Boolean Equations

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Introduction

Figure: George Boole

By George Boole, *The Mathematical Analysis of Logic*, 1847

Overview

Boolean algebra includes **set algebra** and **logical algebra**, etc. There is a one-to-one correspondence between set algebra and logical algebra.

- Set Algebra
  i.The number of elements in a set can be finite or infinite
  ii. Studies operations like union, intersection, complement, and difference under Boolean algebra principles
- Logical Algebra
  i.The values are limited to only two elements: 0 (false) and 1 (true).
  ii. Basic Operations includes AND (Conjunction), OR (Disjunction), NOT (Negation).

Preview: Set Algebra

- Empty Set, denoted by $\emptyset$, can be represented by 0 here.
- Universal Set, denoted by U, can be represented by 1 here.
- Complement
  The complement of a set A is denoted as A'.
  It is defined by
  $$A' = U \setminus A,$$
  where U is the universal set.
  Note that $0' = 1$, $1' = 0$.

Preview: Set Algebra

- **Difference**
  The difference of two set A and B, say $A - B$, is defined as a set, in which all the elements belong to A but not belong to B. i.e.
  $$A - B = A \cap B'$$
  where B' is the complement of set B.
- Symmetric Difference
  The symmetric difference of two set A and B, say $A + B$, is defined as
  $$\begin{aligned} A + B &= (A - B) \cup (B - A) \\ &= (A \cap B') \cup (B \cap A') \end{aligned}$$
  Obviously, $B + A = A + B$

Cross

- **Cross**
  The cross of two set A and B, $A \times B$, is defined as the complement of $A + B$
- We derive that

$$A \times B = (A \cap B) \cup (A' \cap B')$$

- Inference

$$A \times B = A' \times B'$$

Example 1

- **Solve the Equation**

$$A + X = 0$$

- **Solution**

$$\begin{aligned} A + X &= (A - X) \cup (X - A) \\ &= (A \cap X') \cup (X \cap A') = 0 \end{aligned}$$

$$\therefore A \cap X' = 0, A' \cap X = 0$$

$$\therefore A \subset X, X \subset A$$

$$\therefore X = A$$

Example 2

- **Solve the Equation**

$$(A + X) \cup (B + Y) = 0$$

- **Solution**

$$A + X = 0, B + Y = 0$$

$$\therefore X = A, Y = B$$

According to Example 1.

Example 3

- **Solve the Equation**

$$A \times X = 1$$

- **Solution**

  By definition of cross,

$$A + X = 0$$

$$\therefore X = A$$

According to Example 1.

Example 4

- **Solve the Equation**

$$(A \times X) \cap (B \times Y) = 1$$

- **Solution**

$$A \times X = 1, B \times Y = 1$$

$$\therefore A + X = 0, B + Y = 0$$

$$\therefore X = A, Y = B$$

According to Example 2.

Example 5

- **Solve the Equation**

$$A \cup X = 1$$

- **Solution**

$$A' \cap X' = 0$$

$$\therefore X \cup (A' \cap X') = X \cup 0 = X$$

$$\therefore (X \cup A') \cap (X \cup X') = X$$

$$X = A' \cup X$$

It indicates that each solution X is of the form $A' \cup U$. Note that for each solution, U equals to X.
On the other hand, the form $X = A' \cup U$ satisfies the equation. (Remember to verify)

Example 6

- **Prove**
  Equation
  $$X \cup (A \cap Y) = K$$
  has general solution
  $$X = [U \cup (A' \cup V')] \cap K$$
  $$Y = (A' \cup K) \cap V$$

Example 7

- **Solve the Equation**

$$X \cup Y = K$$

- **Solution** We need to prove that the general solution could be expressed as

$$X = K \cap (U \cup V'),$$
$$Y = K \cap (U' \cup V),$$

where U and V are arbitrary sets.

Verify the solution

$$X \cup Y = [K \cap (U \cup V')] \cup [K \cap (U' \cup V)]$$

$$= K \cap [(U \cup V') \cup (U' \cup V)] = K \cap [U \cup V' \cup U' \cup V] = K \cap 1 = K.$$

Example 7

- **Solution(continued)** Since X, Y are subsets of $X \cup Y = K$, we have:

$$X = X \cup (Y \cap Y') = (X \cup Y) \cap (X \cup Y')$$

$$= K \cap (X \cup Y'),$$

$$Y = Y \cup (X \cap X') = (X \cup Y) \cap (X' \cup Y)$$

$$= K \cap (X' \cup Y).$$

$\therefore$

$$X = K \cap (U \cup V'),$$
$$Y = K \cap (U' \cup V),$$

Preview: Logical Algebra - Equivalence

- Set A — Variable x
- Union $A \cup B$ — Logical OR $x \vee y$
- Intersection $A \cap B$ — Logical AND $x \wedge y$
- Complement A^c — Negation $\neg x$
- $\emptyset$ — 0
- U — 1
- Properties
  E.g. Commutativity

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

$$x \vee y = y \vee x, \quad x \wedge y = y \wedge x$$

Boolean 0-1 Equation

- **Definition**

  Let $\langle \mathbb{B}, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra, $f(X)$ and $g(X)$ are boolean functions of n variables. Equation

  $$f(X) = g(X)$$

  is called $n-$ *variable* boolean equation on $\mathbb{B}$

- $$f_i(X) \rho_i g_i(X) \quad (i = 1, \ldots, m)$$

  is called Generalized Boolean Equation System.

- Example

  $$\mathbf{x_1} \oplus \mathbf{x_2} = 1$$

- Solution

  $$(x_1, x_2) \in \{(0,1), (1,0)\}$$

Example 8

- **Solve the Equation**

$$\bar{y}\bar{z} + xz = 0$$

- **Solution**

  Taking the complement on both sides

$$(\bar{y}\bar{z} + xz)' = 1.$$

$$(\bar{y}\bar{z})' = \bar{y}' + \bar{z}' = y + z,$$

$$(xz)' = x' + z'.$$

  Thus, the equation simplifies to:

$$(y + z) \cdot (x' + z') = 1.$$

$$\therefore (y + z) = (x' + z') = 1.$$

  Enumearte all the possible cases, we obtain that the solution set is:

$$S = (x, y, z) \in \{(0,1,0), (0,1,1), (0,0,1), (1,1,0)\}.$$

Example 9

- **Solve the Equation**

$$a + x = 1$$

- **Solution**

$$\bar{a}\bar{x} = 0$$

$$\begin{aligned} x + \bar{a}\bar{x} &= (x + \bar{a})(x + \bar{x}) \\ &= x + 0 \\ &= x \end{aligned}$$

$$\therefore x + \bar{a} = x$$

$$x = \bar{a} + u$$

$$\text{as } u = x$$

Example 9.1

- **Solve the Equation**

$$x + a \cdot y = k$$

- **Solution**

$$x = [u + (\bar{a} + \bar{v})] \cdot k$$

$$y = (\bar{a} + k) \cdot v$$

u, v, k are arbitrary elements.

Example 10

- **Prove the Theorem**
  The necessary and sufficient condition for the Boolean equation
  $$ax + b\bar{x} = 0$$
  (where a, b are boolean constants) to have a solution is
  $$ab = 0.$$

- **Solution** Let x be a solution of $ax + b\bar{x} = 0$, then:
  $$ax + b\bar{x} = 0 \iff \begin{cases} ax = 0 \iff x \leq \bar{a}, \\ b\bar{x} = 0 \iff b \leq x. \end{cases}$$
  This implies:
  $$b \leq x \leq \bar{a} \Rightarrow b \leq \bar{a} \Rightarrow ab = 0.$$

Example 10

- **Solution(continued)**
  Conversely, if $ab = 0$, then:

$$ab(x + \bar{x}) = 0.$$

Expanding the expression:

$$aa\bar{x} + ab\bar{x} + abb + abx + bb\bar{x} = 0.$$

$$a(\bar{a}x + b\bar{x}) + b(ab + ax + b\bar{x}) = 0.$$

Let $x = \bar{a}x + b\bar{x}$, then:

$$\bar{x} = a\bar{b} + ax + \bar{b}\bar{x}.$$

Thus, we obtain:

$$ax + b\bar{x} = 0.$$

$\therefore x = \bar{a}x + b\bar{x}$ is the solution to the equation.

Appendix

$$
\begin{aligned}
&(A \cup B) \cap (A' \cup B') \\
&= [(A \cup B) \cap A'] \cup [(A \cup B) \cap B'] \\
&= (A \cap A') \cup (B \cap A') \cup (A \cap B') \cup (B \cap B') \\
&= (B \cap A') \cup (A \cap B') = A + B.
\end{aligned}
$$

Figure: Derivation for Cross

- **Proper Definition of Boolean Equation**
  If $h(x)$ is boolean function, $k(x) = \bar{h}(x)$,then $h(x) = 0$ and $k(x) = 1$ are respectively called Boolean 0-Equation and Boolean 1-Equation, collectively referred to as 0-1 Boolean Equations.
- Potential Applications:
  Sequential Logic
  CSP(Constraint Satisfaction Problems)
  SAT(Satisfiability Problem)