Georgia Tech
College of
Engineering

COE 3001

MECHANICS OF DEFORMABLE BODIES

Lecture 6 – Completability Equation and Statically Indeterminate Problems

Yazhuo Liu
Georgia Institute of Technology

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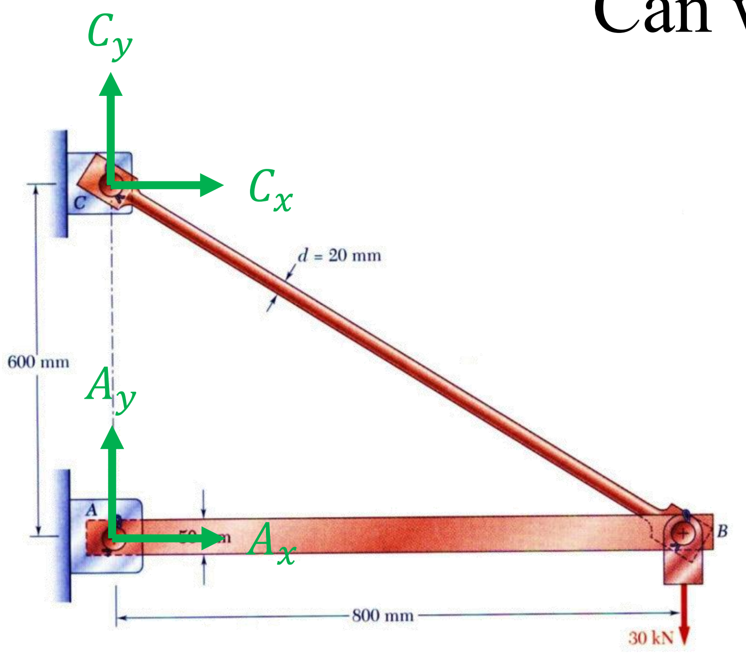
Statically determinate structures

Let's revisit the example in *Lecture 2 – review of statics*.

Can we solve this problem using equilibrium equation?



How many unknowns?

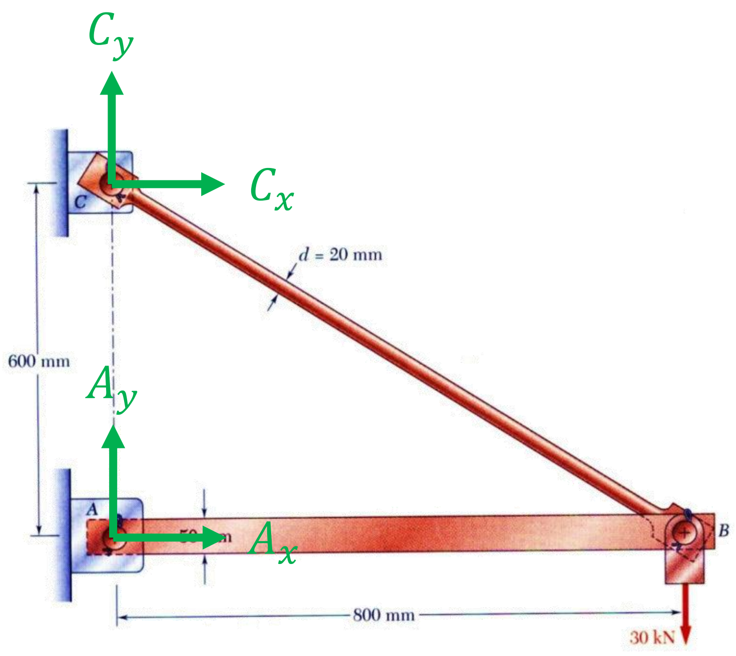
$$C_x, \quad C_y, \quad A_x, \quad A_y$$

How many equations?

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum M_C = 0$$
$$\sum M_B = 0$$

# of unknown = # of equations: YES, we can!

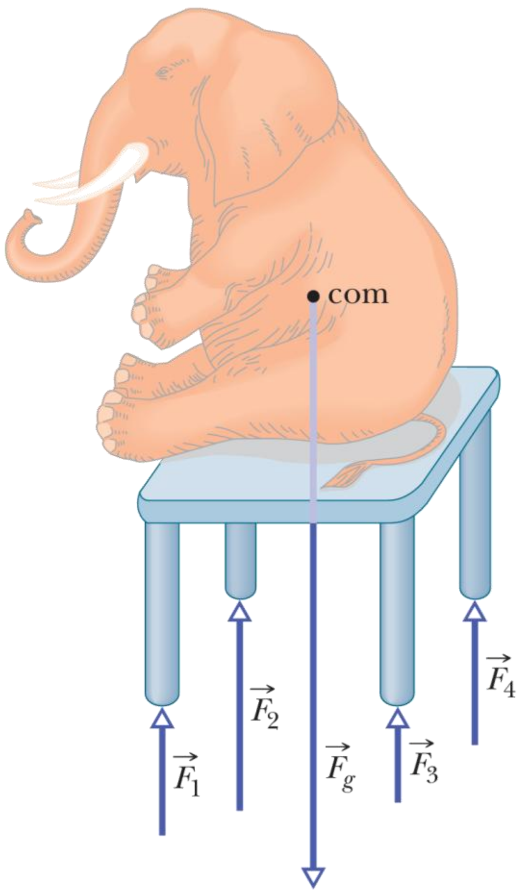
Statically determinate structures



A structural system is considered statically determinate if:

- All the unknown forces (support reactions, internal forces) can be determined using only free-body diagrams and equations of equilibrium.
- No knowledge of material properties or deformations is needed to find the unknowns.

Statically indeterminate structures



But most structures are more complex.

Recall the second problem in *Lecture 2 – review of statics*.

Why can we not solve this problem?

How many unknowns?

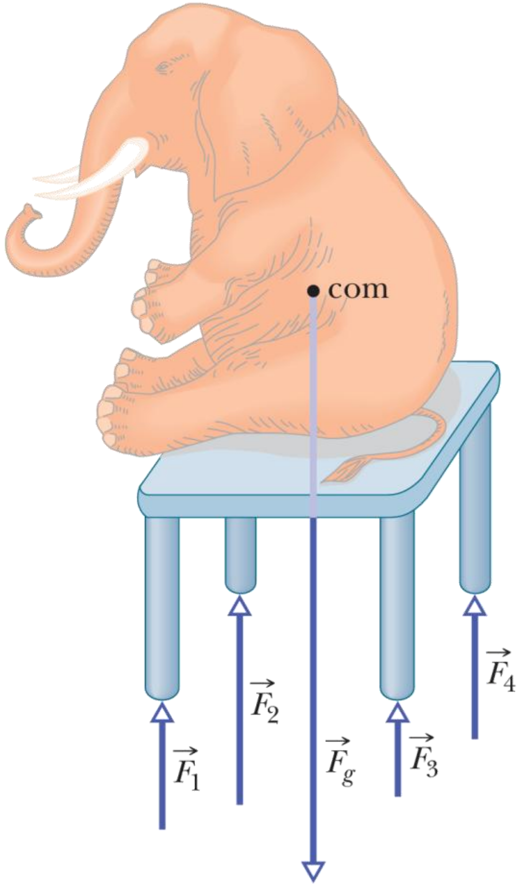
$$F_1, \quad F_2, \quad F_3, \quad F_4$$

How many equations?

$$\sum F_z = 0$$
$$\sum M_x = 0$$
$$\sum M_y = 0$$

# of unknown > # of equations: **indeterminate**

Statically indeterminate structures



A structural system is called **statically indeterminate** when:

- The number of unknown forces exceeds the number of available independent equilibrium equations.
- As a result, equilibrium alone is not sufficient to solve for all unknowns.

How many additional equations we need? How can we find these equations?

To analyze such structures, you must supplement the equilibrium equations with additional equations pertaining to the displacements of the structure.

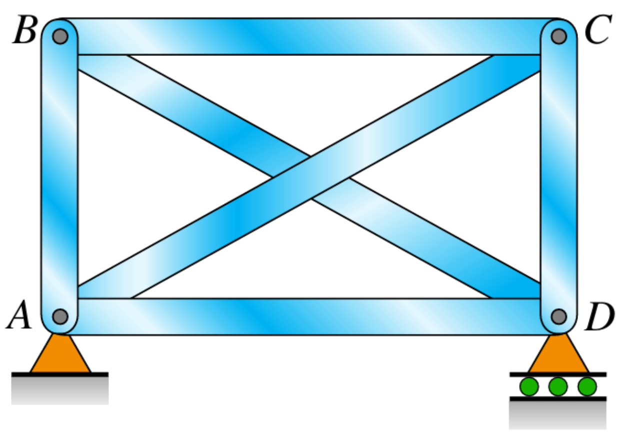
Degree of static indeterminacy

- How many additional equations we need?

# of Unknowns – # of Independent Equilibrium Equations

which is known as *degree of static indeterminacy.*

- It tells us how many extra equations we need to solve the problem



Calculation steps:

1. Identify all unknown reactions in the structure.

   6 internal forces + 3 reactions = 9 unknowns

2. Determine the number of independent equilibrium equations available.

   4 nodes × 2 = 8 equations

3. Subtract the number of equilibrium equations from the number of unknown reactions.

   9 – 8 = 1 degree of static indeterminacy

Compatibility equations

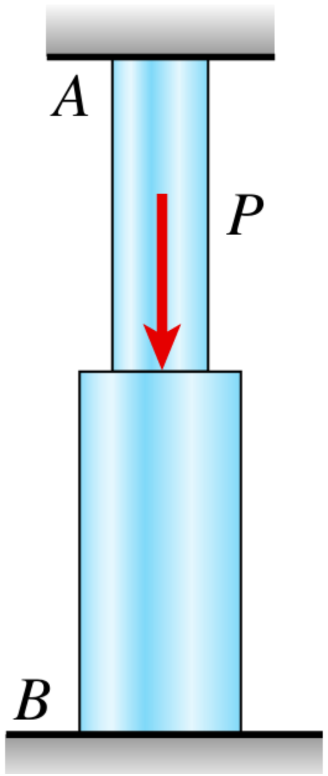
- How can we find the additional equations?

✓A compatibility equation ensures that the displacements and deformations in a structure are consistent with its geometric constraints and boundary conditions.

✓Even though internal forces are unknown, their resulting deformations must satisfy:

  - Fixed supports do not allow displacement.
  - Connected members must deform together (no gaps or overlaps).

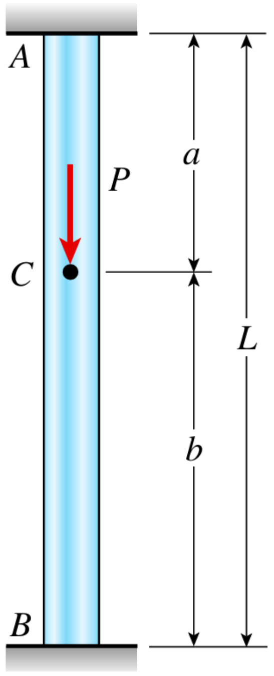
✓These physical constraints provide additional equations needed to solve indeterminate systems.



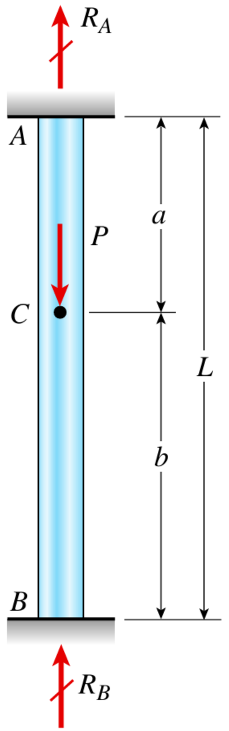
$$\delta_{AB} = 0$$

Flexibility (Force) method

Example: Analysis the statically indeterminate bar, draw its ADD and AFD.



① Draw the FBD



② Write down all equilibrium equation(s)

$$R_A + R_B - P = 0$$

③ Determine the degree of static indeterminacy.

Unknowns:

$$R_A, \quad R_B$$

# of equilibrium equations:

$$1$$

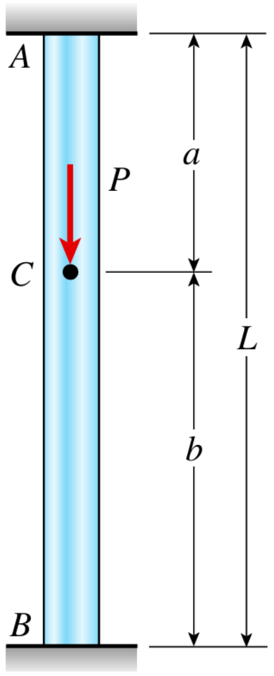
degree of static indeterminacy:

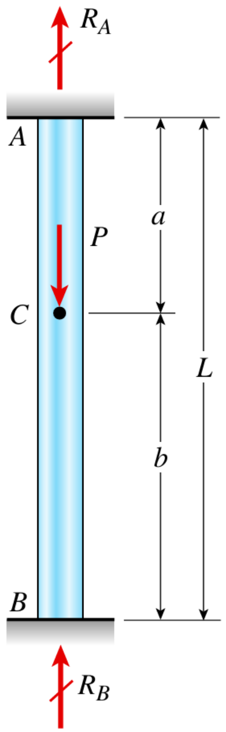
$$2 - 1 = 1$$

We need one more equation to solve the unknowns

Flexibility (Force) method

Example: Analysis the statically indeterminate bar, draw its ADD and AFD.





④ Writing down compatibility equation(s)

The total length of the bar can not be changed due to the constrains

$$\delta_A = 0$$

⑤ Use force-displacement relations to convert compatibility equation(s) into equations for the unknown reaction(s)

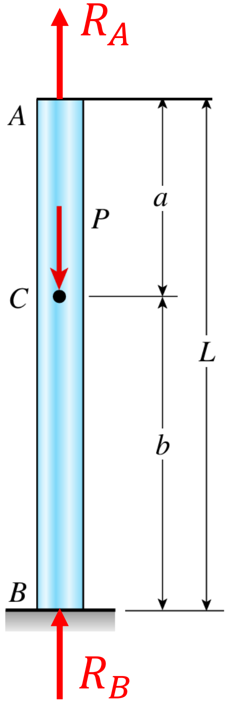
$$\delta = \frac{PL}{EA}$$

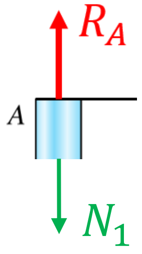
$$\Longrightarrow \delta_A = \delta_1 + \delta_2 = \frac{N_1 a}{EA} + \frac{N_2 b}{EA} = 0$$

Flexibility (Force) method

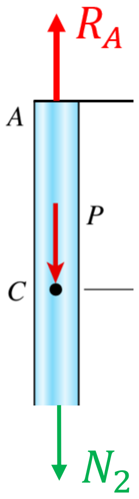
Example: Analysis the statically indeterminate bar, draw its ADD and AFD.

FBD:



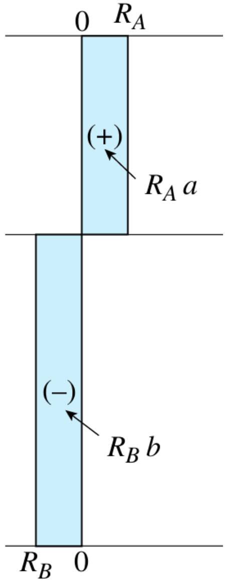


$$R_A - N = 0$$
$$\Longrightarrow N_1 = R_A$$



$$R_A - P - N = 0$$
$$\Longrightarrow N_2 = R_A - P = -R_B$$

AFD



$$\delta_{AC} = \frac{aN_1}{EA} = \frac{R_A a}{EA}$$

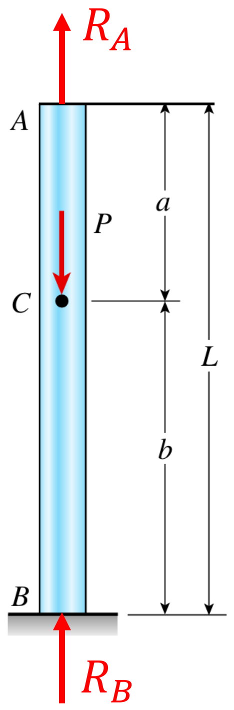
$$\delta_{CB} = \frac{bN_2}{EA} = -\frac{R_B b}{EA}$$

$$\delta_{AC} + \delta_{CB} = 0$$

$$\Longrightarrow \frac{R_A a}{EA} - \frac{R_B b}{EA} = 0$$

Flexibility (Force) method

Example: Analysis the statically indeterminate bar, draw its ADD and AFD.



⑤ Solve for all reaction forces.

Equilibrium equation:

$$R_A + R_B - P = 0$$

Compatibility equation:

$$\frac{R_A a - R_B b}{EA} = 0$$

*Note that flexibilities L/EA appear in this equation. Hence, this approach is called the "flexibility" method.

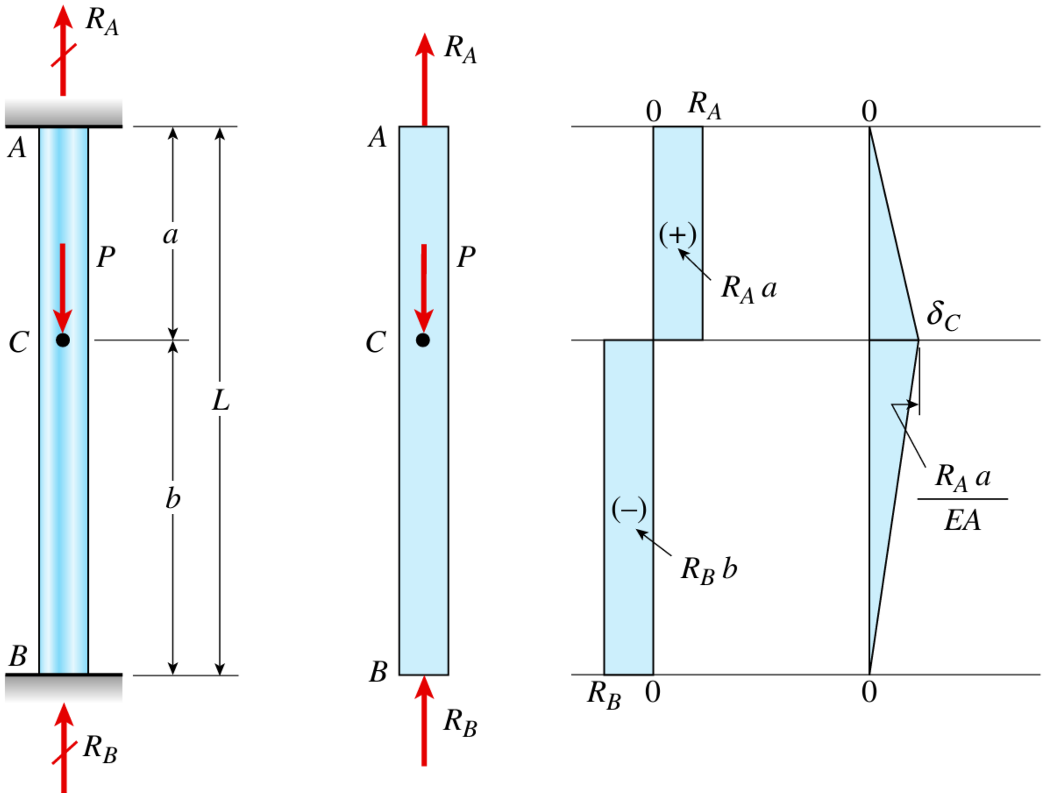
$$R_A + R_B = P$$
$$R_A a - R_B b = 0$$

$$R_A a - (P - R_A) b = 0$$
$$\Rightarrow R_A = \frac{Pb}{L}$$
$$\Rightarrow R_B = \frac{Pa}{L}$$

*Note that we have solved for forces. Hence, this approach is also called the "force" method.

Flexibility (Force) method: Summary

Example: Analysis the statically indeterminate bar, draw its ADD and AFD.



① Draw FBD

② Write down all static equilibrium equation(s)

③ Calculate *degree of static indeterminacy*

④ Find out the compatibility equation(s)

⑤ Using Force-displacement relation, convert compatibility equation(s) into equations of reaction force(s)

⑥ Solve all reaction force(s)

⑦ Calculate all required stress and strains

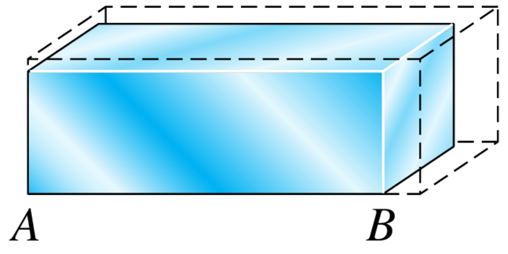
Thermal Strain

- When a material experiences a change in temperature, it undergoes **thermal strain**, which is independent of any external forces.
- If the material is allowed to expand freely, no internal stresses will develop.
- **Thermal Strain Formula**:

$$\varepsilon_T = \alpha \Delta T$$

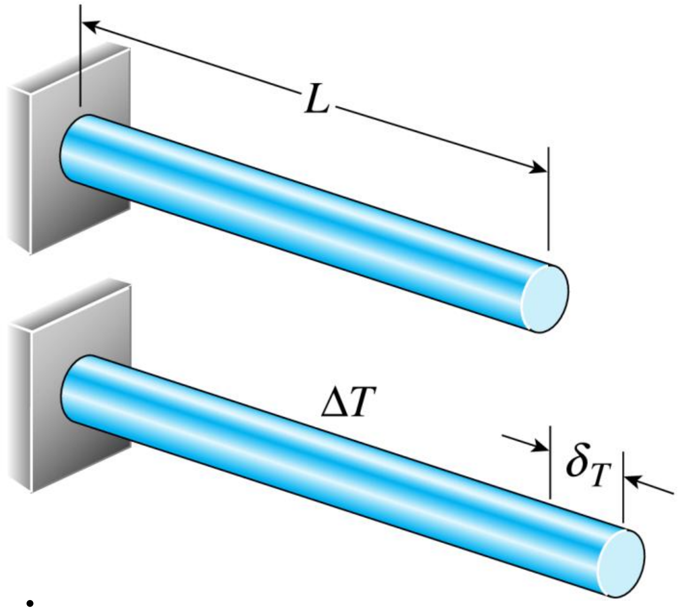
  - α is the **coefficient of thermal expansion**, which varies by material.
  - ΔT is the **change in temperature**.
  - ε_T is the **thermal strain** (positive for expansion, negative for contraction).

Block of material subjected to an increase in temperature



Temperature-displacement relation

- **Deformation due to thermal effect**



$$\varepsilon_T = \alpha \Delta T$$

$$\delta_T = \varepsilon_T L = \alpha(\Delta T)L$$

Example:

For a bar of length $L = 1$ m, $\alpha = 12 \times 10^{-6}$ (°C^{-1}),

If the change in temperature is $\Delta T = 50$ °C, the expansion is:

$$\delta_T = \alpha(\Delta T)L = (12 \times 10^{-6}\ ^\circ\mathrm{C}^{-1}) \times (50\ ^\circ\mathrm{C}) \times (1\ \mathrm{m})$$

$$= 0.6\ \mathrm{mm}$$

Thermal strain vs. mechanical strain

What causes the strain?

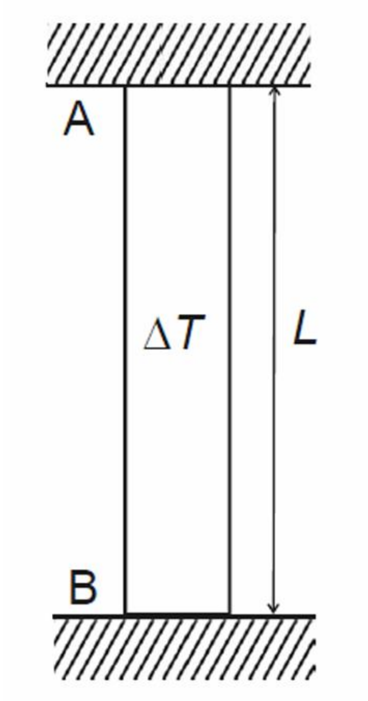
- Thermal strain arises due to temperature changes, not external loads.
- Mechanical strain results from externally applied loads.

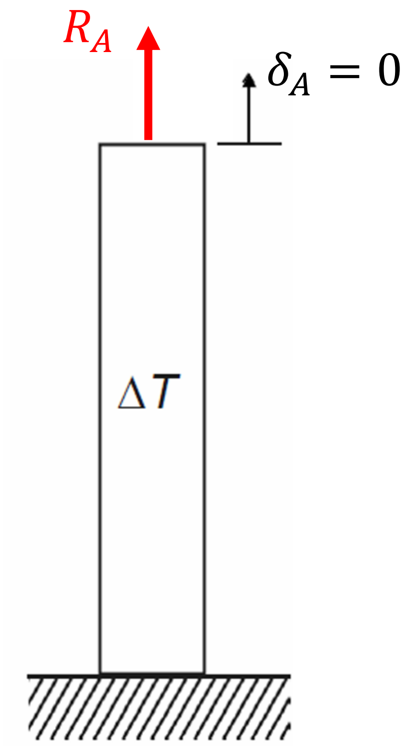
Can thermal strain induce stress?

- **Yes, if constraints are present**:
  When expansion/contraction is prevented, the thermal strain causes internal stress — this is known as thermal stress.
- **No, if unconstrained**:
  A free object expands or contracts without resistance, leading to strain without stress.

Exercise

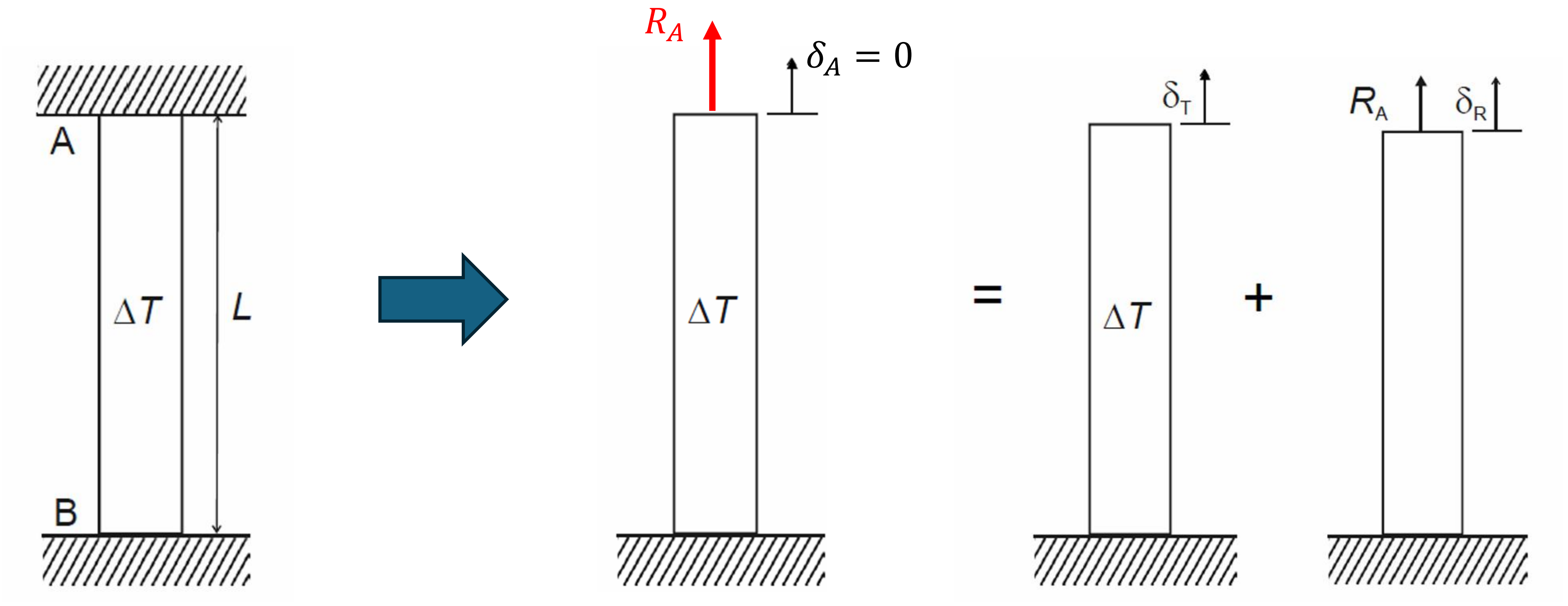
The temperature of the bar is raised by ΔT. Find the stress in the beam.





Method of superposition

For linear systems, we can decompose a complex problem into simpler subproblems, solve them independently, and add the results to get the full solution.



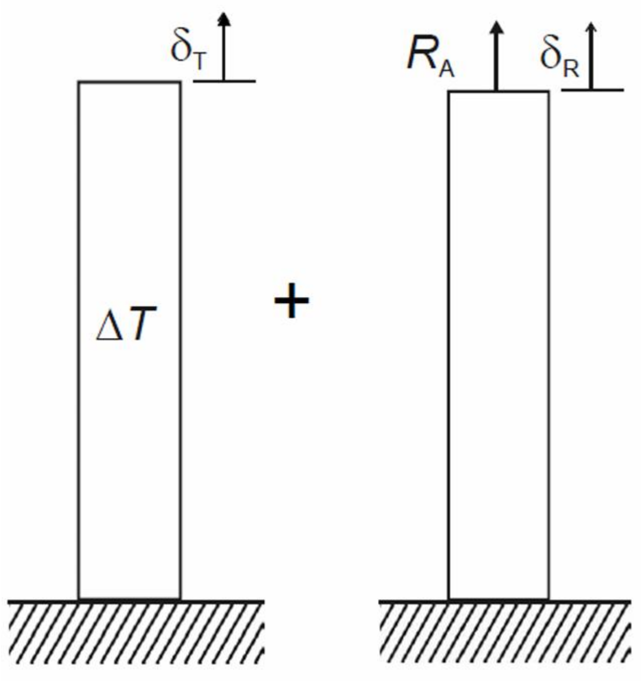
Method of superposition

Step 1: Thermal expansion

- A bar **freely heated** with temperature rise ΔT.
- It **expands freely** by $\delta_T = \alpha(\Delta T)L$
- There is **no stress**, just thermal strain.

Step 2: Apply a reaction force

- Apply an upward force R_A
- The applied force cause an elongation $\delta_R = \frac{R_A L}{EA}$
- This force induces mechanical stress.



Compatibility equation: $\delta_A = \delta_T + \delta_R = 0 \Longrightarrow R_A = -EA\alpha\Delta T \Longrightarrow \sigma = \frac{R_A}{A} = -E\alpha\Delta T$

How we respond to thermal expansion?

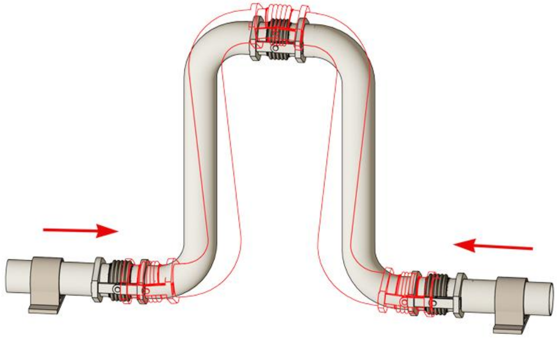
For plain carbon steel: $\alpha = 12.5 \times 10^{-6}\ ^{\circ}\mathrm{C}^{-1}$, $E = 200$ GPa.

$$\sigma_T = E\alpha\Delta T = 12.5 \times 10^{-6} \times 200 \times 10^{3}\ \Delta T\ \mathrm{MPa} = 2.5\Delta T\ \mathrm{MPa}$$

Thermal expansion is a natural behavior of materials. However, uncontrolled expansion can cause stress, deformation, or failure, so engineers design around it.

Expansion joint on a bridge

Expansion joint on the Cornish Main Line, England



Pipe loop with three hinged expansion joints