1. Prove the following identities:

$$(A+B)^T = A^T + B^T,$$

$A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$

$A^T = (a_{ji})_{n \times m}$, $B^T = (b_{ji})_{n \times m}$

$A + B = (a_{ij} + b_{ij})_{n \times n}$

$A^T + B^T = (a_{ji} + b_{ji})_{n \times m}$

$(A+B)^T = (a_{ij} + b_{ij})^T_{n \times m} = (a_{ji} + b_{ji})_{n \times m}$ ✓

$$(AB)^T = B^T A^T,$$

$AB = C$

$(a_{ij})_{n \cdot m} \cdot (b_{ij})_{m \cdot p} = (c_{ij})_{n \cdot p}$, $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

$A^T = (a_{ji})_{m \cdot n}$, $B^T = (b_{ji})_{p \cdot m}$

$A^T \cdot B^T$ is incompatible $\rightarrow$ must do $B^T \cdot A^T$

$(AB)^T = C^T = c_{ji} = \sum_{k=1}^{n} b_{jk} \cdot a_{ki} = B^T \cdot A^T$

$$(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T .$$

By applying commutativity to $(AB)^T = B^T A^T$

$A_1 A_2 = C \rightarrow (A_1 A_2 A_3)^T = (C A_3)^T$

$= A_3^T C^T = A_3^T (A_1 A_2)^T = A_3^T A_2^T A_1^T$

We can repeat this to express any chain

$\prod_{k=1}^{n} A_k$

2. Show that AB is not necessarily symmetric if A and B are symmetric.

$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 8 & 15 \\ 4 & 4 & 18 \\ 5 & 12 & 9 \end{bmatrix}$ not symm. ✓

3. If $A + jB$ is Hermitian, A, B real, then

$$A^T = A, \quad B^T = -B.$$

From defn. of Hermitian, conjugate symm. requires that $b_{ij} = -b_{ji}$ ($b_{ij} = 0$ @ $i = j$)

$\therefore$ transpose equiv to scalar mult by -1

4. For any square matrix $A = \begin{pmatrix} A_1 & * \\ O & A_2 \end{pmatrix}$

with A_1, A_2 two square submatrices,

show that $\det A = \det A_1 \cdot \det A_2$.

$$A = \underbrace{\begin{bmatrix} I_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{A_x} \cdot \underbrace{\begin{bmatrix} A_1 & * \\ 0 & I_2 \end{bmatrix}}_{A_y}$$

$$\mathrm{Det}(A) = \mathrm{Det}(A_x) \cdot \mathrm{Det}(A_y)$$

By column expansion, $\mathrm{Det}(A_x) = \mathrm{Det}(A_2)$
(all 0s except one 1) $\mathrm{Det}(A_y) = \mathrm{Det}(A_1)$

$$\mathrm{Det}(A) = \mathrm{Det}(A_1) \cdot \mathrm{Det}(A_2)$$

Sources: Gemini,
https://www.youtube.com/watch?v=NGwafwLZq7g
Taboga, Marco (2021). "Determinant of a block matrix", Lectures on matrix algebra.
https://www.statlect.com/matrix-algebra/determinant-of-block-matrix.