A Multi-Channel Representation for Images on Quantum Computers using the $RGB\alpha$ Color Space

Bo Sun, Phuc Q. Le, Abdullah M. Iliyasu, Fei Yan, J. Adrian Garcia, Fangyan Dong, and Kaoru Hirota
Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, Tokyo, Japan
{ sun, hirota} @hrt.dis.titech.ac.jp

Abstract– A Multi-Channel Representation for Quantum Image (MCRQI) is proposed to facilitate the further image processing tasks based on the Flexible Representation for Quantum Image (FRQI). Channel Swapping Operation, One Channel Operation, are proposed as basic image processing operations on MCRQI representation. The simulation experiment results on classical computer show that the channel information of R G B (color) and α (transparency) can be carried easily on quantum computer by employing three qubits to represent the color space, and also indicate that this MCRQI is very flexible to realize some classic-like operations. The MCRQI provides a foundation not only express image in $RGB\alpha$ color space, but also to explore theoretical and practical aspects of image processing on quantum computer.

Keywords – Quantum Computation, Image Preparation, Image Processing, Complexity, Color Space

I. INTRODUCTION

Quantum image processing is a new and exciting sub-discipline of quantum computation that aims to extend classical image processing applications on a quantum computer. The first step in this direction involves proposals on representations to capture and store the image on quantum computers. In [1] the images are two dimensions arrays of qubits, called the Qubit Lattice. The image is a normalized quantum state that captures information about the colors and position of every point in [2]. Because of its flexibility in terms the representation and processing operations based on it, this representation was appropriately named the Flexible Representation for Quantum Image or Simply FRQI, and it has started gaining acceptability as evinced by proposals targeting a wide range of applications that rely on it.

Various strategies exploiting the FRQI representation were used to propose processing transformations that target the geometric information, GTQI [3], and the color information CTQI [4] of image encoded using this representation. The possibility of using the restricted versions of these representations to watermark FRQI quantum images was explored in [5].

Inspired by FRQI representation which treats the color as a single channel, a Multi-Channel Representation for Quantum Image is proposed to capture $RGB\alpha$ channels information. This is accomplished by assigning three qubits to encode the image's information of color and transparency. The experimental results show that the MCRQI can be realized by image storage and retrieval simulation. The MCRQI paves the way for high-leveled quantum image processing tasks such as employing channel swapping for image watermarking.

The rest of paper is organized as follows. The MCRQI is presented in 2. Its polynomial preparation is discussed in 3. MCRQI based image processing operations are proposed in 4. Experiments on a few simple image processing tasks based on this representation are reported in 5.

II. MULTI-CHANNEL REPRESENTATION FOR QUANTUM IMAGE

The Multi-Channel Representation for Quantum Image to capture the $RGB\alpha$ channel information is presented in (1).

$$\left|I(\theta)\right\rangle = \frac{1}{2^{n+1}} \sum_{i=0}^{2^{2n}-1} \left|c^i_{RGB\alpha}\right\rangle \otimes \left|i\right\rangle. \tag{1}$$

The color information $\left|c^i_{RGB\alpha}\right\rangle$ carrying the information R, G, B and α channels defined as,

$$\begin{aligned} \left|c^i_{RGB\alpha}\right\rangle = &\cos\theta_{Ri}\left|000\right\rangle + \cos\theta_{Gi}\left|001\right\rangle + \\ &+\cos\theta_{Bi}\left|010\right\rangle + \cos\theta_{\alpha i}\left|011\right\rangle + \\ &+\sin\theta_{Ri}\left|100\right\rangle + \sin\theta_{Gi}\left|101\right\rangle + \\ &+\sin\theta_{Bi}\left|110\right\rangle + \sin\theta_{\alpha i}\left|111\right\rangle. \end{aligned} \tag{2}$$

Where $\theta_{Ri}, \theta_{Gi}, \theta_{Bi}, \theta_{\alpha i} \in [0, \frac{\pi}{2}]$ are the vectors encoding the colors of the R, G, B and α channels respectively; $\otimes$ is the tensor product notation; $\left|000\right\rangle, \left|001\right\rangle, \cdots, \left|111\right\rangle$ are $8-D$ computational basis quantum states; $\left|i\right\rangle$ for $i = 0, 1, \cdots, 2^{2n}-1$ are $2^{2n}-D$ computational basis states.

Fig.1 and Fig.2 show the revised circuit to encode and process FRQI quantum image based on the modified R, G, and B channels, and a simple 2×2 FRQI image and its MCRQI state, respectively.

Like FRQI, the MCRQI state is also a normalized state, i.e. $\left\|\left|I(\theta)\right\rangle\right\| = 1$ as given by

$$\begin{aligned} \left\|\left|I(\theta)\right\rangle\right\| = \frac{1}{2^{n+1}} \sum_{i=0}^{2^{2n}-1} (&\cos^2\theta_{Ri} + \sin^2\theta_{Ri} + \\ &+\cos^2\theta_{Gi} + \sin^2\theta_{Gi} + \\ &+\cos^2\theta_{Bi} + \sin^2\theta_{Bi} + \\ &+\cos^2\theta_{\alpha i} + \sin^2\theta_{\alpha i})^{\frac{1}{2}} = 1. \end{aligned} \tag{3}$$

978-1-4577-1402-3/11/$26.00 ©2011 IEEE

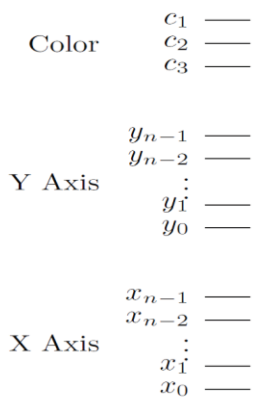


Fig. 1 General Circuit of MCRQI

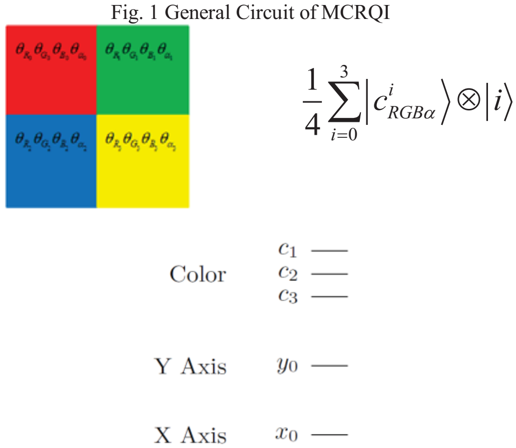


Fig. 2 A simple image and its MCRQI state (α channel is used as an opacity channel, here the value of this channel is 1 that means the image is totally opaque)

III. Polynomial Preparation For MCRQI Quantum Images

In quantum computation, the preparation process transforms an initialized quantum state from to the desired quantum image state required for further processing. This transformation involves unitary transforms described by unitary matrices. The polynomial preparation theorem PPT steers a MCRQI quantum state from its initialized state to the desired quantum image state that captures all the information about the image. Based on the results proved in Lemma 1 and Corollary 1, the PPT can be proved.

Lemma 1 *Given 4 vectors of angles,* $\theta_X=(\theta_{X_0},\theta_{X_1},\dots,\theta_{X_{2^{2n}-1}})$, $X\in\{R,G,B,\alpha\}$ *satisfying (2), where 2n is the number of qubits carrying the position information. Assume the initialized state is* $|0\rangle^{\otimes 2n+3}$, *there is a unitary transformation* P *that turns the quantum computers to the MCRQI state,* $|I(\theta)\rangle$, *composed by Hadamard and controlled rotation transformations.*

Proof There are two steps to achieve the unitary transformation P as shown in Fig.3. Hadamard transformations are used in step 1 and then controlled-rotation transformations are used in step 2 to change from $|H\rangle$ to $|I(\theta)\rangle$.

Let us consider the 2-D identity matrix I and the 2-D Hadamard matrix,

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad H=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}.$$

The tensor product of (2n+2) Hadamard matrices is denoted by $H^{\otimes 2n+2}$. Applying the transformation $A=I\otimes H^{\otimes 2n+2}=I\otimes H^{\otimes 2}\otimes H^{\otimes 2n}$ on $|0\rangle^{\otimes 2n+3}=|0\rangle\otimes|0\rangle^{\otimes 2}\otimes|0\rangle^{\otimes 2n}$ to produce the state $|H\rangle$,

$$\begin{aligned}A(|0\rangle^{\otimes 2n+3})&=I|0\rangle\otimes\frac{1}{2}\sum_{l=0}^{3}|l\rangle\otimes\frac{1}{2^n}\sum_{i=0}^{2^{2n}-1}|i\rangle\\&=\frac{1}{2^{n+1}}|0\rangle\otimes\sum_{l=0}^{3}|l\rangle\otimes\sum_{i=0}^{2^{2n}-1}|i\rangle\\&=|H\rangle\end{aligned}\tag{4}$$

Remember the rotation matrices (the rotations about y axis by the angle 2θ),

$$R_y(2\theta)=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix},\quad \theta\in\{\theta_{Ri},\theta_{Gi},\theta_{Bi},\theta_{\alpha i}\}.\tag{5}$$

Based on the above transforms, we can construct four 8×8 control rotation matrices R_{Ri},R_{Gi},R_{Bi},

$$\begin{aligned}R_{Ri}&=\left(I\otimes\sum_{j=0,j\neq 0}^{7}|j\rangle\langle j|\right)+R_y(2\theta_{Ri})\otimes|0\rangle\langle 0|,\\R_{Gi}&=\left(I\otimes\sum_{j=0,j\neq 1}^{7}|j\rangle\langle j|\right)+R_y(2\theta_{Gi})\otimes|1\rangle\langle 1|,\\R_{Bi}&=\left(I\otimes\sum_{j=0,j\neq 2}^{7}|j\rangle\langle j|\right)+R_y(2\theta_{Bi})\otimes|2\rangle\langle 2|,\\R_{\alpha i}&=\left(I\otimes\sum_{j=0,j\neq 2}^{7}|j\rangle\langle j|\right)+R_y(2\theta_{\alpha i})\otimes|3\rangle\langle 3|.\end{aligned}\tag{6}$$

Then we can obtain $R_i^{'}=R_{Bi}R_{Gi}R_{Ri}R_{\alpha i}$, and from $R_i^{'}$, the R_i transform can be constructed, with $i=0,1,\dots,2^{2n}-1$,

$$R_i=\left(I^{\otimes 3}\otimes\sum_{j=0,j\neq i}^{2^{2n}-1}|j\rangle\langle j|\right)+R_i^{'}\otimes|i\rangle\langle i|\tag{7}$$

It is clear that R_i is a unitary matrix since $R_iR_i^{\dagger}=I^{\otimes 2n+3}$. Applying R_k and R_mR_k on $|H\rangle$ gives us

$$\begin{aligned}R_k(|H\rangle)=\frac{1}{2^{n+1}}\Bigg\{&\left[I|0\rangle\otimes\left(I^{\otimes 2}\sum_{l=0}^{3}|l\rangle\right)\right]\otimes\left[\left(\sum_{j=0,j\neq k}^{2^{2n}-1}|j\rangle\langle j|\right)\left(\sum_{i=0}^{2^{2n}-1}|i\rangle\right)\right]+\\&+\left[R_k^{'}\left(|0\rangle\otimes\sum_{l=0}^{3}|l\rangle\right)\right]\otimes\left[\left(|k\rangle\langle k|\right)\left(\sum_{i=0}^{2^{2n}-1}|i\rangle\right)\right]\Bigg\}\end{aligned}\tag{8}$$

$$R_m R_k |H\rangle = R_m(R_k|H\rangle)$$
$$= \frac{1}{2^{n+1}}\left(|0\rangle \otimes \sum_{l=0}^{3}|l\rangle \otimes \sum_{j=0,\, j\neq k,\, j\neq m}^{2^{2n}-1} |j\rangle + \right. \tag{9}$$
$$\left. + |c_{RGB\alpha k}\rangle \otimes |k\rangle + |c_{RGB\alpha m}\rangle \otimes |m\rangle \right)$$

From (9), it is clearly that

$$B|H\rangle = \left(\prod_{i=0}^{2^{2n}-1} R_i \right) |H\rangle = |I(\theta)\rangle \tag{10}$$

Therefore, the unitary transform $P = BA$ is the operation turning quantum computer from the initialized state, $|0\rangle^{\otimes 2n+3}$ to the MCRQI state $|I(\theta)\rangle$. □

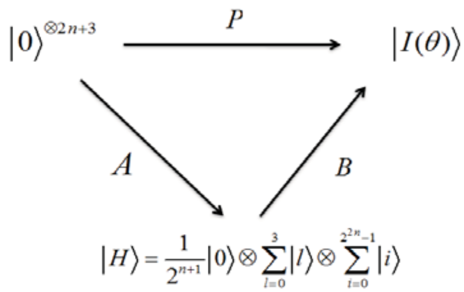


Fig. 3 Two steps to achieve P

In the quantum circuit model, a complex transformation is broken down into simple gates, i.e., single qubit and controlled two qubit gates, such as NOT, Hadamard, and CNOT gates which are shown in Fig. 3 and Fig. 4.

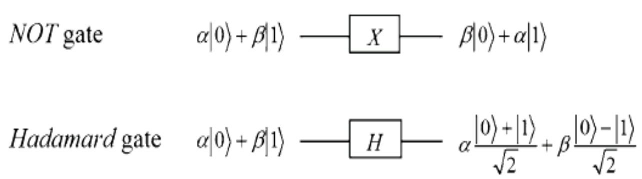


Fig. 4 NOT gate and Hadamard gate

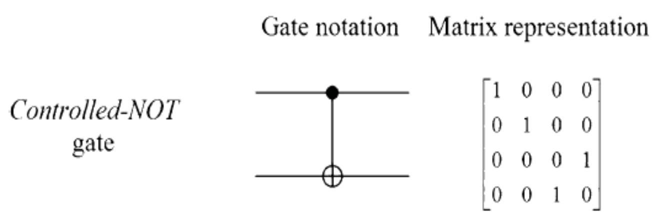


Fig. 5 CNOT gate

Corollary 1 *The unitary transform* P *described in the Lemma 1, for three given vectors of angles,* $\theta_X = (\theta_{X_0}, \theta_{X_1}, ..., \theta_{X_{2^{2n}-1}})$, $X \in \{R, G, B, \alpha\}$*, can be implemented by Hadamard, CNOT and* $C^{2n+2}\left(R_y\left(\frac{2\theta_i}{2^{2n}-1}\right)\right)$ *gates, where* $R_y\left(\frac{2\theta_i}{2^{2n}-1}\right)$ *are the rotations about y axis by the angle* $\frac{2\theta_i}{2^{2n}-1}$, $i = 0, 1, ..., 2^{2n}-1$.

Proof From the proof of Lemma 1, the transform P is composed of BA. The transform A can be directly implemented by one 2×2 identity matrix and (2n+2) Hadamard gates; and the B is constructed by $\prod_{i=0}^{2^{2n}-1} R_i$, where

$$R_i = \left(I^{\otimes 3} \otimes \sum_{j=0,\, j\neq i}^{2^{2n}-1} |j\rangle\langle j| \right) + R_i^{'} \otimes |i\rangle\langle i|,$$
$$R_i^{'} = R_{Bi} R_{Gi} R_{Ri} R_{\alpha i},$$
$$R_{Xi} = C^2\left(R_y\left(2\theta_{Xi}\right)\right),\ X \in \{R, G, B, \alpha\}$$

Hence, R_i can be implemented by, $C^{2n+2}\left(R_y\left(2\theta_{Xi}\right)\right)$, and NOT operations [9]. Furthermore, the result in [10] implies that $C^{2n+2}\left(R_y\left(2\theta_i\right)\right)$ operations can be broken down into $2^{2n+2}-1$ simple operations, $R_y\left(\frac{2\theta_i}{2^{2n}-1}\right)$, $R_y\left(-\frac{2\theta_i}{2^{2n}-1}\right)$, and $2^{2n+2}-2$ CNOT operations. The example in the case of n=1 is shown in Fig. 5.

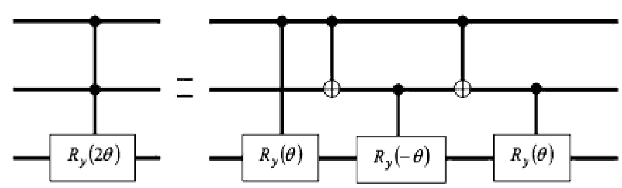


Fig. 5 $C^2\left(R_y\left(2\theta\right)\right)$ gates can be built from $C\left(R_y(\theta)\right)$, $C\left(R_y(-\theta)\right)$, and CNOT gates

The total number of simple operations used to prepare MCRQI state is

$$\begin{aligned} & 2(n+1) + 4\times 2^{2n} \times \left(2^{2(n+1)} - 1 + 2^{2(n+1)} - 2\right) \\ & = 4\times 2^{4(n+1)} - 6\times 2^{2(n+1)} + 2(n+1) \end{aligned} \tag{10}$$

This number is quadratic to the total 4×2^{2n} angle values. This indicates the efficient of the preparation process. □

Theorem 1 (Polynomial Preparation Theorem) *Given four vectors,* $\theta_X = (\theta_{X_0}, \theta_{X_1}, ..., \theta_{X_{2^{2n}-1}})$, $X \in \{R, G, B, \alpha\}$ *of angles, there is a unitary transform* P *that turns quantum computers from the initial state,* $|0\rangle^{\otimes 2n+3}$ *to the CRQI state,*

$$|I(\theta)\rangle = \frac{1}{2^{n+1}} \sum_{i=0}^{2^{2n}-1} |c_{RGB\alpha i}\rangle \otimes |i\rangle,$$

composed of polynomial number of simple gates.

Proof Coming from Lemma 1 and Corollary 1. □

IV. MCRQI BASED IMAGE PROCESSING OPERATIONS

In classical image processing, there are many fundamental operations, such as image geometric transformation, color transformation, etc. In quantum image processing, based on FRQI representation [2], many operations were also defined, such as GTQI [3] and CTQI [4]. However, FRQI treats the color as a single channel, therefore, if we want to process image on only one channel (R G B or α) of color, this representation is not convenient. The proposed MCRQI representation can solve this problem and based on it, two channel-based operations are proposed which show this representation is very flexible for channel based image processing.

A. Channel Swapping Operation

Definition 1. The Channel Swapping Operation (CSO) on MCRQI quantum images are the operations CS_{RG} , CS_{RB} , CS_{BG} (swapping the value of R and G or R and B or G and B) which when apply on $|I(\theta)\rangle$ in (1) produce the output of the following form

$$CS_{XY}(|I(\theta)\rangle) = |I(\theta)\rangle = \frac{1}{2^{n+1}}\sum_{i=0}^{2^{2n}-1} CS_{XY}(|c^{i}_{RGB\alpha}\rangle)\otimes|i\rangle \tag{11}$$

where, $X, Y \in \{R, G, B\}$, $X \neq Y$, and

$$\begin{aligned} CS_{RG}(|c^{i}_{RGB\alpha}\rangle) &= |c^{i}_{GRB\alpha}\rangle; \\ CS_{RB}(|c^{i}_{RGB\alpha}\rangle) &= |c^{i}_{BGR\alpha}\rangle; \\ CS_{BG}(|c^{i}_{RGB\alpha}\rangle) &= |c^{i}_{RBG\alpha}\rangle. \end{aligned} \tag{12}$$

The expression of $|c^{i}_{RGB\alpha}\rangle$ is shown in (2), and

$$\begin{aligned} |c_{GRB\alpha i}\rangle = &\cos\theta_{Gi}|000\rangle + \cos\theta_{Ri}|001\rangle + \\ &+\cos\theta_{Bi}|010\rangle + \cos\theta_{\alpha i}|011\rangle + \\ &+\sin\theta_{Gi}|100\rangle + \sin\theta_{Ri}|101\rangle + \\ &+\sin\theta_{Bi}|110\rangle + \sin\theta_{\alpha i}|111\rangle. \end{aligned} \tag{13}$$

$$\begin{aligned} |c_{BGR\alpha i}\rangle = &\cos\theta_{Bi}|000\rangle + \cos\theta_{Gi}|001\rangle + \\ &+\cos\theta_{Ri}|010\rangle + \cos\theta_{\alpha i}|011\rangle + \\ &+\sin\theta_{Bi}|100\rangle + \sin\theta_{Gi}|101\rangle + \\ &+\sin\theta_{Ri}|110\rangle + \sin\theta_{\alpha i}|111\rangle. \end{aligned} \tag{14}$$

$$\begin{aligned} |c_{RBG\alpha i}\rangle = &\cos\theta_{Ri}|000\rangle + \cos\theta_{Bi}|001\rangle + \\ &+\cos\theta_{Gi}|010\rangle + \cos\theta_{\alpha i}|011\rangle + \\ &+\sin\theta_{Ri}|100\rangle + \sin\theta_{Bi}|101\rangle + \\ &+\sin\theta_{Gi}|110\rangle + \sin\theta_{\alpha i}|111\rangle. \end{aligned} \tag{15}$$

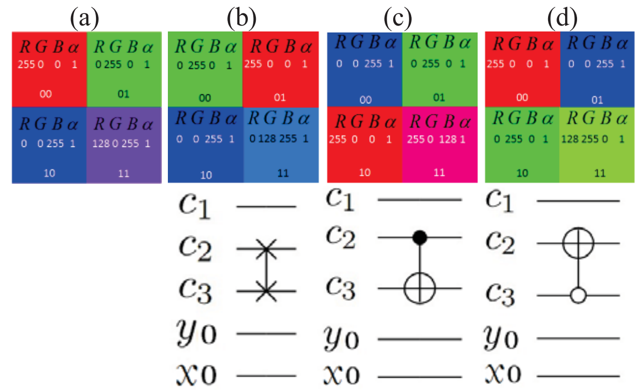


Fig. 6 An example of two-channel swapping operation and realization quantum circuits ((a) is the original image; (b) is the R and G swapping output image from (a); (c) is the R and B swapping output image from (a); (d) is the G and B swapping output image from (a))

Fig. 6 shows an example of two-channel swapping operation.

B. One Channel Operation

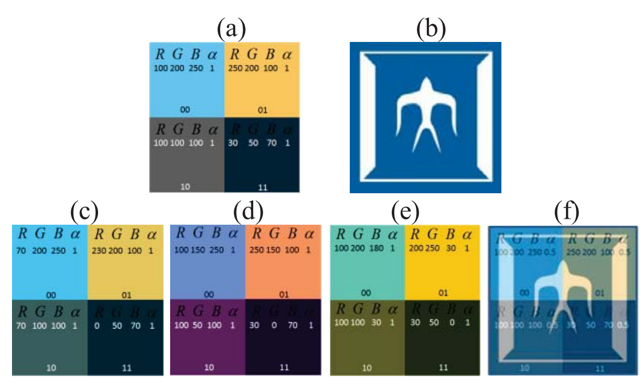
Definition 2. The One Channel Operation (OCO) on MCRQI quantum images are the operations OC_R , OC_G , OC_B (changing the value of R-Channel, G-Channel, and B-Channel respectively) which when apply on $|I(\theta)\rangle$ produce the output of the following form

$$OC_{Z}(|I(\theta)\rangle) = |I(\theta)\rangle = \frac{1}{2^{n+1}}\sum_{i=0}^{2^{2n}-1} OC_{Z}(|c^{i}_{RGB\alpha}\rangle)\otimes|i\rangle \tag{16}$$

where, $Z \in \{R, G, B, \alpha\}$, and

$$\begin{aligned} OC_{R}(|c^{i}_{RGB\alpha}\rangle) &= |c^{i}_{R'GB\alpha}\rangle; \\ OC_{G}(|c^{i}_{RGB\alpha}\rangle) &= |c^{i}_{RG'B\alpha}\rangle; \\ OC_{B}(|c^{i}_{RGB\alpha}\rangle) &= |c^{i}_{RGB'\alpha}\rangle; \\ OC_{\alpha}(|c^{i}_{RGB\alpha}\rangle) &= |c^{i}_{RBG\alpha'}\rangle. \end{aligned} \tag{17}$$

$|c^{i}_{R'GB\alpha}\rangle, |c^{i}_{RG'B\alpha}\rangle, |c^{i}_{RGB'\alpha}\rangle, |c^{i}_{RBG\alpha'}\rangle$ are the new quantum states which carrying the color information after applying OCO on $RGB\alpha$ channels of original image.



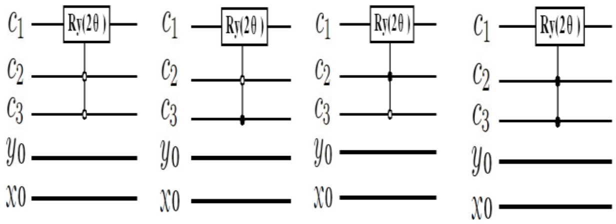


Fig. 6 An example of one channel operation and realization quantum circuits ((a) is the original image; (b) is the background image behind (a); (c), (d), (e), (f) are the output images after changing the value of R, G, B, and α channels respectively.

V. EXPERIMENTS ON THE MCRQI

The experiments reported were carried out on desktop computer with Intel Core 2 Duo 1.86 GHz CPU and 2 GB RAM with the Matlab simulation environment.

A 32x32 colored image was used and transformations focusing on two or more of the $RGB\alpha$ (value of α channel is set to 1 which means the images are total opaque) color channels were carried out. To represent the image on Matlab, a vector of 8192 real-valued coefficients is stored and the operations to process this vector are represented by 8192x8192 matrices. Fig. 7 (a) shows the input image and the figures in (b)-(e) show transformed images with the circuit to realize each output image.

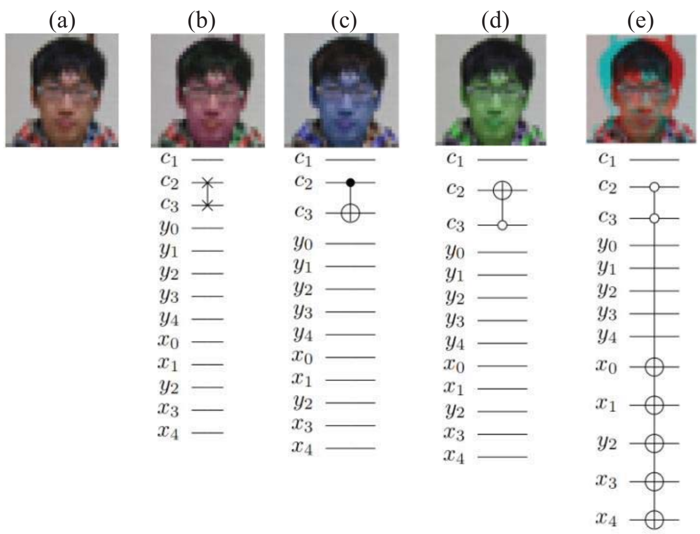


Fig. 7 Simulation experiment to demonstrate the feasibility of the MCRQI representation

These experiments indicate that transformations that focus on the channel content, $RGB\alpha$ color space of each point in the image can be accomplished by operations on the color information of the MCRQI representation. These are reported in Fig. 7 (b), (c), and (d) for a swap between B and G, R and G, and R and B.

The image in Fig.7 (e) is obtained by operations that target both color and position information of the 3C-FRQI representation. As seen from the circuit to realize this image, it requires slightly more resources because in terms of the control-operations to obtain the required image transformation.

From the experiments, the complexity of each circuit is low disrespect to the size of the input image, this mean the speed of the quantum transformation is fast for all input image. Meanwhile, the classical operations with the same performance are depended on the size of the input images and become slower on big images.

VI. CONCLUSIONS

A Multi-Channel Representation for Quantum Image has been proposed. This proposal allows the treatment of color transparency information of the FRQI quantum image as a four channel $RGB\alpha$ space by dedicating three qubits to encode the channels.

The preparation method turn quantum computer from initialized state to the MCRQI state. The Polynomial Preparation Theorem (PPT) shows that there is a unitary transformation which turns quantum computers from the initialized state to the MCRQI state efficiently, and this transformation can be constructed using the Hadamard and control rotation gates. The experiment results show that MCRQI is available by image storage and retrieval simulation.

Experiment results showing how various combinations of the information encoded in the $RGB\alpha$ color space can be manipulated were proposed.

High-level image processing tasks that require some amount of detail can be realized using the proposed MCRQI representation.

Used with other processing transformations such as the GTQI, CTQI, and their restricted variants, the proposed MCRQI representation can be used to improve the available image processing tasks such as quantum image watermarking and quantum movie production.

REFERENCES

[1] S. E. Venegas-Andraca, S. Bose, , "Storing, processing and retrieving an image using quantum mechanics, Proc. of the SPIE Conf. *Quantum Information and Computation*, pp.137-147, 2003.

[2] P. Q. Le, F. Dong, K. Hirota, "A flexible representation of quantum images for polynomial preparation, image compression, and processing operations," *Quantum Information Processing*. DOI-10.1007/s11128-010-0177-y, April, 2010.

[3] P. Q. Le, A. M. Iliyasu, F. Dong, K. Hirota, Fast geometric transformations on quantum images, *IAENG International Journal of Applied Mathematics*, Vol. 40, No 3, pp. 113-123. June, 2010

[4] P.Q. Le, A.M. Iliyasu, F. Dong, and K. Hirota. "Efficient Color Transformations on Quantum Image", *J. of Adv. Comp. Int. Inf. (JACIII)*, Vol. 45, No. 4, To appear July, 2011

[5] P.Q. Le, A.M. Iliyasu, F. Dong, and K. Hirota. "Strategies for designing Geometric Transformations on Quantum Images", *Elsevier Thei. Comp. Sci.*, 412, pp. 1406-1418, 2011

[6] A.M. Iliyasu, P.Q. Le, F. Dong, and K. Hirota, "Restricted Geometric Transformations and their applications for Quantum Image Watermarking and Authentication", *Proc. Of the 10th Asian Conf. on Quantum Inf. Sci. (AQIS 2010)*, pp. 212-214, Aug., 2010

[7] J. I. Latorre, "Image compression and entanglement." URL arXiv:quant-ph/0510031, 2005.

[8] S.E. Venegas-Andraca, J.L. Ball.: Processing images in entangled quantum systems, *Quantum Information Processing 9(1)*, pp.1-11, 2010. DOI-10.1007/s11128-009-0123-z.
[9] M. Nielsen, I. Chuang, : Quantum Computation and Quantum Information. Cambridge UniversityPress, New York (2000)
[10] A. Barenco, C.H. Bennett, R. Cleve, D.P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J.A. Smolin, H. Weinfurter, : Elementary gates for quantum computation. Phys. Rev. A **52**, 3457 (1995)