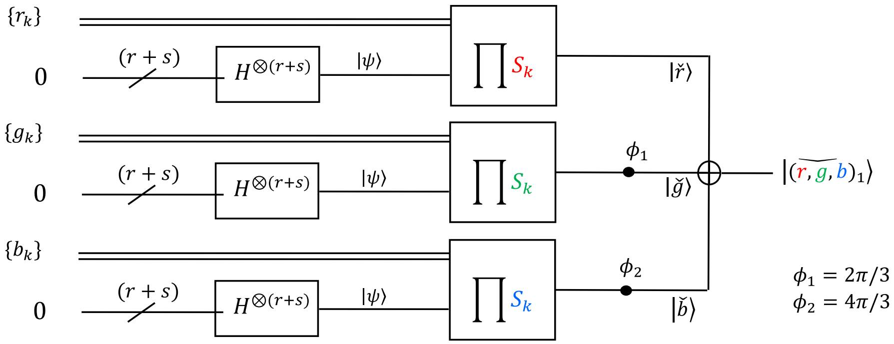

Fig. 10 The circuit for the color image superposition in Model 1

where $B_1 = A/\sqrt{NM}$. This quantum superposition is the sum of three color superpositions. If we consider that the preparation of each color state is accomplished separately, then the circuit for the above image superposition can be implemented, as shown in Fig. 10. However, it uses two phase gates ϕ_1 and ϕ_2 and requires $(r + s)$ qubits for each color component.

Up to A coefficient, the amplitude of the state $|n, m\rangle$, which is denoted as

$$F_1(n, m) = \left(T\left[r_{n,m}\right] + T\left[g_{n,m}\right]e^{-\frac{i2\pi}{3}} + T\left[b_{n,m}\right]e^{-\frac{i4\pi}{3}}\right), \tag{56}$$

is the component of the 3-point discrete Fourier transform (DFT) of the signal

$$f_1 = \left(T\left[r_{n,m}\right], T\left[g_{n,m}\right], T\left[b_{n,m}\right]\right) = \left(e^{i\alpha r_{n,m}}, e^{i\alpha g_{n,m}}, e^{i\alpha b_{n,m}}\right). \tag{57}$$

The quantum superposition of colors

$$\left|\left(\check{r}, \check{g}, \check{b}\right)_1\right\rangle = \frac{1}{B_1} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} F_1(n, m)|n, m\rangle \tag{58}$$

requires $(r + s)$ qubits.

To restore the transforms $T\left[r_{n,m}\right]$, $T\left[g_{n,m}\right]$, and $T\left[b_{n,m}\right]$ when the pixel state $|n, m\rangle$ is measured, the completed 3-point DFT is needed. In other words, together with the component in (56), two more components are required,

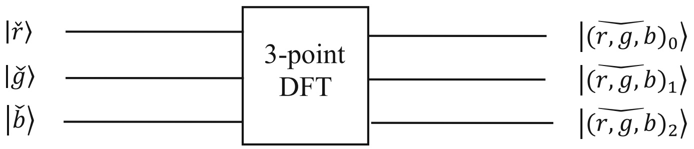

Fig. 11 The circuit for the 3-point DFT of the color states in Model 1

$$F_0(n,m) = \left(T\left[r_{n,m}\right] + T\left[g_{n,m}\right] + T\left[b_{n,m}\right]\right)$$

and

$$F_2(n,m) = \left(T\left[r_{n,m}\right] + T\left[g_{n,m}\right]e^{-\frac{i4\pi}{3}} + T\left[b_{n,m}\right]e^{-\frac{i2\pi}{3}}\right).$$

The corresponding two additional $(r+s)$-quantum states are

$$\left|\left(\check{r},\check{g},\check{b}\right)_k\right\rangle = \frac{1}{A_k}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} F_k(n,m)|n,m\rangle, \quad k = 0, 2,$$

where the coefficients

$$A_k = \sqrt{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|T\left[r_{n,m}\right] + T\left[g_{n,m}\right]e^{-i2\pi k/3} + T\left[b_{n,m}\right]e^{-i4\pi k/3}\right|^2}.$$

Thus, we can define the qubits

$$\left|(\check{r},\check{g},\check{b})_k\right\rangle = \frac{1}{B_k}\left(\left|\check{r}\right\rangle + e^{-i2\pi k/3}\left|\check{g}\right\rangle + e^{-i4\pi k/3}\left|\check{b}\right\rangle\right), \quad k = 0, 1, 2, \tag{59}$$

where $B_k = A_k/\sqrt{NM}$, and call it *the 3-point DFT of qubits of colors*. It not difficult to see that

$$\frac{1}{3}\sum_{k=0}^{2} B_k^2 = \frac{1}{NM}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(r_{n,m}^2 + g_{n,m}^2 + b_{n,m}^2\right).$$

The circuit of implementation of the 3-point DFT of color states is given in Fig. 11.

Now, we consider another concept of the 3-point DFT of qubits for color images in the RGB model.

Model 2. When applying the approach given in (48) to the normalized values of color components, i.e., when using the concept of the qubit pixel for each color, the following superposition state can be considered:

$$\left|\left(\check{r}, \check{g}, \check{b}\right)\right\rangle = \frac{1}{A}\left(|\check{r}\rangle + e^{-i2\pi/3}|\check{g}\rangle + e^{-i4\pi/3}\left|\check{b}\right\rangle\right), \tag{60}$$

where the quantum color states are

$$|\check{r}\rangle = \frac{1}{E_r}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} r_{n,m}|n, m\rangle, \quad |\check{g}\rangle = \frac{1}{E_g}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g_{n,m}|n, m\rangle,$$
$$\left|\check{b}\right\rangle = \frac{1}{E_b}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} b_{n,m}|n, m\rangle.$$

Here, the coefficients E_r, E_g, and E_b are the square roots of energies of the colors,

$$E_k = \sqrt{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} k_{n,m}^2}, \quad k = r, g, b.$$

The normalization coefficient A is calculated by

$$A = \sqrt{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|\frac{r_{n,m}}{E_r} + \frac{g_{n,m}}{E_g}e^{-i2\pi/3} + \frac{b_{n,m}}{E_b}e^{-i4\pi/3}\right|^2}.$$

Thus, we consider the superposition of states

$$\left|\left(\check{r}, \check{g}, \check{b}\right)\right\rangle = \frac{1}{A}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(\frac{r_{n,m}}{E_r} + \frac{g_{n,m}}{E_g}e^{-i2\pi/3} + \frac{b_{n,m}}{E_b}e^{-i4\pi/3}\right)|n, m\rangle. \tag{61}$$

For simplicity of calculations, we can assume that all three color components of the image have the same energy, i.e., $E_r = E_g = E_b = E$. Then, the superposition in (61) can be written as

$$\left|\left(\check{r}, \check{g}, \check{b}\right)\right\rangle = \frac{1}{AE}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(r_{n,m} + g_{n,m}e^{-i2\pi/3} + b_{n,m}e^{-i4\pi/3}\right)|n, m\rangle \tag{62}$$

and the coefficient A is

$$A = \frac{1}{E}\sqrt{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|r_{n,m} + g_{n,m}e^{-i2\pi/3} + b_{n,m}e^{-i4\pi/3}\right|^2}.$$

If the energies of the image color components are different, these components can be linearly amplified, in order to equalize their energies, and then represented by (62).

(a) (b)

Fig. 12 **a** The flower image and **b** energy equalization of the image

5.2 Algorithm of energy equalization in model 2

1. Calculate the average of energy of three color components,

$$E^2 = \frac{1}{3}\left(E_r^2 + E_g^2 + E_b^2\right).$$

2. Calculate the coefficients

$$\alpha_1 = E/E_r, \alpha_2 = E/E_g, \alpha_3 = E/E_b.$$

3. Renew the color components as

$$r'_{n,m} = \alpha_1 r_{n,m}, g'_{n,m} = \alpha_2 g_{n,m}, b'_{n,m} = \alpha_3 b_{n,m},$$

   where $n = 0 : (N-1), m = 0 : (M-1)$.

The new color image $\left(r'_{n,m}, g'_{n,m}, b'_{n,m}\right)$ has color with equal energy, for instance, for the red component

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(r'_{n,m}\right)^2 = \frac{E^2}{E_r^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} r_{n,m}^2 = E^2.$$

As an example, Fig. 12 shows the original flowers image in part (a) and the image after energy equalization in part (b). The similar result for the tree image is given in Fig. 13. The energies of colors were equated to the average energy of colors. One can note that the equalized image has a good quality, as the original image. Moreover, the image can be reconstructed from the equalized one, by using the ratios of energies E_k, $k = r, b, g$.

At each quantum pixel $|n, m\rangle$, the amplitude in the above superposition (62) is the component of the 3-point DFT, up to the constant $1/AE$,

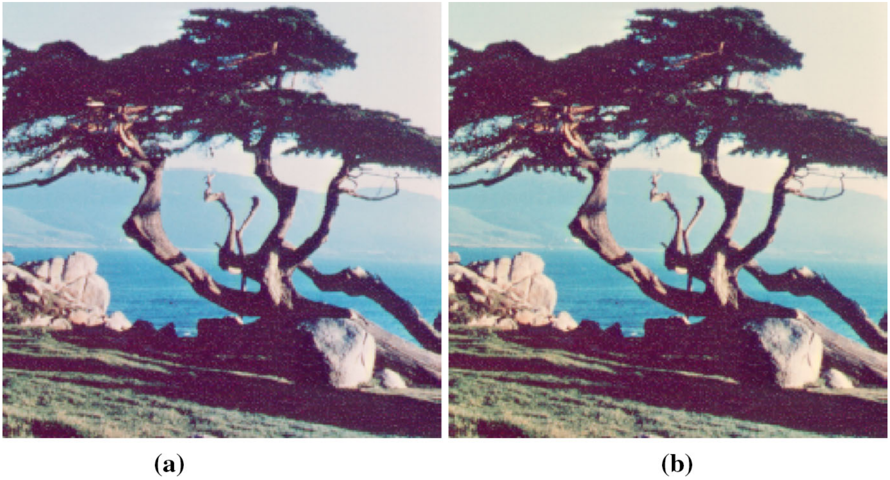

Fig. 13 **a** Tree image, **b** energy equalization of the image

$$F_1 = F_1(n, m) = \left(r_{n,m} + g_{n,m}e^{-i2\pi/3} + b_{n,m}e^{-i4\pi/3}\right).$$

The coefficient A can be written as

$$A = \frac{1}{E}\sqrt{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}|F_1(n, m)|^2}.$$

The data are real, and for the 3-point DFT, the component

$$F_2 = \bar{F}_1 = \left(r_{n,m} + g_{n,m}e^{i2\pi/3} + b_{n,m}e^{i4\pi/3}\right).$$

Thus, the component F_1 together with the gray value $v_{n,m} = (r_{n,m} + g_{n,m} + b_{n,m})/3$ determines the complete 3-point DFT of the color,

$$f_{n,m} = (r_{n,m}, g_{n,m}, b_{n,m}) \to \left(F_0 = 3v_{n,m}, F_1, F_2\right).$$

The inverse 3-point DFT results in the original colors.

The color image $(r + s)$-state quantum representation in Eq. 62 can be written as

$$\left|\left(\check{r}, \check{g}, \check{b}\right)\right\rangle = \frac{1}{AE}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} r_{n,m}|n, m\rangle + \frac{1}{AE}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g_{n,m}e^{-i2\pi/3}|n, m\rangle + \frac{1}{AE}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} b_{n,m}e^{-i4\pi/3}|n, m\rangle,$$

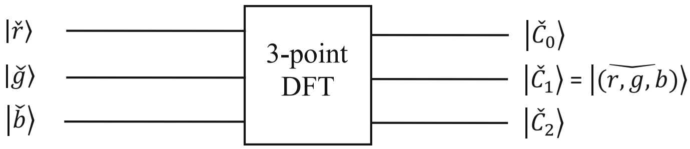

Fig. 14 The circuit for the color image representation in Model 2

or in short form as

$$\left|\left(\check{r}, \check{g}, \check{b}\right)\right\rangle = \frac{1}{A}\left(|\check{r}\rangle + |\check{g}\rangle e^{-i2\pi/3} + \left|\check{b}\right\rangle e^{-i4\pi/3}\right). \tag{63}$$

This representation can be considered as one of the components of the 3-point DFT of the color-qubit state. Namely, we can write that $\left|\left(\check{r}, \check{g}, \check{b}\right)\right\rangle = \left|\check{C}_1\right\rangle$, where

$$\left|\check{C}_p\right\rangle = \left(|\check{r}\rangle + |\check{g}\rangle e^{-i2\pi p/3} + \left|\check{b}\right\rangle e^{-i4\pi p/3}\right), \quad p = 0, 1, 2. \tag{64}$$

It is the 3-point DFT of the color state superpositions, and it requires $(r + s)$ qubits for each $p = 0, 1$, and 2. This model differs from Model #1, which is determined by the phases in the basis states. However, both models are similar in the sense that color components, namely color states, are processed together by the 3-point DFT. The circuit for implementation of superposition (63) is given in Fig. 14.

6 Comments

1. Information of the signal at each point or image in each pixel is written in the phase of the basis state in the quantum superposition. There is no constraint on the size and range of the signal and image when using the FTQR.
2. FTQR possesses properties, such as the sum, amplification, and shifting, that are not available for the known methods of quantum signal and image representation.
3. The calculation of phase-type amplitudes in signal and image representation is simple. These amplitudes can be calculated in advance and used by the lookup table method.
4. In the FTQR, all states in the superposition have an equal probability.
5. FTQR requires $(r + s)$ qubits for images of size $2^r \times 2^s$ pixels.
6. FTQR has a simple algorithm for signal and image reconstruction after measuring the superposition states.
7. FTQR is not the classical discrete Fourier transform (DFT). The inverse DFT requires knowledge of all components of the DFT. For the FTQR, the inverse transform or image reconstruction is defined by measuring the state of the superposition, as indicated in Comment 6.
8. The models with the FTQR can be implemented by using $(r + s)$ qubits, as well as by circuits with the separate calculation of qubit states for color components of the image. These circuits use more qubits; however, we believe that in a quantum computer there will be enough qubits to represent color images, even for color

components individually. Therefore, we consider such schemes effectively, especially because many well-known methods in digital image processing are based on separate color processing.

9. In the presented Models 1 and 2, the described concept of the 3-point DFT on qubits allows for processing colors as units in the frequency domain. These models are similar in the sense that color components, namely color states, are processed together by the 3-point DFT. The circuits of implementation in Figs. 11 and 14 can be considered as the circuits of the 3-state DFT of the color image in quantum computing.

7 Conclusion

Methods of image representation have been described, including a new approach for representing discrete signals and images in quantum computing, by mapping the input data into the unit circle. For color images, the primary color components are mapped into different parts of the unit circle to get the quantum image representation. Such representation leads to the concept of the Fourier transform qubit representation of images. For color images in the RGB model, we introduce the concept of the 3-point DFT of color-qubit states. It is the representation with only phase-shifted base states. The similar Fourier transform qubit representation of images in quantum computing can be described in other color models, such as the XYZ, CMY(K), and YCbCr models [19]. We believe that proposed models can be used in quantum image processing together with the existent methods of image representation. Our future work is aimed at exploring more practical applications, such as image segmentation, image matching, and image quality measurement in the open-source quantum environment, and image cryptography [20, 21].

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