Image storage, retrieval, compression and segmentation in a quantum system

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Image storage, retrieval, compression and segmentation in a quantum system

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Abstract A set of quantum states for M colors and another set of quantum states for N coordinates are proposed in this paper to represent M colors and coordinates of the N pixels in an image respectively. We design an algorithm by which an image of N pixels and m different colors is stored in a quantum system just using $2N + m$ qubits. An algorithm for quantum image compression is proposed. Simulation result on the Lena image shows that compression ratio of lossless is 2.058. Moreover, an image segmentation algorithm based on quantum search quantum search which can find all solutions in the expected times in $O(t\sqrt{N})$ is proposed, where N is the number of pixels and t is the number of targets to be segmented.

Keywords Quantum image processing · Image storage and retrieval · Image compression · Image segmentation · Quantum search algorithms · Quantum computation

1 Introduction

Quantum computation [1,2] which has the unique computing performance of quantum coherence, entanglement, superposition of quantum states and other inherent

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characteristics becomes quickly an international research focus. Quantum image processing represents an emerging image processing technology by taking advantage of quantum computation. Quantum image processing faces two questions: How an image is stored in and retrieved from a quantum system? How to process the image stored in a quantum system?

In a classical computer, a color of 24-bit RGB True Color is stored with 24bit space. While in a quantum system, Venegas-Andraca et al. presented that frequency of the physical nature of color could represent a color instead of the RGB model or the HIS model, so a color could be represented by only a 1-qubit quantum state [3] and an image could be stored in a quantum array [3,4]. Le et al. [5] presented that a quantum composite state (FRQI state) could store the colors and the coordinates of an image with N pixels, where FRQI state was compounded by $\mathrm{N} * (1 + \mathrm{logN})$ 1-qubit quantum states, the color and the coordinate of a pixel were represented respectively by a 1-qubit state and $logN$ 1-qubit states.

In recent years, some algorithms of quantum image processing have been proposed, such as quantum image transformation [6], color transformation [7], quantum image compression [5,8,9] and quantum image segmentation [4]. In order to improve efficiency of image processing, some efficient quantum algorithms can be applied to image processing, such as quantum Fourier transform [10,11], quantum search algorithms [12–15]. Grover's quantum search algorithm [12,13] showed that we could find the unique solution in iteration times in $O(\sqrt{N})$. For the number of unknown solutions, BBHT [16] algorithm can search out a solution in expected time in $O(\sqrt{N/t})$ when the number of solutions $t \leq 3N/4$, where specific number of solutions is unknown.

First, this paper proposes that M colors are represented by $QSMC$ in image processing. $QSMC$ ipggs composed of M 1-qubit quantum states, and each color in an image is represented by using a quantum state of $QSMC$. Second, we present that $QSNC$ represents the coordinates of N pixels where the coordinate of a pixel is represented by a 1-qubit quantum state. And then we design an algorithm based on $QSMC$ and $QSNC$ to store an image in a quantum system, and discuss how to retrieve images stored in a quantum system. Then again, we propose a quantum compression algorithm and apply the algorithm to compress the Lena image. Last, we design a color image segmentation algorithm based on a quantum search algorithm, of which the core sub-algorithm is the quantum search algorithm for all the solutions.

The paper is organized as follows: In Sect. 2, we propose a method for image storage and retrieval. In Sect. 3, we design a quantum compression algorithm for lossless compression. In Sect. 4, we present a color image segmentation algorithm, and its sub-algorithm—"the quantum search algorithm for all solutions" is shown in Sect. 5. Summary is proposed in Sect. 6.

2 Image storage and retrieval

Inspired by articles [3–5], we extend the image storage algorithm presented in [3] and propose that $QSMC$ represents M colors and $QSNC$ represents the coordinates of N pixels in an image. The procedure of creating $QSMC$ is explained in Algorithm 1. And the procedure to create $QSNC$ is shown in Algorithm 2.

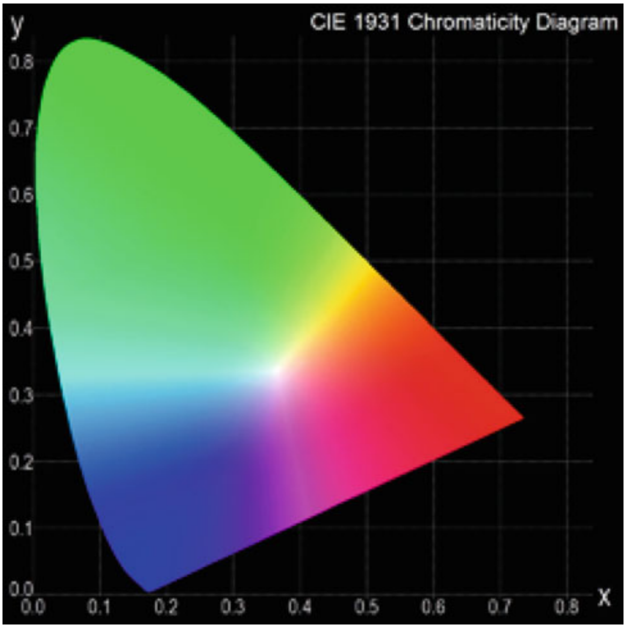


Fig. 1 CIE 1931 chromaticity diagram

Algorithm 1 Creating a set of quantum states for M colors ($QSMC$)

1. Sort M different colors of RGB True Color.
2. Create a bijective function from a color to an angle $F_1 : color \rightleftarrows \phi$, where $color = \{color_1, color_2, \ldots, color_M\}$, $color_i$ corresponds to the ith color in M colors, $\phi = \{\phi_1, \phi_2, \ldots, \phi_M\}$, $\phi_i = \frac{\pi(i-1)}{2(M-1)}$, $i \in \{1, 2, \ldots, M\}$.
3. Create M quantum states $|v_1>, |v_2>, \ldots, |v_M>$, where $|v_i> = \cos\phi_i|0> + \sin\phi_i|1>$. Specifically, quantum state $|v_i>$ is created by applying the rotation operator $R_y(2\phi_i) = \begin{bmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{bmatrix}$ to $|0>$. That is, we store a color by using a quantum gate.

Images are divided into grayscale images and color images for classical image processing. For grayscale images, $M = 256$, colors are sorted in ascending order by the grayscale values. For example, $color_1$ and $color_{256}$ correspond to grayscale values 0 and 255 respectively. For color images of 24-bit RGB True Color, $M = 2^{24}$, color distribution is shown in Fig. 1. We sort 2^{24} colors by a strategy which is described below, so that the colors $color_i$ and $color_{i+1}$ are similar which correspond to two adjacent angles ϕ_i and ϕ_{i+1}. First, let $color_1$ be black and $color_{2^{24}}$ be white. Second, we sort solid colors corresponding to the edge of the part of the Fig. 1 in ascending order according to the value of the wavelength. Each solid color corresponds to a color in the RGB color model, and the solid color and the corresponding color in RGB can't be distinguished or are very similar to the human eye. Last, for a color like brown that isn't a solid color in visible light, so it is placed in the front or rear of the color which is found from color sequence with minimum distance (RGB distance, weighted RGB

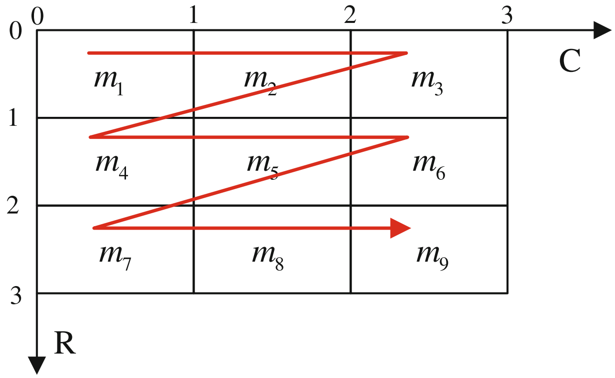


Fig. 2 Representation of the image pixel coordinates

distance or angle distance) to brown. For example, let $X_i = (R_i, G_i, B_i)$ be a color to be sorted, C is the set of colors which have been sorted, $X_j = (R_j, G_j, B_j) \in C$. Let us define the weighted distance as follow

$$D(X_i, X_j) = \sqrt{w_r(R_i - R_j)^2 + w_g(G_i - G_j)^2 + w_b(B_i - B_j)^2} \tag{1}$$

Since $X_k \in C$ is searched out by $\min_{X_j \in C} D(X_i, X_j)$, $D(X_i, X_k)$ is the minimum weighted distance to X_i. If $D(X_{k-1}, X_i) < D(X_{k-1}, X_k)$, X_i is inserted between X_{k-1} and X_k, otherwise, X_i is inserted between X_k and X_{k+1}. And then subscripts of the elements in the set C are updated.

Algorithm 2 Creating a set of quantum states for N coordinates ($QSNC$) of the pixels in an square image

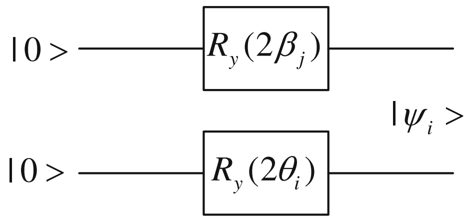
1. Create a bijection function from a coordinate to an angle $F_2 : coordinate \rightleftarrows \theta$, where coordinate $= \{\text{coo}_1, \text{coo}_2, \ldots, \text{coo}_N\}, \theta = \{\theta_1, \theta_2, \ldots, \theta_N\}, \theta_i = \frac{\pi(i-1)}{2(N-1)}, i \in \{1, 2, \ldots, N\}$, coo_i corresponds to the coordinate of the ith pixel of N pixels in an square image.
2. Create N quantum states $|u_1>, |u_2>, \ldots, |u_N>$, where $|u_i>= \cos\theta_i|0> + \sin\theta_i|1>$. Specifically, the quantum state $|u_i>$ is created by applying the rotation operator $R_y(2\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$ to $|0>$.

We explain Algorithm 2 with a 3×3 image in Fig. 2 as an example. Pixel coordinate system is used in Fig. 2, and quantum state $|u_1>= \cos\theta_1|0> + \sin\theta_1|1>$ represents coo_1 which is the coordinate of the first pixel(m_1) in the image, namely, $\text{coo}_1 = (1, 1)$. Quantum state $|u_9>= \cos\theta_9|0> + \sin\theta_9|1>$ represents coo_9 which is the coordinate of the ninth pixel(m_9) in the image, namely, $\text{coo}_9 = (3, 3)$.

2.1 Storing an image in a quantum system

Because measurement makes a quantum superposition state collapse into a basis state, in order to retrieve successfully the color, many identical quantum superposition states

Fig. 3 The realization of quantum circuit to store the ith pixel



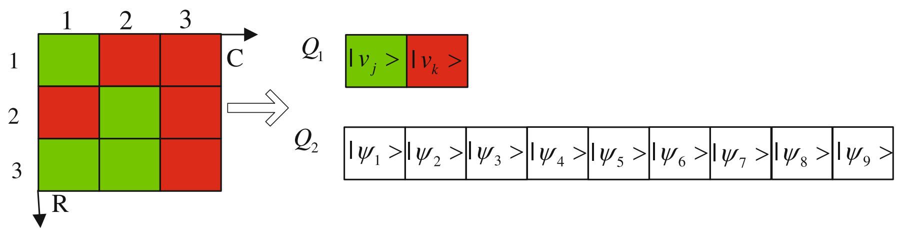


Fig. 4 A 3×3 image storage

which represent the color must be generated when a color is stored in a quantum state. In order to reduce the number of the identical quantum states used in retrieving an image, we design algorithm 3 to store an image in a quantum system.

Algorithm 3 Storing an image in a quantum system

1. Suppose that m is the number of different colors of an image, the quantum states of $QSMC$ corresponding to these different colors are stored in a quantum queue Q_1.
2. Create a bijective function $F_3 : pos \rightleftarrows \beta$, where pos $= \{1, 2, \ldots, m\}$ is a set of the positions in the quantum queue Q_1, and $\beta = \{\beta_1, \beta_2, \ldots, \beta_m\}$ is an angle set, $\beta_i = \frac{\pi(i-1)}{2(m-1)}, i \in \{1, 2, \ldots, m\}$. if $m = 1$, let $\beta_1 = 0$. And then, apply the rotation $R_y(2\beta_i) = \begin{bmatrix} \cos\beta_i & -\sin\beta_i \\ \sin\beta_i & \cos\beta_i \end{bmatrix}$ $(i = 1, 2, \ldots, m)$ on $|0>$ to create successively m quantum states $|w_1>, |w_2>, \ldots |w_m>$.
3. Assume that $|v_k>$ corresponds to the color of the ith pixel in the image and is stored in the jth position of Q_1 which is represented by the quantum state $|w_j>$.
4. $|u_i>$ represents the ith pixel's coordinate in the image.
5. A quantum composite state $|\psi_i>=|w_j>\otimes|u_i>$ represents the ith pixel's color and coordinate, the realization of quantum circuit shown in Fig. 3, and $|\psi_i>$ is stored in another quantum queue Q_2.
6. Successively take $i = 1, 2, \ldots, N$, repeating Step 3 to Step 5 to complete the entire image storage.

We explain Algorithm 3 by using a 3×3 image shown in Fig. 4 as an example. There are only two colors (green and red) and nine coordinates in the image. Assume that green and red are represented by the quantum states of $QSMC$ $|v_j>$ and $|v_k>$, the positions of which in Q_1 are represented respectively by $|w_1>$ and $|w_2>$. From Algorithm 3, we know that $|w_1>=|0>$, $|w_2>=|1>$, $|u_i>=\cos\theta_i|0>+\sin\theta_i|1>$,

$\theta_i \in \{0, \frac{\pi}{16}, \frac{2\pi}{16}, \ldots, \frac{\pi}{2}\}, |\psi_i>= |0> \otimes |u_i>, i \in \{1, 5, 7, 8\}, |\psi_j>= |1> \otimes |u_j>, j \in \{2, 3, 4, 6, 9\}$.

Applying Algorithm 3, the colors and coordinates of a 24-bit RGB True Color image of N pixels are stored in a quantum system with $2N + m$ qubits. A pixel's color and coordinate can be stored averagely with $2 + m/N$ qubits even if RGB True Color becomes 48 bits even more or the pixels of an image increase.

For some applications which simply need the colors without the coordinates of an image , Algorithm 3′ is created by the following changes on Algorithm 3: Delete the Step 4 of Algorithm 3 and define $|\psi_i>$ in Step 5 as $|\psi_i>= |w_j>$. Applying algorithm 3′ on the image in Fig. 4, the quantum states in Q_2 are that $|\psi_i>= |0>$ ($i \in \{1, 5, 7, 8\}$) and $|\psi_j>= |1>$ ($j \in \{2, 3, 4, 6, 9\}$). Because the quantum states in Q_2 are some basis states in this case, a quantum state is retrieved by a single measurement. The image of N pixels are stored in quantum system with $N + m$ qubits by Algorithm 3′. A pixel's color is stored averagely with $1 + m/N$ qubits.

2.2 Retrieving an image from a quantum system

We define an observable operator on projection measurement $P = \sum_{i=1}^{2} m_i P_i$, where $P_1 = |0><0|$, $P_2 = |1><1|$, $m_1 = +1$, $m_2 = -1$. And then we retrieve the image in quantum queues Q_1 and Q_2 by applying the observable operator P. For example, applying P to measure the quantum state $|v_j>= \cos\phi_j|0> + \sin\phi_j|1>$ in quantum queue Q_1(shown in Fig. 4):

Result is $+1$ with probability

$$p(+1) = \cos^2\phi_j \tag{2}$$

According to (2), we calculate

$$\phi_j = \arccos\left(\sqrt{p(+1)}\right) \tag{3}$$

Result is -1 with probability

$$p(-1) = \sin^2\phi_j \tag{4}$$

By (4), we obtain

$$\phi_j = \arcsin\left(\sqrt{p(-1)}\right) \tag{5}$$

The number of measurements is finite, so the statistical results for the probability of $+1$ or -1 are only estimates of the true probability. Suppose that the numbers of $+1$ and -1 are respectively n_1 and n_2 in n measurements. Set $\hat{p}(+1) = \frac{n_1}{n}$ and $\hat{p}(-1) = \frac{n_2}{n}$, so we know that $\hat{p}(+1)$ is the estimate of $p(+1)$ and $\hat{p}(-1)$ is the estimate of $p(-1)$.

Substituting $\hat{p}(+1)$ for $p(+1)$ in (3) gives

$$\hat{\phi}_j = \arccos\left(\sqrt{\hat{p}(+1)}\right) \tag{6}$$

Substituting $\hat{p}(-1)$ for $p(-1)$ in (5), we obtain

$$\hat{\phi}_j = \arcsin\left(\sqrt{\hat{p}(-1)}\right) \tag{7}$$

We can calculate $\hat{\phi}_j$ by formula (6) or (7). Assuming that a quantum state is measured successfully with the probability of $1-\alpha$ with n measurements at most, now, we describe how to get the value of n (For example , how many measurements are necessary to measure successfully the quantum state $|v_j >= \cos\phi_j|0> + \sin\phi_j|1>$).

Let us define

$$X = \begin{cases} 0, & \text{result of measurment is } +1 \\ 1, & \text{result of measurment is } -1 \end{cases} \tag{8}$$

Results of measurements are only divided into two categories (+1 and −1), so X is either 1 or 0. Thus X is a Bernoulli random variable. The probability mass function of random variable X is given by

$$\begin{cases} p(0) = P\{X = 0\} = 1 - p \\ p(1) = P\{X = 1\} = p \end{cases} \tag{9}$$

p is the probability of $X = 1$, so $p = p(-1) = \sin^2\phi_j$. The expectation μ and variance σ of X are respectively $\mu = p$ and $\sigma^2 = p(1-p)$. Suppose that $X_1, X_2, \ldots X_n$ are n samples of X and n is sufficiently large, then $\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}} = \frac{n\bar{X}-np}{\sqrt{np(1-p)}}$ has approximately a standard normal distribution by the Central Limit Theorem. Thus $P\{|\frac{n\bar{X}-np}{\sqrt{np(1-p)}}| < Z^2_{\alpha/2}\} \approx 1-\alpha$ where $1-\alpha$ is a confidence level (the value of $Z_{\alpha/2}$ can be found in standard normal distribution look-up tables: e.g., when $1-\alpha = 0.95$, we can obtain $Z_{\alpha/2} = 1.96$). Solving the inequality $|\frac{n\bar{X}-np}{\sqrt{np(1-p)}}| < Z^2_{\alpha/2}$, we obtain that the confidence interval of p is $[p_{\min}, p_{\max}]$ with approximate confidence level $1-\alpha$. $p_{\min}$ and $p_{\max}$ are expressed as follows:

$$p_{\min} = \frac{2n\bar{X} + m - \sqrt{(2n\bar{X}+m)^2 - 4(n+m)n\bar{X}^2}}{2n+2m} \tag{10}$$

$$p_{\max} = \frac{2n\bar{X} + m + \sqrt{(2n\bar{X}+m)^2 - 4(n+m)n\bar{X}^2}}{2n+2m} \tag{11}$$

where $m = Z^2_{\alpha/2}$.

The size of the confidence interval $[p_{\min}, p_{\max}]$ is

$$\Delta p = p_{\max} - p_{\min} = \frac{\sqrt{4nm\bar{X} - 4nm\bar{X}^2 + m^2}}{n+m} \tag{12}$$

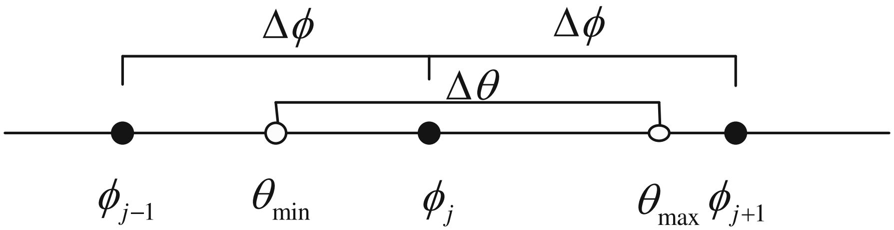


Fig. 5 The relation of $\Delta\theta$, $\Delta\phi$, ϕ_j, $\theta_{\min}$ and $\theta_{\max}$

Let

$$p_{\max} = \sin^2\theta_{\max},\ p_{\min} = \sin^2\theta_{\min} \tag{13}$$

From (13), we obtain

$$\Delta\theta = \theta_{\max} - \theta_{\min} = \arcsin\sqrt{p_{\max}} - \arcsin\sqrt{p_{\min}} \tag{14}$$

The sine function acts on the two ends of the formula (14), therefore we derive

$$\begin{aligned} \sin(\Delta\theta) &= \sqrt{p_{\max}(1-p_{\min})} - \sqrt{p_{\min}(1-p_{\max})} \\ &\le \sqrt{p_{\max}(1-p_{\min}) - p_{\min}(1-p_{\max})} = \sqrt{p_{\max} - p_{\min}} = \sqrt{\Delta p} \end{aligned} \tag{15}$$

The confidence interval of p is $[p_{\min}, p_{\max}]$ with the approximate probability of $1-\alpha$, therefore we know that $p \in [p_{\min}, p_{\max}]$. Thus we get

$$\phi_j \in [\theta_{\min}, \theta_{\max}] \tag{16}$$

According to Algorithm 1, we see

$$\Delta\phi = |\phi_{i+1} - \phi_i| = \frac{\pi}{2(M-1)} \tag{17}$$

where $i \in \{1, 2, \ldots (M-1)\}$ and $M = 2^{24}$.

Suppose

$$\sqrt{\Delta p} < \sin(\Delta\phi) \tag{18}$$

By (15) and (18), we obtain

$$\Delta\theta < \Delta\phi \tag{19}$$

When $\phi_j \le \theta_{\max} \le \phi_{j+1}$ and $\Delta\theta = \theta_{\max} - \theta_{\min} < \Delta\phi$, we can derive that $\phi_{j-1} \le \theta_{\min} \le \phi_j$. Thus $\Delta\theta$, $\Delta\phi$, ϕ_j, $\theta_{\min}$ and $\theta_{\max}$ are shown in Fig. 5. According to Fig. 5, We see that the quantum state which is measured in this case is $|v_j>\equiv \cos\phi_j|0> + \sin\phi_j|1>$.

The formula (12) is replaced into formula (18), we obtain

$$\sin^4(\Delta\phi)n^2 + \left[2m\sin^4(\Delta\phi) - 4m(\bar{X} - \bar{X}^2)\right]n + \sin^4(\Delta\phi)m^2 - m^2 > 0 \quad (20)$$

Solving

$$\sin^4(\Delta\phi)n^2 + \left[2m\sin^4(\Delta\phi) - 4m(\bar{X} - \bar{X}^2)\right]n + \sin^4(\Delta\phi)m^2 - m^2 = 0 \quad (21)$$

we obtain that two solutions are

$$n_- = \frac{1}{2}\left[\frac{4m(\bar{X} - \bar{X}^2)}{\sin^4(\Delta\phi)} - 2m - \sqrt{\left(\frac{4m(\bar{X} - \bar{X}^2)}{\sin^4(\Delta\phi)} - 2m\right)^2 + 4m^2\left(1 - \sin^8(\Delta\phi)\right)}\right] \quad (22)$$

$$n_+ = \frac{1}{2}\left[\frac{4m(\bar{X} - \bar{X}^2)}{\sin^4(\Delta\phi)} - 2m + \sqrt{\left(\frac{4m(\bar{X} - \bar{X}^2)}{\sin^4(\Delta\phi)} - 2m\right)^2 + 4m^2\left(1 - \sin^8(\Delta\phi)\right)}\right] \quad (23)$$

From (22), we know that $n_- \leq 0$, so n_- isn't selected. Thus, we obtain the correct quantum state after at most $n_{\max} = \lceil n_+ \rceil$ measurements with the approximate probability of $1-\alpha$. By (23), we see that $n_{\max} = \lceil n_+ \rceil$ increases when $\Delta\phi$ decreases.

Using the same method, we can retrieve quantum states in quantum queue Q_2(shown in Fig. 4). For example, $|\psi_i> = |w_j> \otimes |u_i>$ where $|w_j> = \cos\beta_j|0> + \sin\beta_j|1>$ and $|u_i> = \cos\theta_i|0> + \sin\theta_i|1>$. First, we measure the quantum state $|w_j>$ after $n_+(w_j)$ measurements at most where $n_+(w_j)$ can be calculated by substituting $\Delta\beta = \frac{\pi}{2(m-1)}$ for $\Delta\phi = \frac{\pi}{2(M-1)}$ in (23). Generally $m \ll M$, so we derive that $\Delta\beta \gg \Delta\phi$ and $n_+(w_j) \ll n_+(v_j)$. Second, we measure the quantum state $|u_i>$ and calculate $n_+(u_i)$ by substituting $\Delta\theta = \frac{\pi}{2(N-1)}$ for $\Delta\phi = \frac{\pi}{2(M-1)}$ in (23).

3 Image compression

Image compression solves the problem how to reduce the amount of digital image data. We can use Algorithm 3 to store an image. An image may have many of the same colors, it means that the image has color redundant data. The coordinates of the image are continuous and relate to each other, it means that coordinate redundant data exist in the image. We design Algorithm 4 to reduce the two aspects of data redundancy. Data redundancy is defined as $R_D = 1 - \frac{1}{r}$, where r is the compression ratio. r is defined as

$$r = \frac{n_1}{n_2} \quad (24)$$

where the original image is represented by n_1 qubits and the compressed image is represented by n_2 qubits.

Algorithm 4 The quantum image compression algorithm

1. A square image or a square sub-block of a square image to be compressed is named newImage. Let m be the number of different colors of the image newImage, and store the quantum states of $QSMC$ which correspond to these different colors in the quantum queue Q_1.
2. According to a scanning direction, if there are some consecutive pixels of the same color, only the first pixel is stored. Process is implemented as follows (shown in Fig. 6): The image newImage is scanned by rows from $(1, 2)$ or by columns from (2, 1) and comes to a suspension when encountering different colors so that the color and the number of the same color are stored before further scanning is conducted by taking the last pixel as the new starting point. s denotes the maximum number of consecutive pixels of the same color and n denotes the total of pixels in the compressed image.
3. Create a bijective function $F_4 : posNum \rightleftarrows \gamma$, where posNum $= \{1, 2, \ldots, m, m+1, \ldots, m+s\}$ and $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{m+s}\}(\gamma_i = \frac{\pi(i-1)}{2(m+s-1)}, i \in \{1, 2, \ldots, m+s\})$. And then, the rotation operator $R_y(2\gamma_i) = \begin{bmatrix} \cos\gamma_i & -\sin\gamma_i \\ \sin\gamma_i & \cos\gamma_i \end{bmatrix}(i = 1, 2, \ldots, m+s)$ is applied on $|0>$ to create successively $m+s$ quantum states $|\bar{w}_1>, |\bar{w}_2>, \ldots |\bar{w}_m>, |x_1>, |x_2>, \ldots, |x_s>$, where $|\bar{w}_j>$ represents the jth position of quantum queue Q_1 and $|x_i>$ represents an integer i, the definitions of $|\bar{w}_j>$ and $|x_i>$ are

$$\begin{cases} |\bar{w}_i> = \cos\gamma_i|0> + \sin\gamma_i|1>, & i \in \{1, 2, \ldots, m\} \\ |x_i> = \cos\gamma_{m+i}|0> + \sin\gamma_{m+i}|1>, & i \in \{1, 2, \ldots, s\} \end{cases} \tag{25}$$

4. $|\bar{\psi}_i>$ represents the ith pixel and is defined as

$$|\bar{\psi}_i> = \begin{cases} |\bar{w}_j> \otimes |u_x>, \ i = 1, |u_x> \in QSNC, x \in \{1, 2, \ldots, N\} \\ |\bar{w}_j> \otimes |u_y>, \ i = n, |u_y> \in QSNC, y \in \{1, 2, \ldots, N\} \\ |\bar{w}_j> \otimes |x_k>, \ i \in \{2, 3, \ldots, n-1\}, k \geq 2 \\ |\bar{w}_j>\ , \ i \in \{2, 3, \ldots, n-1\}, k = 1 \end{cases} \tag{26}$$

   where u_x and u_y respectively represent coordinates of the first pixel and last pixel of the image newImage in the original image, k is the number of consecutive pixels of the same color according to the scanning direction.
5. $|\bar{\psi}_i>$ is stored in another quantum queue Q_2.
6. Successively take $i = 1, 2, \ldots, n$, repeating Step 4 to Step 5.

We can compress an square image or a square sub-block of an image by Algorithm 4. In order to explicitly explain Algorithm 4, we use a 8×8 image shown in the right of Fig. 6 as an example. And assume that the 8×8 image is the first square sub-block of a 16×16 image, from Algorithm 2, we derive that coordinates of the first pixel and the last pixel in the sub-block are represented respectively by u_1 and u_{120},

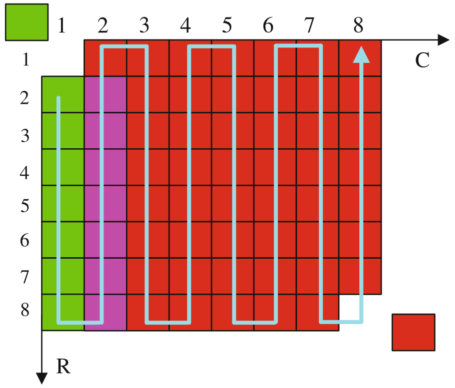


Fig. 6 Scanning by *rows* and scanning by *columns*

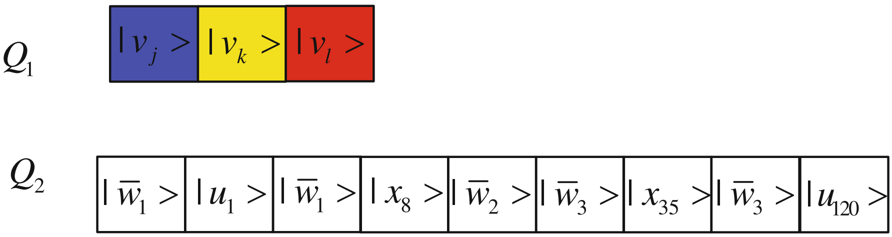


Fig. 7 The compressed image is stored in quantum queues Q_1 and Q_2

that is, $|u_x>=|u_1>$ and $|u_y>=|u_{120}>$. From Step 1, we know that $m=3$, it means that three colors are stored in Q_1. We suppose that three colors are represented respectively by $|v_j>$,$|v_k>$ and $|v_l>$. From Step 2 to 6, we see that $s=53$ (it means the maximum number of consecutive pixels of the same color is 53 by according to the scanning direction in the right of Fig. 6) and $n=5$ (it means that quantum states of five pixels are stored in Q_2). The five quantum states in Q_2 are listed as follows: $|\bar{\psi}_1>=|\bar{w}_1>\otimes|u_1>$, $|\bar{\psi}_2>=|\bar{w}_1>\otimes|x_8>$, $|\bar{\psi}_3>=|\bar{w}_2>$, $|\bar{\psi}_4>=|\bar{w}_3>\otimes|x_{53}>$, $|\bar{\psi}_5>=|\bar{w}_3>\otimes|u_{120}>$. The compressed image stored in Q_1 and Q_2 is shown in Fig. 7. The compression ratio $r=\frac{3+64*2}{3+5+4}=10.9$.

3.1 Lossless compression simulation experiment

In many applications, it is required that the compressed image does not lose any information. The compression is called lossless compression. The grayscale is a special color. $|v_1>,|v_2>,\ldots,|v_{256}>$ are created when $M=256$ by Algorithm 1, where $color_1, color_2,\ldots,color_{256}$ correspond to grayscale values $0,1,\ldots,255$. We compress the 256×256 Lena image in Fig. 8 by Algorithm 4 starting from (2, 1) scanning by columns. The number of different colors in the image $m=212$, the maximum number of the same color $s=10$, the number of pixels in the compressed image $n=56,377$, the compression ratio $r=2.058$. The distribution of the same color of adjacent pixels is shown in Fig. 9.

Fig. 8 The 256 × 256 Lena image

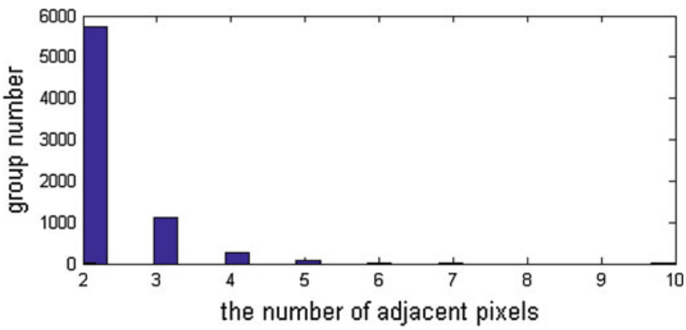


Fig. 9 The distribution of the same color of adjacent pixels in the Lena image

4 Image segmentation

Image segmentation subdivides an image into its constituent regions or objects. The level to which the subdivision is carried depends on the problem being solved [17]. By image segmentation and the target feature extraction, the original image is transformed into a more abstract and compact form of expression which is conducive to target expression, so that high-level image analysis, image understanding, automatic identification of computer model are possible. We design Algorithm 5 for quantum image segmentation. In a classical computer, there are differences in storing and encoding between color images and grayscale images, and it is difficult or massive calculation to determine whether color is similar. Therefore, the algorithms are efficient for grayscale images, which may not work for color image segmentation. In this paper, a color is

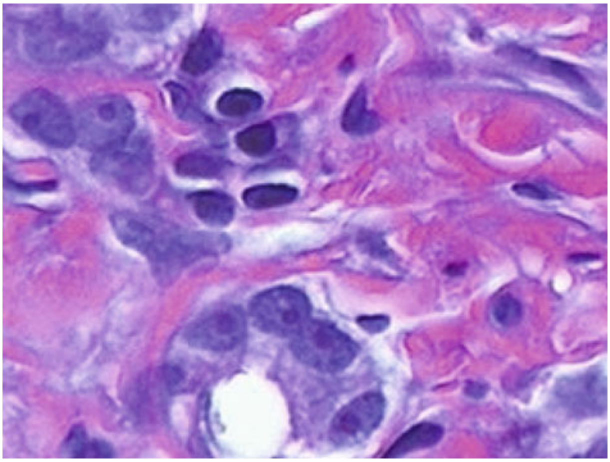


Fig. 10 A 227×303 image for segmentation

represented by the 1-qubit quantum state of $QSMC$, so Algorithm 5 is universal for grayscale images and color images.

Algorithm 5 Image segmentation based on quantum search

1. Sample the colors to be segmented and calculate color sample mean which corresponds to a new color, and suppose that the new color is represented by c_m, where c_m is the mth element in the set $color = \{color_1, \ldots, color_M\}$.
2. Store the colors of the image and the new color c_m in a quantum system by Algorithm 1 and Algorithm $3'$, so the new color c_m can be represented by the quantum state $|v_m>$.
3. Let f be a threshold, searching out colors to satisfy the condition $|v_i>\in \{|v_{m-f}>, |v_{m-f+1}>, \ldots, |v_{m+f}>\}$ by an extended Grover's quantum search algorithm, where $|v_i>$ corresponds to the color to be found and is the ith element in the set $\{|v_1>, |v_2>, \ldots, |v_M>\}$ corresponding to the color set $\{color_1, color_2, \ldots, color_M\}$ (see Algorithm 1).
4. If the searched-out colors are too few and there are still some colors left to be found out, it means the segmentation is "too fine" [18], and then the threshold $f = f + a, a < f$, is to be increased and go back to Step 2. However, if the searched-out colors are too many and some unwanted colors are scanned, it means the segmentation is "too coarse" [18], and then the threshold $f = f - b, b < f$, is to be reduced and go back to Step 2. If none of the above cases, exit. Where f, a and b are three integers.

In Step 1, Matlab convert an image into an image matrix and calculate the mean of the colors in sampling area by formula $m = \lfloor(\sum_{c_i} i)/size(A)\rfloor$, where A is a sampling area, $size(A)$ is the number of colors in A and c_i is the color of a pixel in A.

In order to improve the efficiency of Algorithm 5, an extended Grover's quantum search algorithm — "the quantum search algorithm for all solutions" is adopted in Step 3, detailed design of which is in Sect. 5.

We segment a 227×303 image in Fig. 10 by Algorithm 5 and sample a small area around (87, 226). Segmentation result is shown in Fig. 11.

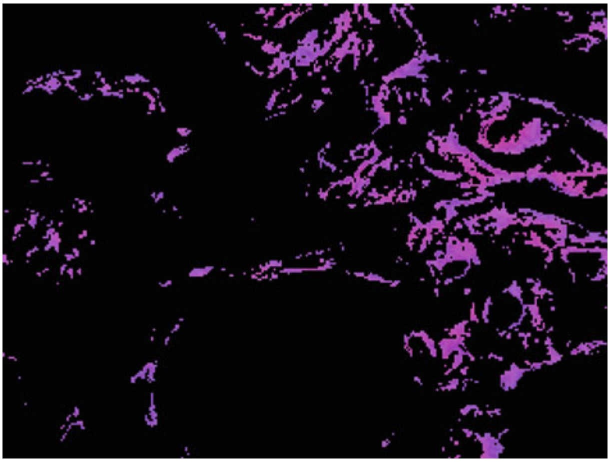


Fig. 11 Image segmentation result for the 227 × 303 image

5 The quantum search algorithm for all the solutions (the unknown number of solutions)

All of colors less than a certain threshold value should be found in image segmentation. The Grover search algorithm is expanded to solve such problems in this paper. The core conception of this algorithm is to search out all solutions in decreased iteration times but not only to improve the probability of search success for one solution.

In order to make the algorithm explicit, we divide the algorithm into three parts: Quantum counting algorithm in Sect. 5.1, another core sub-algorithm in Sect. 5.2 and the complete algorithm in Sect. 5.3.

5.1 Quantum counting algorithm

The phase of the initial state in the Grover search algorithm [12] can be estimated by using phase estimation algorithm for the number of unknown solutions [10,19,20]. On the basis of the previous algorithm ideas, this paper is to find the eigenvectors and eigenvalues of iteration operator G, then to estimate the phase of the initial state by quantum counting algorithm algorithm.

Let us rewrite the initial state of Grover algorithm

$$|\psi>=\sum_{x\in M}\frac{1}{\sqrt{N}}|x>+\sum_{x\in \tilde{M}}\frac{1}{\sqrt{N}}|x>=\sqrt{\frac{t}{N}}|a>+\sqrt{\frac{N-t}{N}}|b> \tag{27}$$

where

$$|a>=\frac{1}{\sqrt{t}}\sum_{x\in M}|x>,|b>=\frac{1}{\sqrt{N-t}}\sum_{x\in \tilde{M}}|x> \tag{28}$$

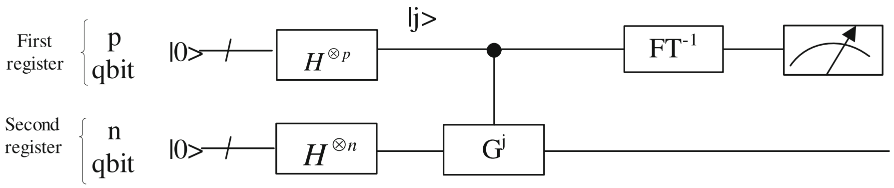


Fig. 12 Quantum circuit of quantum counting algorithm

Set

$$|\psi_0> = 1/2(|a> + i|b>) \tag{29}$$
$$|\psi_1> = 1/2(|a> - i|b>) \tag{30}$$

Since $<\psi_0|\psi_0>= 1, <\psi_1|\psi_1>= 1, <\psi_0|\psi_1>= 0$, we see that $|\psi_0>$ and $|\psi_1>$ constitute a set of orthogonal bases. So initial state is restated as

$$|\psi>=\left(\sqrt{\frac{t}{N}}-i\sqrt{\frac{N-t}{N}}\right)|\psi_0> + \left(\sqrt{\frac{t}{N}}+i\sqrt{\frac{N-t}{N}}\right)|\psi_1> \tag{31}$$

Setting $e^{2\pi i\alpha} = \sqrt{\frac{t}{N}} + i\sqrt{\frac{N-t}{N}}$, Eq. (31) is simplified to

$$|\psi>= e^{-2\pi i\alpha}|\psi_0> + e^{2\pi i\alpha}|\psi_1> \tag{32}$$

The G operator is applied respectively to Eqs. (29) and (30), getting the following equations

$$G|\psi_0> = \left(\left(1-\frac{2t}{N}\right)+\frac{2i\sqrt{t(N-t)}}{N}\right)|\psi_0> \tag{33}$$
$$G|\psi_1> = \left(\left(1-\frac{2t}{N}\right)-\frac{2i\sqrt{t(N-t)}}{N}\right)|\psi_1> \tag{34}$$

Let us define

$$e^{2\pi i\phi} = \left(1-\frac{2t}{N}\right)+\frac{2i\sqrt{t(N-t)}}{N} \tag{35}$$

From Eqs. (28), (31) and (35), we derive

$$\phi = 2\alpha \tag{36}$$

By Eqs. (33) and (34), we know that $|\psi_0>$ and $|\psi_1>$ are G's eigenvectors, corresponding eigenvalues are $e^{2\pi i\phi}$ and $e^{2\pi i(1-\phi)}$. Therefore, quantum counting estimation algorithm shown in Fig. 12 can be used to estimate ϕ.

FT^{-1} is quantum inverse Fourier transform in Fig. 12 as

$$FT^{-1}\left(\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1} e^{2\pi ij\phi/N}|j>\right)=|\phi> \tag{37}$$

After applying j times G operator in second register in Fig. 12, quantum states become

$$e^{-2\pi i\theta}\sum_{j=0}^{2^p-1} e^{2\pi(1-\phi)ij}|j>|\psi_0> +e^{2\pi i\theta}\sum_{j=0}^{2^p-1} e^{2\pi ij\phi}|j>|\psi_1> \tag{38}$$

And then applying the inverse Fourier transform to the first register in Fig. 12, and finally measuring the first register, measurement results $\hat{\phi}$ is the ϕ's estimation value to m bits accuracy with the success probability of at least $1-\varepsilon$. Because $\alpha=\phi/2$, $\hat{\alpha}$ is the α's estimation value to $m+1$ bits accuracy. We know that $p=m+\lceil\log(2+1/(2\varepsilon))\rceil$ was derived by phase estimation algorithm presented in [10] where p is the number of the base states in the first register (see Fig. 12) , the number of calling Oracle in the algorithm is at most $k\leq 2^p$.

Letting $\theta=\frac{\pi}{2}-2\pi\alpha$, we obtain that $\sin\theta=\sqrt{\frac{t}{N}}$ and $\cos\theta=\sqrt{\frac{N-t}{N}}$ by $e^{2\pi i\alpha}=\sqrt{\frac{t}{N}}+i\sqrt{\frac{N-t}{N}}$, thus θ is the phase of the initial state in the Grover search algorithm. Setting $\hat{\theta}=\frac{\pi}{2}-2\pi\hat{\alpha}$, hence

$$|\Delta\theta|=|\theta-\hat{\theta}|=2\pi\left|\alpha-\hat{\alpha}\right|\leq\pi 2^{-m} \tag{39}$$

Quantum counting algorithm can determine whether there is the solution for the search problem, specific analysis as follows:

Since the quantum search algorithm in this paper is used in areas such as image segmentation, the search space N is a large number. Thus ,we assume that $n=\log N\geq 10$. When $m=\lceil\frac{n}{2}\rceil+3$ and $\gamma=2\pi\times 2^{-(m+1)}=\frac{\pi}{8}2^{-\lceil\frac{n}{2}\rceil}$, from (39) we yield $|\Delta\theta|\leq\gamma$. Since $\hat{\theta}$ is the θ's estimation value to $m+1$ bits accuracy, $\hat{\theta}/\gamma$ is an integer. if $\hat{\theta}=\gamma$, $\hat{\theta}=2\gamma$, and $\hat{\theta}=3\gamma$, we have

$$\begin{cases} 0\leq\theta\leq\frac{\pi}{4}2^{-\lceil\frac{n}{2}\rceil},\ if\ \hat{\theta}=\gamma \\ \frac{\pi}{8}2^{-\lceil\frac{n}{2}\rceil}\leq\theta\leq\frac{3\pi}{8}2^{-\lceil\frac{n}{2}\rceil},\ if\ \hat{\theta}=2\gamma \\ \frac{\pi}{4}2^{-\lceil\frac{n}{2}\rceil}\leq\theta\leq\frac{\pi}{2}2^{-\lceil\frac{n}{2}\rceil},\ if\ \hat{\theta}=3\gamma \end{cases} \tag{40}$$

When $0<x<\frac{\sqrt{2}}{2}$, we see that $f(x)=\frac{\sqrt{2}\pi}{4}x>f(x)=\arcsin(x)$, so $\theta=\arcsin(\sqrt{\frac{t}{N}})<\frac{\sqrt{2t}\pi}{4}2^{-\frac{n}{2}}$. And $\theta>\sin\theta=\sqrt{\frac{t}{N}}=\sqrt{t}2^{-\frac{n}{2}}$, hence, we derive

$$\begin{cases} 2^{-\frac{n}{2}}<\theta<\frac{\sqrt{2}\pi}{4}2^{-\frac{n}{2}},\ if\ t=1 \\ 2^{-\frac{n}{2}+\frac{1}{2}}<\theta<\frac{\pi}{2}2^{-\frac{n}{2}},\ if\ t=2 \end{cases} \tag{41}$$

For $\hat{\theta} = 2\gamma$, we assume that n is even with a probability of 0.5 and θ uniformly distributes in the interval $[\frac{\pi}{8}2^{-\lceil\frac{n}{2}\rceil}, \frac{3\pi}{8}2^{-\lceil\frac{n}{2}\rceil}]$. If n is odd, we know that $t = 0$. If n is even , we see that $t = 1$ with a probability of $\frac{3\pi-8}{2\pi}$ and $t = 0$ with a probability of $\frac{8-\pi}{2\pi}$. From (40) and (41), we conclude that

$$\begin{cases} t = 0,\ if\ \hat{\theta} \le \gamma \\ t = 0,\ with\ a\ probability\ at\ least\ 0.88\ if\ \hat{\theta} = 2\gamma \\ t = 1,\ with\ a\ probability\ at\ most\ 0.12\ if\ \hat{\theta} = 2\gamma \\ t \ge 1,\ if\ \hat{\theta} \ge 3\gamma \end{cases} \tag{42}$$

5.2 Quantum counting Grover's algorithm (QCG)

Known conditions: The initial state $|\psi_0 > \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x>$, $\hat{\theta}$ is an estimation of θ, $\sin^2\theta = t/N$(t is unknown), $3\gamma \le \hat{\theta} \le \pi/4 - \gamma$, $j_1 = \left\lfloor \frac{\pi}{4(\hat{\theta}+\gamma)} \right\rfloor$, $j_2 = \left\lfloor \frac{\pi}{4(\hat{\theta}-\gamma)} \right\rfloor$.

Output: i.

Procedure:

1. Choose an integer uniformly at random from $[j_1, j_2]$ as j.
2. Apply j iterations of Grover's algorithm starting from $|\psi_0>$.
3. Measure the first register: output i.

Lemma 1 $\sin(2d\theta) \ge 0$ *when* $3\gamma \le \hat{\theta} \le \pi/4 - \gamma$ *and* $d = j_2 - j_1 + 1$, *where* $\gamma = \frac{\pi}{8}2^{-\lceil\frac{n}{2}\rceil}$, $j_1 = \left\lfloor \frac{\pi}{4(\hat{\theta}+\gamma)} \right\rfloor$ *and* $j_2 = \left\lfloor \frac{\pi}{4(\hat{\theta}-\gamma)} \right\rfloor$.

Proof If $\hat{\theta} \ge \pi/8$, we see that $\left\lfloor \frac{\pi}{4(\pi/4-2\gamma)} \right\rfloor \le j_2 \le \left\lfloor \frac{\pi}{4(\pi/8-\gamma)} \right\rfloor$ and $\left\lfloor \frac{\pi}{4(\pi/4)} \right\rfloor \le j_1 \le \left\lfloor \frac{\pi}{4(\pi/8+\gamma)} \right\rfloor$, namely, $1 \le j_2 \le 2$ and $j_1 = 1$. Hence, $1 \le d \le 2$, we have

$$2d\theta \le 4(\hat{\theta} + \gamma) < \pi \tag{43}$$

where $\hat{\theta} \ge \pi/8$.

If $\hat{\theta} < \pi/8$, now

$$\begin{aligned} 2d\theta &= 2\theta\left(\left\lfloor \frac{\pi}{4(\hat{\theta}-\gamma)} \right\rfloor - \left\lfloor \frac{\pi}{4(\hat{\theta}+\gamma)} \right\rfloor + 1\right) \le \frac{\theta\pi}{2}\left(\frac{1}{\hat{\theta}-\gamma} - \frac{1}{\hat{\theta}+\gamma}\right) + 4\theta \\ &\le \frac{\pi\gamma}{(\hat{\theta}-\gamma)} + 4(\hat{\theta}+\gamma) \end{aligned} \tag{44}$$

Setting $\hat{\theta} = x\gamma$, so formula (44) is rewritten as

$$2d\theta \le \left(\frac{\pi}{x-1} + 4(x+1)\gamma\right) \tag{45}$$

By known conditions $3\gamma \le \hat{\theta} < \pi/8$ and $\hat{\theta} = x\gamma$, we derive that $3 \le x < \frac{\pi}{8\gamma}$. Letting function $f(x) = \frac{\pi}{x-1} + 4(x+1)\gamma$, where $3 \le x < \frac{\pi}{8\gamma}$, we solve that the maximum value of the function $f(x)$ is $\max(f(3), f(\frac{\pi}{8\gamma}))$. $f(x)$ substitutes for $\frac{\pi}{x-1} + 4(x+1)\gamma$ in (45) , so we yield that

$$2d\theta \leq f(x) \leq \max\left(f(3), f\left(\frac{\pi}{8\gamma}\right)\right) < \pi \tag{46}$$

where $3\gamma \leq \hat{\theta} < \pi/8$.

From (43) and (46), we obtain that $2d\theta < \pi$ when $3\gamma \leq \hat{\theta} \leq \pi/4 - \gamma$, so $\sin(2d\theta) > 0$. □

Lemma 2 $\cos(4j_1\theta + 2d\theta) \leq 0$ *when* $4\gamma \leq \hat{\theta} \leq \pi/4 - \gamma$ *and* $d = j_2 - j_1 + 1$, *where* $\gamma = \frac{\pi}{8}2^{-\lceil\frac{n}{2}\rceil}$, $j_1 = \left\lfloor\frac{\pi}{4(\hat{\theta}+\gamma)}\right\rfloor$ *and* $j_2 = \left\lfloor\frac{\pi}{4(\hat{\theta}-\gamma)}\right\rfloor$.

Proof Setting $\hat{\theta} = x\gamma$ and $\omega = 4j_1\theta + 2d\theta = 2\theta(j_1 + j_2 + 1)$, now

$$\omega \geq 2\theta\left(\frac{\pi}{4(\hat{\theta}-\gamma)} + \frac{\pi}{4(\hat{\theta}+\gamma)} - 1\right) \geq \frac{\pi\hat{\theta}}{\hat{\theta}+\gamma} - 2(\hat{\theta}-\gamma) = \frac{x\pi}{x+1} - 2(x-1)\gamma \tag{47}$$

$$\omega \leq 2\theta\left(\frac{\pi}{4(\hat{\theta}-\gamma)} + \frac{\pi}{4(\hat{\theta}+\gamma)} + 1\right) \leq \pi\frac{\hat{\theta}}{\hat{\theta}-\gamma} + 2(\hat{\theta}+\gamma) = \frac{x\pi}{x-1} + 2(x+1)\gamma \tag{48}$$

From known conditions $4\gamma \leq \hat{\theta} \leq \pi/4 - \gamma$ and $\hat{\theta} = x\gamma$, we derive that $4 \leq x \leq (\frac{\pi}{4\gamma} - 1)$. Letting $f(x) = \frac{x\pi}{x+1} - 2(x-1)\gamma$ where $4 \leq x \leq (\frac{\pi}{4\gamma} - 1)$. $f(x)$ substitutes for $\frac{x\pi}{x+1} - 2(x-1)\gamma$ in (47) gives

$$\omega \geq f(x) \geq \min(f(4), f(\pi/(4\gamma) - 1)) \geq \pi/2 \tag{49}$$

where $4 \leq x \leq (\frac{\pi}{4\gamma} - 1)$, namely, $4\gamma \leq \hat{\theta} \leq \pi/4 - \gamma$.

If $\hat{\theta} \geq (\pi/8 + 2\gamma)$, we have that $j_2 = 1$, $j_1 = 1$ and $d = 1$. Now

$$\omega = 4j_1\theta + 2d\theta = 6\theta \leq 6(\hat{\theta} + \gamma) \leq 3\pi/2 \tag{50}$$

where $(\pi/8 + 2\gamma) \leq \hat{\theta} \leq (\pi/4 - \gamma)$.

If $4\gamma \leq \hat{\theta} < (\pi/8 + 2\gamma)$, Setting $f(x) = \frac{x\pi}{x-1} + 2(x+1)\gamma$ where $4 \leq x < (\frac{\pi}{8\gamma} + 2)$, $f(x)$ substitutes for $\frac{x\pi}{x-1} + 2(x+1)\gamma$ in (48), we obtain

$$\omega \leq f(x) \leq \max\left(f(4), f\left(\frac{\pi}{8\gamma} + 2\right)\right) < 3\pi/2 \tag{51}$$

where $4 \leq x < (\frac{\pi}{8\gamma} + 2)$, namely, $4\gamma \leq \hat{\theta} < (\pi/8 + 2\gamma)$.

From (49), (50) and (51), we obtain $\pi/2 \leq \omega \leq 3\pi/2$ where $4\gamma \leq \hat{\theta} < (\pi/4 - \gamma)$, namely, $\cos(4j_1\theta + 2d\theta) \leq 0$.

Theorem 1 *The QCG algorithm searches out a solution with expected probability at least 1/2; Times of G operator iteration is at most* $\frac{\pi}{2}\sqrt{\frac{N}{t}}$ *in the algorithm.*

Proof When $\hat{\theta} \geq 3\gamma$, we know that $t \geq 1$ from (42). Let j be the number of G operator iteration and $\hat{\theta} = x\gamma$ where $x \geq 3$. Now, $\theta \leq (x+1)\gamma$, we have that $j \leq j_2 \leq \frac{\pi}{4(\hat{\theta}-\gamma)} = \frac{\pi}{4\gamma(x-1)} \leq \frac{\pi}{4\theta}\frac{x+1}{x-1} \leq \frac{\pi}{2}\sqrt{\frac{N}{t}}$, that is, the number of G operator iteration is at most $\frac{\pi}{2}\sqrt{\frac{N}{t}}$.

There is

$$\sum_{l=0}^{d-1}\cos(\alpha+2\beta l) = \frac{\sin(l\beta)\cos(\alpha+(d-1)\beta)}{\sin\beta} \tag{52}$$

j is chosen by uniformly at random from $[j_1, j_2]$, so

$$\begin{aligned} P_d &= \sum_{l=0}^{d-1}\frac{1}{d}\sin^2((2(j_1+l)+1)\theta) \\ &= \frac{1}{2} - \frac{1}{2d}\sum_{l=0}^{d-1}\cos((4j_1\theta+2\theta)+2l(2\theta)) \end{aligned} \tag{53}$$

By (52), we rewrite (53) as

$$P_d = \frac{1}{2} - p \tag{54}$$

where $p = \frac{\sin(2d\theta)\cos(4j_1\theta+2d\theta)}{2d\sin(2\theta)}$.

When $4\gamma \leq \hat{\theta} \leq \pi/4-\gamma$, from Lemmas 1 and 2 , we derive

$$p \leq 0 \tag{55}$$

When $3\gamma \leq \hat{\theta} < 4\gamma$, namely, $\hat{\theta} = 3\gamma$, we see that $w = 4j_1\theta + 2d\theta = 2\theta(j_1+j_2+1) = \frac{3\theta\pi}{8\gamma} + 2\theta$, in this case, $\frac{3\pi}{4} + 2\theta \leq w \leq \frac{3\pi}{2} + 2\theta$. Supposing $4j_1\beta + 2d\beta = \frac{3}{2}\pi$, for $\theta \leq \beta$, we have that $\frac{3\pi}{4} < w \leq \frac{3}{2}\pi$, so $\cos(4j_1\theta+2d\theta) \leq 0$. And we know that $\sin(2d\theta) > 0$ from Lemma 1. Thus we get $p \leq 0$ when $3\gamma \leq \hat{\theta} < 4\gamma$ and $\theta \leq \beta$. For $\beta < \theta \leq 4\gamma$, we obtain that $\frac{3}{2}\pi < w \leq \frac{3}{2}\pi + 2\theta$ and $p \leq \frac{1}{2d} \approx 0$.

So $P_d \geq \frac{1}{2}$ when $3\gamma \leq \hat{\theta} \leq \pi/4-\gamma$.

5.3 Search algorithm to search for all solutions with the unknown number of solutions

Known conditions: The initial state $|0\rangle^{\otimes p}|0\rangle^{\otimes n}$, $N = 2^n (n \geq 10)$ is a size of state space, while t is the number of solutions (unknown), $M = \{i|T[i] = x\}$ is the set of marked states; $\tilde{M} = \{i|T[i] \neq x\}$ is the set of unmarked states. $\varepsilon = 1/6, m = \lceil n/2 \rceil + 3, p = \lceil n/2 \rceil + 5$.

Output: all solutions.

Procedure:

1. Apply quantum counting algorithm (Sect. 5.1) to the initial state.
2. Measure the first register: let $\hat{\theta}$ be output and calculate $\hat{t} = N\sin^2\hat{\theta}$.
3. If $\hat{\theta} > \pi/4-\gamma$, choose i at random from search space. If $T[i] \neq x$, take randomly i from search space again, and judge whether i is the solution. If a solution in five times in a row is not found, go back to Step 1. If $T[i] = x$, delete the marked state $|i>$ from M before being added $\tilde{M}$, in this case, $\hat{t} = \hat{t}-1$ and $\hat{\theta} = \arcsin(\sqrt{(\hat{t}/N)})$, go back to Step 3.
4. If $\hat{\theta} < 3\gamma$, the number of solutions is zero, exit.
5. If $\hat{\theta} \geq 3\gamma$, calculate $j_1 = \left\lfloor \frac{\pi}{4(\hat{\theta}+\gamma)} \right\rfloor$, $j_2 = \left\lfloor \frac{\pi}{4(\hat{\theta}-\gamma)} \right\rfloor$.
6. Apply QCG algorithm, output i.
7. If $T[i] \neq x$, apply QCG algorithm again, and judge whether i is the solution. If can't find a solution in $\max(5, (j_2 - j_1))$ times in a row, go back to Step 1.
8. If $T[i] = x$, delete the marked state $|i>$ from M and add it to $\tilde{M}$, at the same time, $\hat{t} = \hat{t} - 1$ and $\hat{\theta} = \arcsin(\sqrt{(\hat{t}/N)})$, go back to Step 4.

Theorem 2 *This algorithm searches out all solutions in expected times in $O(t\sqrt{N})$.*

Proof Applying quantum counting algorithm to the initial state, $\hat{\theta}$ is obtained with probability of success at least $1 - \varepsilon = 5/6$ and the number of call Oracle is $k \leq 2^p \leq 32\sqrt{N}$.

When $\hat{\theta} > \pi/4 - \gamma$, we derive $\theta \geq \hat{\theta} - \gamma > \pi/4 - 2\gamma \approx \frac{\pi}{4}$ and $t \geq N/2$. Here, take randomly an item from search space again, and then check whether it is a solution using Oracle. The approach has a success probability at least 1/2.

If $\hat{\theta} < 3\gamma$, we see that the number of solutions is zero with at least $\frac{5}{6} \times 0.88 > 0.73$ probability of success by (42). If $\hat{\theta} \geq 3\gamma$, we determine that the number of solutions isn't zero with at least $5/6$ probability of success. So the algorithm can search out $t-1$ solutions with 100 % probability of success and find the last solution with at least 0.73 probability of success.

The expected times is estimated for this algorithm to search out all solutions as follows:

$t \geq N/2$ when $\hat{\theta} > \pi/4-\gamma$. Setting $t_1 = t - N/2$, the expected times is to search out t_1 solutions for the algorithm

$$E(t_1) \leq 2t_1 = 2t - N \tag{56}$$

$0 < t \leq N/2$ when $3\gamma \leq \hat{\theta} \leq \pi/4-\gamma$. We know that the QCG algorithm searches out a solution with expected probability at least 1/2 and the number of iterations $k \leq \frac{\pi}{2}\sqrt{\frac{N}{t}}$ from Theorem 1. So the number of expectation is to search out t solutions

$$\sum_{j\leq t, t\leq N/2} E(K_j) \leq \sum_{j\leq t, t\leq N/2} \frac{\pi}{2}\sqrt{\frac{N}{j}} \leq \min\left(\frac{N}{2}, t\right)\frac{\pi}{2}\sqrt{N} \tag{57}$$

When $\hat{\theta}(|\Delta\theta| \leq \gamma)$ is obtained by applying quantum counting algorithm with success probability of 100 %, the number of expectation is at most

$$\frac{6}{5} \times 32\sqrt{N} < 39\sqrt{N} \tag{58}$$

By (56) + (57) + (58), thus the number of expectation is at most to search out all solutions

$$\min\left(\frac{N}{2}, t\right)\frac{\pi}{2}\sqrt{N} + \max(2t - N, 0) + 9\sqrt{N} \tag{59}$$

So the algorithm searches out all solutions in expected times of $O(t\sqrt{N})$. □

6 Summary

In this paper, $QSMC$ represents M different colors and $QSNC$ represents the coordinates of N pixels. In a classical computer, a color of 2^{24} True Color is represented with 24-bits , and a color of 2^{48} with 48-bits, while a color is represented with only 1-qubit for 2^{24} or 2^{48} True Color. An image with N pixels and m different colors is stored in a quantum system by algorithm 3 only using $2N+m$ quantum states , while the image is stored by FRQI [5] with $(1+logN)N$ quantum states. An image of N pixels is stored a quantum system with $N+m$ quantum states by Algorithm 3′. Since m different colors are stored in the quantum queue Q_1 by Algorithm 3, and m quantum states instead of N states are retrieved for the image colors, the few number of identical states are prepared when $N \gg m$. On the basis of storing an image by Algorithm 3, lossless compression experiment on the Lena image shows that the compression ratio is 2.058 by Algorithm 4. The sub-algorithm of the image segmentation algorithm, the quantum search algorithm for all solutions(the unknown number of solutions), searches out a solution with expected probability at least 1/2 and the number of iterations at most $\frac{\pi}{2}\sqrt{\frac{N}{t}}$ and searches out all solutions in the expected times in $O(t\sqrt{N})$.

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References

1. Feynman, R.P.: Simulating physics with computers. Int. J. Theor. Phys. **21**, 467–488 (1982)
2. Deutsch, D.: Quantum theory, the Church-Turing principle and the universal quantum computer. Proc. Roy. Soc. Lond. A **400**, 97–117 (1985)
3. Venegas-Andraca, S.E., Bose, S.: Storing, processing and retrieving an image using quantum mechanics. In: Proceeding of SPIE Conference Quantum Information and Computation, vol. 5105, pp. 137–147 (2003)

4. Venegas-Andraca, S.E., Ball, J.L.: Processing images in entangled quantum systems. Quantum Inf. Process. **9**(1), 1–11 (2010)
5. Le, P.Q., Dong, F., Hirota, K.: A flexible representation of quantum images for polynomial preparation, image compression, and processing operations. Quantum Inf. Process. **10**, 63–84 (2011)
6. Le, P.Q., Iliyasu, A.M., Dong, F., Hirota, K.: Strategies for designing geometric transformations on quantum images. Theor. Comput. Sci. **412**, 1406–1418 (2011)
7. Le, P.Q., Iliyasu, A.M., Dong, F., Hirota, K.: Efficient color transformations on quantum images. J Adv Comput Intell Intell Inf **15**(6), 698–706 (2011)
8. Beach, G., Lomont, C., Cohen, C.: Quantum image processing (quip). In: Proceedings of Applied Imagery Pattern Recognition Workshop, pp. 39–44 (2003)
9. Chao-Yang, P., Zheng-Wei, Z., Guang-Can, G.: A hybrid quantum encoding algorithm of vector quantization for image compression. Chin. Phys. **15**(12), 3039–3043 (2006)
10. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
11. Shor, P.W.: Algorithms for quantum computation: Discrete logarithms and factoring. In: Proceedings of the 35th Annual IEEE Symposium on Foundations of Computer Science, pp. 124–134 (1994)
12. Grover, L.K.: A fast quantum mechanical algorithm for database search. In: Proceedings of the 28th Annual ACM Symposium on Theory of Computing (1996)
13. Grover, L.K.: Quantum mechanics helps in searching for a needle in a haystack. Phys. Rev. Lett. **79**, 325–328 (1997)
14. Zhou, R.G., Wang, H., Wu, Q., Yang, S.: Quantum associative neural network with nonlinear search algorithm. Int. J. Theor. Phys. **51**(3), 705–723 (2012)
15. Zhou, R.G., Nan, J., Qiulin, D.: Multi-pattern recognition based on Grover's algorithmChinese. J. Electron. **16**(4), 679–682 (2007)
16. Boyer, M., Brassard, G., Hyer, P., Tapp, A.: Tight bounds on quantum searching. Fortsch. Phys. **46**, 493–500 (1998)
17. Gonzalez, R.C., Woods, R.E.: Digital Image Processing, 2/E. Publishing House of Electronics Industry, Beijing (2007)
18. Felzenszwalb, P.F.: Efficient graph-based image segmentation. Int. J. Comput. Vis. **59**(2), 167–181 (2004)
19. Mosca, M.: Quantum searching, counting and amplitude amplification by eigenvector analysis. In: Proceedings of Randomized Algorithms, Workshop of Mathematical Foundations of Computer Science, pp. 90–100 (1998)
20. Brassard, G., Hϕyer, P., Tapp, A.: Quantum Counting. Quant-ph/9805082V1, 1–12