# Differential Equations Tutorial 3

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Week 5

# Contents

# Details are significant

- $$y' = \frac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$$
- $$y = \pm\sqrt{2C - 2\cos x}$$

  If do not consider i.c.
- $$y' = ye^x - 2e^x + y - 2$$
- $$y = 2 + Ce^{e^x + x}.$$

# Recall

- Homogeneous ? General Solution ?
- Inhomogeneous ? General Solution ?
- Integrating Factor
- Variation of Parameters
  Yes but how to derive? Do not memorize.

# Integrating Factor

- Core Step:
$$(uy)' = u'y + uy' = uy' - upy$$
- Why? By ATTEMPT.
- **Example**
Solve
$$y' + 2xy = e^{x-x^2}, \quad y(0) = -1.$$
- $p(x) = ? \quad q(x) = ? \quad u(x) = ?$
- $$y = e^{x-x^2} - 2e^{-x^2}.$$

# Variation of Parameters

- Structure of Solution
- Core Step:

$$v'y_h + vy_h' = pvy_h + q.$$

$$y_h' = py_h$$

- Integrating we arrive at

$$v = \int \frac{q}{y_h}\,dx + C = \int qe^{-\int p(x)\,dx}\,dx + C.$$

- Exactly the same with Integrating Factor Method.

# Singular Points in Linear Differential Equations

- Consider the equation:

$$(x^2 - 9)\frac{dy}{dx} + xy = 0 \quad (*)$$

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$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0$$

- Singular Points: $x = \pm 3$, where the function is discontinuous.
- Integrating Factor:

$$u(x) = \exp\left(\int \frac{x}{x^2 - 9}dx\right) = \sqrt{|x^2 - 9|}$$

- General Solution (on intervals excluding singularities)

$$y(x) = \frac{C}{\sqrt{|x^2 - 9|}}, \quad \text{valid for } x \neq \pm 3$$

- Even if a solution family is defined on large intervals, it may not be valid across singular points.

# Singular Points in Linear Differential Equations

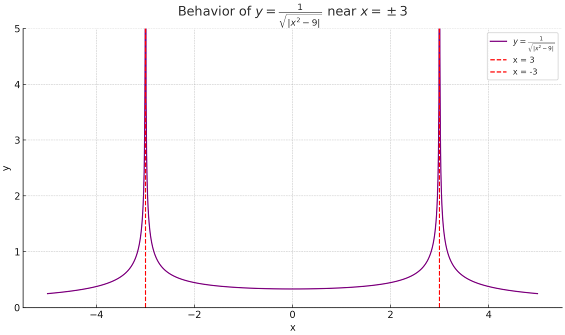


Figure: Solution Curve for Equation $(*)$, $C = 1$

# Modeling and Applications

- The Malthusian Model/The Logistic Model (Mentioned in Lecture)
- Personal Finance
  Let $P(t)$ be the balance at time $t$, $r$ be the interest rate, $W$ be the withdraw.

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{rP(t)\Delta t - W\Delta t}{\Delta t}$$

$$= rP - W.$$

# Modeling and Applications

- Example - Planning for Retirement
- You will need \$50,000 each year to live on after you retire, and that you should plan on living 30 years after your retirement. Assuming that your retirement account will earn 5% interest. You decide that you should put a fixed percentage $\rho$ of your salary into your retirement account. Assuming that your current salary is \$35,000 per year. Your salary will grow at 4% per year. The question is, what value of $\rho$ will achieve your goal?

# Planning for Retirement

- Let $P(t)$ denote the balance in thousands of dollars in your retirement account at time $t$.
- After Retirement:

$$P' = 0.05P - 50.$$

$$P(t) = 1000 + (P_0 - 1000)e^{0.05t}.$$

$$0 = P(30) = 1000 + (P_0 - 1000)e^{1.5}.$$

- Before Retirement:

$$\begin{aligned} P'(t) &= \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} \\ &= 0.05P(t) + \rho S(t) \\ &= 0.05P(t) + 35\rho e^{0.04t} \end{aligned}$$

$$i.c. \quad P(0) = 0$$

$$P(t) = 3500\rho\left(e^{0.05t} - e^{0.04t}\right)$$

# Mixing Problem

- Case1. Constant Volume
- **Example**
  The tank initially holds 100 gal of pure water. At time $t = 0$, a solution containing 2 lb of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant. How much salt is in the tank after 60 min?
- Denote

$$x(t) = \text{ the number of pounds of salt in the tank after } t \text{ min.}$$

$$\frac{dx}{dt} = \text{ the rate at which the amount of salt is changing w.r.t time.}$$

# Mixing Problem

- Splits naturally into two parts since there is a flow into the tank and another out.

$$\text{rate of change} = \text{rate in} - \text{rate out}.$$

- Measured in pounds per gallon (lb/gal)
- The concentration at time $t$, $c(t)$, is given by

$$c(t) = \frac{x(t)}{100} \quad \text{lb/gal}.$$

- We can now determine the rate at which salt is leaving the tank.

$$\text{rate out} = 3\ gal/min \times \frac{x(t)}{100}\ lb/gal = \frac{3x(t)}{100}\ lb/min.$$

# Mixing Problem

- $$\begin{aligned}\frac{dx}{dt} &= \text{rate of change}\\ &= \text{rate in} - \text{rate out}\\ &= 6 - \frac{3x}{100}\end{aligned}$$

- $$e^{3t/100}x = \int 6e^{3t/100}\,dt = \frac{600}{3}e^{3t/100} + C$$

- It follows from the I.C. $x(0) = 0$ that
$$0 = x(0) = 200 + Ce^{-3(0)/100} = 200 + C.$$
Consequently, $C = -200$ and our solution is
$$x(t) = 200 - 200e^{-3t/100}.$$
The amount of salt present in the tank after 60 minutes is
$$x(60) = 200 - 200e^{-3(60)/100} \approx 167 \text{ lb.}$$

# Mixing Problem

- Case2. Nonconstant Volume
- **Example**
  The 600-gal tank is filled with 300 gal of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb* of salt per gallon of solution begins flowing into the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?
- Since the initial amount of solution in the tank is 300 *gal*, the volume of solution in the tank, at any time $t$, is given by

$$V(t) = \underline{300 + 2t}.$$

# Mixing Problem

- $$\text{rate in} = 3 \; gal/min \times 1.5 \; lb/gal = 4.5 \; lb/min.$$
- The concentration of the solution in the tank becomes
$$c(t) = \frac{x(t)}{V(t)} = \frac{x(t)}{300+2t} \quad lb/gal.$$
- Therefore,
$$\text{rate out} = 1 \; gal/min \times \frac{x(t)}{300+2t} \quad lb/gal = \frac{x(t)}{300+2t} \quad lb/min.$$
- $$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}.$$
The corresponding I.C. is $x(0) = 0$.

# Mixing Problem

- Case3. System
- **Example**

  Consider two tanks, labeled tank A and tank B. Tank A contains 100 *gal* of solution in which is dissolved 20 *lb* of salt. Tank B contains 200 *gal* of solution in which is dissolved 40 *lb* of salt.

  Pure water flows into Tank A at a rate of 5 *gal/s*. There is a drain at the bottom of Tank A. Solution leaves Tank A via this drain at a rate of 5 *gal/s* and flows immediately into Tank B at the same rate.

  A drain at the bottom of Tank B allows the solution to leave Tank B, also at a rate of 5 *gal/s*.

  What is the salt content in Tank B after 1 minute?

# Mixing Problem

- Denote

  $x(t) =$ the number of pounds of salt in Tank A after $t$ seconds.

  $y(t) =$ the number of pounds of salt in Tank B after $t$ seconds.

- ..........
- The given example can be described with the following **system** of first order ODEs

$$\frac{dx}{dt} = -\frac{1}{20}x,$$

$$\frac{dy}{dt} = \frac{1}{20}x - \frac{1}{40}y,$$

with initial conditions $x(0) = 20$ and $y(0) = 40$.

# Bernoulli's Equation

- Form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

- Special Case: $n = 0, n = 1$
- The substitution

$$u = y^{1-n}$$

  reduce it to a linear DE
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$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1-n}y^n\frac{du}{dx}$$

# Bernoulli's Equation

- $$\frac{1}{1-n}y^n\frac{du}{dx}+p(x)y=q(x)y^n$$
  $$\frac{1}{1-n}\frac{du}{dx}+p(x)u=q(x)$$
- **Example 1**
  Solve
  $$x\frac{dy}{dx}+y=x^2y^2.$$
- **Example 2**
  Solve
  $$xy'+y(x+1)+xy^5=0,\quad y(1)=1.$$

# Bernoulli's Equation Solution

**Solution**

*Dividing through x we obtain*

$$y' + \frac{x+1}{x}y + y^5 = 0.$$

*Hence, the equation is a Bernoulli equation, where*

$$p(x) = \frac{x+1}{x}, \quad q(x) = -1, \quad n = 5.$$

*We substitute*

$$u = y^{1-5} = y^{-4}, \quad u' = -4y^{-5}y'.$$

*Now dividing the equation through via $y^5$ we have*

$$\frac{-x}{4}u' + u(x+1) + x = 0.$$

# Bernoulli's Equation Solution

*Finally, we change the form to a linear equation*

$$u' - \frac{4(x+1)}{x} u = 4.$$

*Then the integrating factor is*

$$v = e^{\int -4(1+1/x)\,dx} = e^{-4(x+\ln|x|)} = \frac{1}{x^4} e^{-4x}.$$

*Integrating gives*

$$u(x) = e^{4x} x^4 \int \frac{4}{x^4} e^{-4x}\,dx + C e^{4x} x^4, \quad C \in \mathbb{R}.$$

We still need to determine the value of $C$ since this is an initial value problem. for $y$ via substituting $u = y^{-4}$, that is,

$$y^{-4} = e^{4x} x^4 \int \frac{4}{x^4} e^{-4x}\,dx + C e^{4x} x^4.$$

# Bernoulli's Equation Solution

In order to employ the i.c. $y(1) = 1$, we need to rewrite the integral in the form of a definite integral using the FTC because we cannot substitute values in the present indefinite integral. Hence, we have

$$y^{-4} = e^{4x}x^4 \int_1^x \frac{4}{s^4} e^{-4s}\, ds + Ce^{4x}x^4.$$

The lower limit is 1 because $y(1) = 1$. Then using $y(1) = 1$ we have

$$1 = e^4(1)^4 \int_1^1 \frac{4}{s^4} e^{-4s}\, ds + Ce^4(1)^4 = Ce^4,$$

hence, $C = e^{-4}$.

# Bernoulli's Equation Solution

Hence, the final solution is

$$y^{-4} = e^{4x}x^4 \int_1^x \frac{4}{s^4} e^{-4s}\, ds + e^{4x-4}x^4.$$

Collecting terms of $e^{4x-4}x^4$ gives

$$y^{-4} = e^{4x-4}x^4 \left( \int_1^x \frac{4}{s^4} e^{-4s+4}\, ds + 1 \right).$$

Finally, the solution for $y$ is

$$y = \frac{e^{-x+1}}{x\left(4\int_1^x \frac{e^{-4s+4}}{s^4}\, ds + 1\right)^{1/4}}.$$

# Quiz 3-a

In economics, the effectiveness of advertising can be modeled using the following differential equation:

$$\frac{dS}{dt} + \lambda S = \beta A(t)$$

where:

- $S(t)$ represents the product sales amount over time.
- $\lambda$ is the natural decay rate of sales.
- $\beta$ is the advertising response coefficient.
- $A(t)$ is the advertising expenditure as a function of time.

Given that the initial sales amount is $S_0$, solve for $S(t)$.

# Quiz 3-b

In psychology, the strength of memory $M(t)$ over time is modeled by the following differential equation:

$$\frac{dM}{dt} + kM = \alpha S(t)$$

where:

- $M(t)$ represents the memory strength at time $t$.
- $k$ is the decay (forgetting) rate.
- $\alpha$ is the learning efficiency coefficient.
- $S(t)$ is the learning stimulus function.

Given that the initial memory strength is $M_0$, solve for $M(t)$.