# Lecture 7 - Problem solving tutorial

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# 1 Review:

## 1.1 Statics

In this lecture, we mainly solve problems in static equilibrium: a body is in equilibrium when the resultant force and the resultant moment on it are both zero (so it has no linear or angular acceleration).

- Force equilibrium:

$$\sum \vec{F} = 0 \iff \sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$

- Moment equilibrium:

$$\sum \vec{M} = 0 \iff \sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

- In 2D: If we define the out-of-plane direction as $z$, then there is no force along $z$ (all forces are in-plane), so $F_z = 0$. Also, all forces and position vectors lie in the $xy$-plane. Since $\vec{M} = \vec{r} \times \vec{F}$ is perpendicular to the plane spanned by $\vec{r}$ and $\vec{F}$ (i.e., the in-plane), only $M_z$ can be nonzero (so $M_x = M_y = 0$).

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_z = 0$$

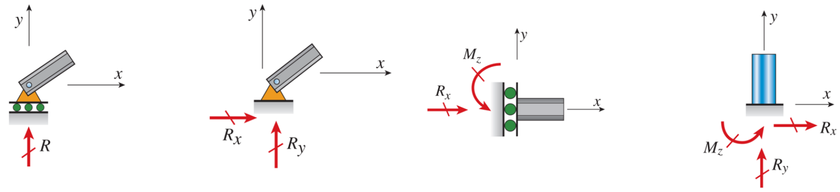
- In 1D: If all forces act along a single line (e.g., the $x$-axis) and all points of application lie on that same line, then $\vec{r}$ and $\vec{F}$ are collinear so $\vec{r} \times \vec{F} = \vec{0}$; hence moments are identically zero and the only equilibrium condition is the scalar force balance along that axis.

$$\sum F_x = 0$$

## 1.2 Supporting types:

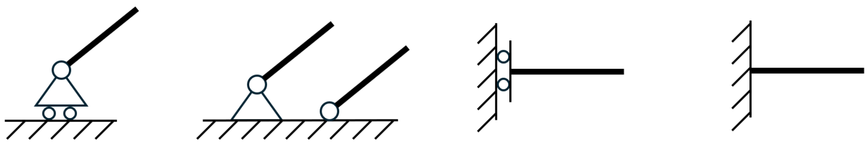
If a support **prevents** motion in a certain direction, it creates a force in that direction. If a support **prevents** rotation, it creates a moment.

Reactions:

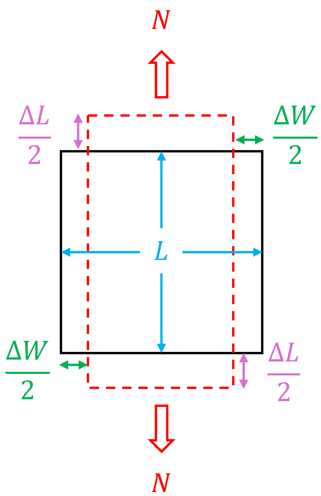



Roller | Pin | Sliding | Fixed

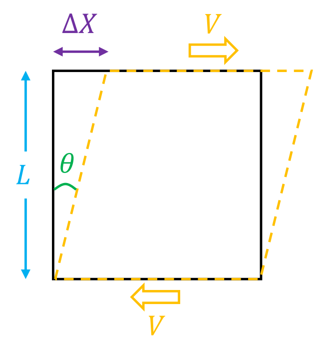
Symbols:



## 1.3 Mechanical Properties of Materials.




| | Normal | Shear |
|---|---|---|
| Stress | $\sigma = \dfrac{N}{A}$ | $\tau = \dfrac{V}{A}$ |
| Strain | $\varepsilon = \dfrac{\Delta L}{L}$ | $\gamma = \frac{\Delta X}{L} = \tan\theta \approx \theta$ |




Here, $N$ and $V$ are the normal force and shear (tangential) force, respectively; $A$ is the cross-sectional area; and $L$ is the original length. $\theta$ is the rotation angle caused by shear deformation (defined as the change in angle between two edges that were originally perpendicular).

**Nominal vs True Stress/Strain:**

Nominal (engineering) quantities use the **original** dimensions ($A_0$, $L_0$):

$$\sigma_{\text{eng}} = \frac{F}{A_0}, \qquad \varepsilon_{\text{eng}} = \frac{L - L_0}{L_0}$$

True quantities use the **instantaneous** dimensions ($A$, $L$):

$$\sigma_{\text{true}} = \frac{F}{A}, \qquad \varepsilon_{\text{true}} = \int_{L_0}^{L} \frac{d\ell}{\ell} = \ln\left(\frac{L}{L_0}\right)$$

Why from $L_0$ to $L$? The integral is taken over the *actual deformation path*: at each instant the incremental true strain is $d\varepsilon = d\ell/\ell$, and the length starts from the initial value $L_0$ and ends at the current value $L$. Integrating from 0 would be unphysical (the specimen never has zero length) and would make $\int_0^L d\ell/\ell$ diverge.

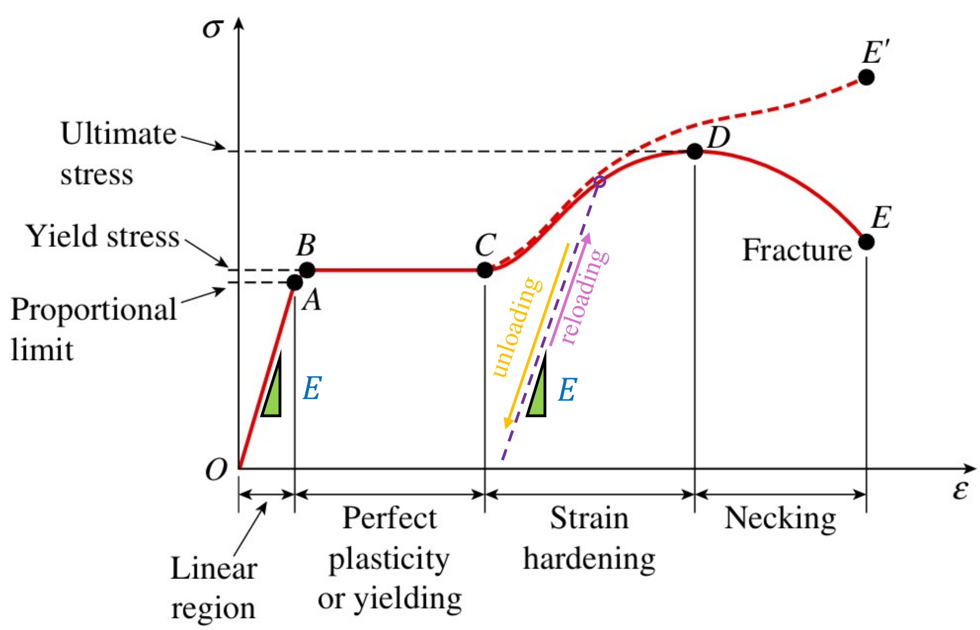
Relationship between them:

$$\frac{L}{L_0} = \frac{L_0 + (L - L_0)}{L_0} = 1 + \frac{L - L_0}{L_0} = 1 + \varepsilon_{\text{eng}},$$
$$\varepsilon_{\text{true}} = \ln\left(\frac{L}{L_0}\right) = \ln(1 + \varepsilon_{\text{eng}}).$$

If the material is approximately incompressible during plastic deformation (constant volume $AL \approx A_0 L_0$), then

$$A \approx \frac{A_0 L_0}{L} = \frac{A_0}{1 + \varepsilon_{eng}},$$
$$\sigma_{\text{true}} = \frac{F}{A} \approx \frac{F}{A_0/(1 + \varepsilon_{\text{eng}})} = \sigma_{\text{eng}}(1 + \varepsilon_{\text{eng}}).$$

**Stress-Strain curve of mild steel (uniaxial tension):**




1. **Linear elastic region:** stress is proportional to strain and the slope is $E$ (Young's modulus).
2. **Yielding / yield plateau:** mild steel shows upper/lower yield and a plateau where strain increases at nearly constant stress.
3. **Strain (work) hardening:** after yielding, continued plastic deformation increases dislocation density, so higher stress is required to continue deforming. If we unload from this region, the

unloading path is approximately a straight line with slope $E$ because the unloading response is mainly elastic; the accumulated plastic strain stays as a permanent offset.

4. **Necking to fracture:** after the ultimate tensile strength, deformation localizes (necking), and the engineering stress drops until fracture.

**Why do the true-stress curve and the engineering-stress curve diverge?** Engineering stress uses the constant original area $A_0$, while true stress uses the instantaneous area $A$ that decreases as the specimen elongates (especially during necking). Even if the force $F$ does not increase, $\sigma_{\text{true}} = F/A$ can keep rising because $A$ is shrinking; meanwhile $\sigma_{\text{eng}} = F/A_0$ can peak and then drop after necking.

**Hooke's Law:**

Normal (axial) and shear behaviors are written separately:

$$\sigma = E\,\varepsilon,$$
$$\tau = G\,\gamma.$$

**Poisson's ratio:**

$$\nu = -\frac{\varepsilon_{\text{trans}}}{\varepsilon_{\text{axial}}}$$

For an isotropic, linear-elastic material, the elastic constants are not all independent because they describe the same underlying stiffness in different loading modes. In particular, $E$, $G$, and $\nu$ must satisfy:

$$G = \frac{E}{2(1+\nu)}$$

So once any two of them are known, the third is determined.

**Limit-based (local) definition:**

These are point-wise (local) definitions. We take the limit as the area/length element shrinks to zero, so the result does not depend on how we choose a finite "average" area or length.

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} \qquad \varepsilon = \frac{du}{dx}$$

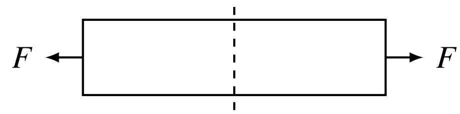
$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A} \qquad \gamma = \frac{du_t}{dx} \approx \theta$$

## 1.4 Axially loaded members:

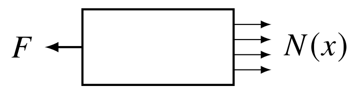
### 1.4.1 Method of Sections

To determine the internal force in a member, we use the method of sections:

1. Set imaginary cuts:




2. Draw Free Body Diagram (FBD):




3. Equilibrium equations:

$$\sum F_x = 0 \implies -F + N(x) = 0 \implies N(x) = F$$

### 1.4.2 Force-Displacement Relations

The relationship between the external force $P$, geometry $(L, A)$, material $(E)$, and displacement $\delta$ is given by:

$$P = \frac{EA}{L}\delta \quad \text{or} \quad \delta = \frac{L}{EA}P$$

This is derived from the integration of local stress-strain relationships:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\textbf{Force} \arrow[rr, leftrightarrow, dashed] \arrow[d, leftrightarrow, "\begin{array}{r}\sigma = \frac{F}{A} \\ \text{or } \sigma(x) = \frac{N(x)}{A(x)}\end{array}"'] & & \textbf{Displacement} \arrow[d, leftrightarrow, "\begin{array}{l}\varepsilon = \frac{\delta}{L} \\ \text{or } \varepsilon(x) = \frac{d\delta(x)}{dx}\end{array}"] \\
\text{Stress} \arrow[rr, leftrightarrow, "\begin{array}{c}\text{Hooke's Law} \\ \sigma = E\varepsilon\end{array}"'] & & \text{Strain}
\end{tikzcd}$$

If we start from the differential (local) form, we can derive the integral form by summing the incremental elongations along the bar.

$$d\delta(x) = \varepsilon(x)\,dx,$$

$$\varepsilon(x) = \frac{\sigma(x)}{E},$$

$$\sigma(x) = \frac{N(x)}{A(x)}.$$

Therefore,

$$d\delta(x) = \frac{N(x)}{EA(x)}\,dx \quad \implies \quad \delta = \int_0^L \frac{N(x)}{EA(x)}\,dx.$$

If $E$, $A$, and $N$ are constant along the member, this reduces to $\delta = \dfrac{NL}{EA}$.

### 1.5 Thermal strain

When the temperature changes by $\Delta T$, a material tends to expand/contract. The corresponding **free thermal strain** is

$$\varepsilon_{\text{th}} = \alpha\,\Delta T,$$

where $\alpha$ is the coefficient of thermal expansion.

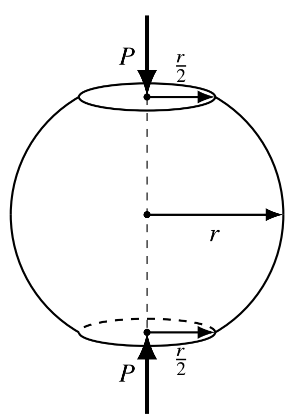
If the member is free to deform (no restraint), this thermal strain does **not** create stress: the body simply changes length. Thermal **stress** appears mainly when boundary conditions (supports, constraints, or compatibility requirements) prevent the free expansion/contraction, forcing a mechanical strain to develop so that equilibrium/compatibility can still be satisfied.

### 1.6 Statically indeterminate problem

1. Draw the free-body diagram (FBD).
2. Write down all equilibrium equation(s).
3. Determine the degree of static indeterminacy.
4. Write the compatibility (deformation) equation(s).
5. Use force–displacement relations to convert compatibility equation(s) into equations for the unknown reaction(s).
6. Solve for all reaction forces.
7. Compute the required internal forces, stresses, and strains.

## 2 Example Problems

### 2.1 Calculate the compression of a variable cross-section bar



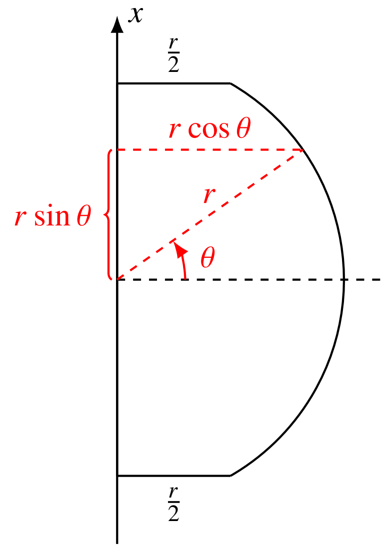

**Geometry.** Consider a bar with a varying circular cross-section as shown. The radius changes from $r/2$ at the top to $r$ at mid-height and back to $r/2$ at the bottom.

**Material.** The bar is linear-elastic with Young's modulus $E$.

**Loading.** The bar is subjected to an axial compressive load $P$.

**Task.** Compute the axial elongation $\delta$ under the load $P$.

**Solution**




From the figure, the cross-section radius at angle $\theta$ is

$$a(\theta) = r\cos\theta, \qquad \theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right].$$

Hence the cross-sectional area is

$$A(\theta) = \pi a(\theta)^2 = \pi r^2 \cos^2\theta.$$

The axial coordinate (along the $x$-axis) is

$$x = r\sin\theta \quad \Longrightarrow \quad dx = r\cos\theta\, d\theta.$$

Under the axial compressive load $P$, the internal force is constant: $N = -P$. The axial compression is

$$\delta = \int \frac{N}{EA}\, dx = \frac{-P}{E} \int_{-\pi/3}^{\pi/3} \frac{1}{\pi r^2 \cos^2\theta} \left(r\cos\theta\, d\theta\right) = \frac{-P}{E\pi r} \int_{-\pi/3}^{\pi/3} \sec\theta\, d\theta.$$

Using $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$,

$$\int_{-\pi/3}^{\pi/3} \sec\theta\, d\theta = 2\ln\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) = 2\ln\left(2 + \sqrt{3}\right).$$

Therefore,

$$\boxed{\delta = -\frac{2P}{E\pi r} \ln\left(2 + \sqrt{3}\right).}$$

The minus sign indicates compression.

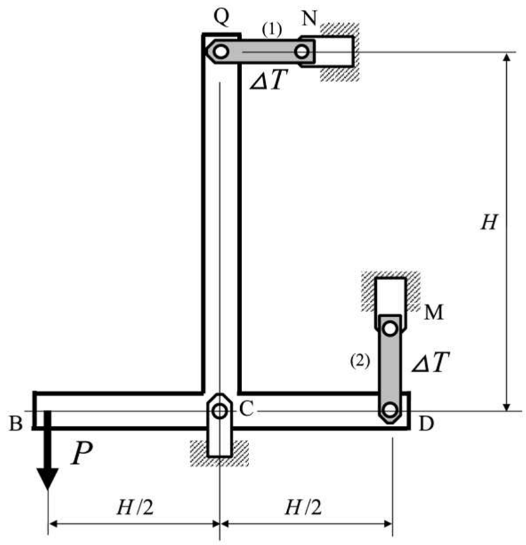
## 2.2 Statically indeterminant problem

A rigid member BCDQ is supported by two deformable supports, and it is pinned at C. Deformable support (1) consists of rod QN, and deformable support (2) consists of rod DM. Both rods have length $L$, cross-sectional area $A$, Young's modulus $E$, and coefficient of thermal expansion $\alpha$. The assembly is loaded at B with a vertical force $P$, and rods (1) and (2) experience an increase in temperature equal to $\Delta T$.

(a) Calculate the axial loads in rods (1) and (2).

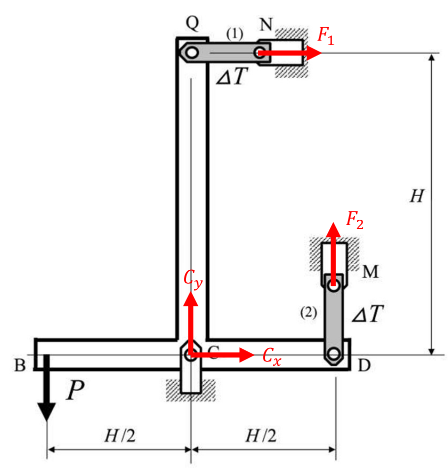
(b) If there was no increase in temperature, that is if $\Delta T = 0$, state whether each rod is in tension or compression.

(c) If the vertical force is removed, that is if $P = 0$, state whether each rod is in tension or compression (due to the thermal loads).




**Solution**

1. FBD



2. Write down all equilibrium equation(s).

$$\sum F_y = 0 \Longrightarrow C_y - P + F_2 = 0$$

$$\sum F_x = 0 \Longrightarrow C_x + F_1 = 0$$

$$\sum M_z|_C = 0 \Longrightarrow -F_1 \cdot H + F_2 \cdot \frac{H}{2} + P \cdot \frac{H}{2} = 0 \implies 2F_1 - F_2 = P$$

3. Determine the degree of static indeterminacy.

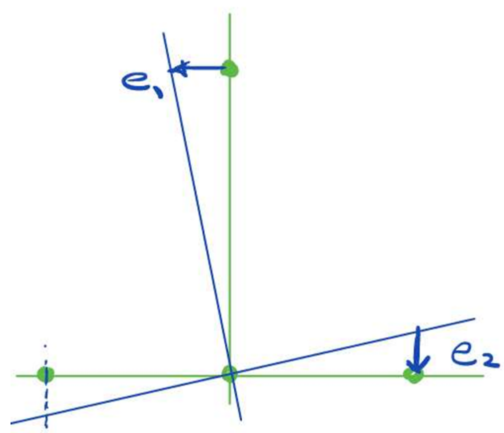
   There are 4 unknown reactions ($F_1, F_2, C_x, C_y$) and 3 equations of equilibrium.

$$\text{Degree} = 4 - 3 = 1$$

   The structure is statically indeterminate to the $1^{st}$ degree.

4. Write the compatibility (deformation) equation(s).

   Based on the geometry of the rigid body rotation about point $C$:




$$\frac{e_1}{H} = -\frac{e_2}{H/2} \implies e_1 = -2e_2$$

5. Use force–displacement relations to convert compatibility equation(s) into equations for the unknown reaction(s).

   The total elongation consists of elastic deformation and thermal expansion:

$$e_1 = \frac{F_1 L}{EA} + \alpha \Delta T L$$
$$e_2 = \frac{F_2 L}{EA} + \alpha \Delta T L$$

   Substitute these into the compatibility equation:

$$\left[\frac{F_1 L}{EA} + \alpha \Delta T L\right] = -2\left[\frac{F_2 L}{EA} + \alpha \Delta T L\right]$$

Divide by $L$ and multiply by $EA$:

$$F_1 + EA\alpha\Delta T = -2F_2 - 2EA\alpha\Delta T$$

$$F_1 + 2F_2 = -3EA\alpha\Delta T$$

6. Solve for all reaction forces.

   Solve the system of linear equations formed by (Eq. 1) and (Eq. 3):

$$\begin{cases} 2F_1 - F_2 = P \\ F_1 + 2F_2 = -3EA\alpha\Delta T \end{cases}$$

   Substituting $F_1 = \frac{1}{2}(P + F_2)$ into the second equation (or solving simultaneously) yields:

$$F_2 = -\frac{P}{5} - \frac{6}{5}EA\alpha\Delta T$$
$$F_1 = \frac{2P}{5} - \frac{3}{5}EA\alpha\Delta T$$

7. Compute the required internal forces, stresses, and strains.

   **Case 1: No temperature change ($\Delta T = 0$)**

$$F_2 = -\frac{P}{5} \quad \text{(Compression)}$$
$$F_1 = \frac{2P}{5} \quad \text{(Tension)}$$

   **Case 2: No mechanical load ($P = 0$)**

$$F_2 = -\frac{6}{5}EA\alpha\Delta T \quad \text{(Compression)}$$
$$F_1 = -\frac{3}{5}EA\alpha\Delta T \quad \text{(Compression)}$$