Differential Equations Tutorial 5

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Week 7

Contents

One Simplification

Exact Differential Equations

Exact Differential Equations

!Theorem-Proof

(b) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \omega = P(x, y)dx + Q(x, y)dy$ is exact

We have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

And substitute this equation into (11), it yields

$$\frac{\partial F}{\partial y} = \int \frac{\partial P}{\partial y} dx + \phi'(y) = \int \frac{\partial Q}{\partial x} dx + \phi'(y)$$

The first integral term in RHS is Q.

$$\frac{\partial F}{\partial y} = Q + \phi'(y) \tag{12}$$

Figure: Here is a simplification.

Existence & Uniqueness

Theorems of existence and uniqueness for first-order ODEs
A rectangular region

$$R = \{(t, x) \mid a < t < b,\ c < x < d\}$$

in the *tx*-plane

- "locally"
- continuity implies the existence

Existence & Uniqueness

In conclusion, suppose we know

1. The equation is in normal form $y' = f(t, y)$,
2. The right-hand side $f(t, y)$ and its derivative $\partial f / \partial y$ are both continuous in the rectangle
$$R = \{(t, y) \mid a < t < b,\ c < y < d\},$$
3. The initial point (t_0, y_0) is in the rectangle R,

we can get:

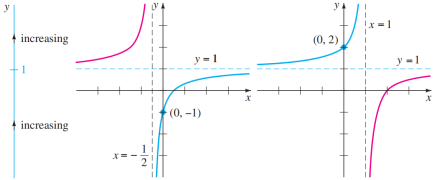
1. There is one and only one solution to the initial value problem,
2. The solution exists until the solution curve $t \to (t, y(t))$ leaves the rectangle R.

Example

EXAMPLE 5 Solution Curves of an Autonomous DE

The autonomous equation $dy/dx = (y - 1)^2$ possesses the single critical point 1. From the phase portrait in Figure 2.1.8(a) we conclude that a solution $y(x)$ is an increasing function in the subregions defined by $-\infty < y < 1$ and $1 < y < \infty$, where $-\infty < x < \infty$. For an initial condition $y(0) = y_0 < 1$, a solution $y(x)$ is increasing and bounded above by 1, and so $y(x) \to 1$ as $x \to \infty$; for $y(0) = y_0 > 1$ a solution $y(x)$ is increasing and unbounded.

Now $y(x) = 1 - 1/(x + c)$ is a one-parameter family of solutions of the differential equation. (See Problem 4 in Exercises 2.2) A given initial condition determines a value for c. For the initial conditions, say, $y(0) = -1 < 1$ and $y(0) = 2 > 1$, we find, in turn, that $y(x) = 1 - 1/(x + \frac{1}{2})$, and $y(x) = 1 - 1/(x - 1)$. As shown in Figures 2.1.8(b) and 2.1.8(c), the graph of each of these rational functions possesses



(a) phase line **(b)** xy-plane $y(0) < 1$ **(c)** xy-plane $y(0) > 1$

FIGURE 2.1.8 Behavior of solutions near $y = 1$

Figure: Solution Curves and Phase Portrait.

Autonomous System

General Form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \text{where } \mathbf{x} \in \mathbb{R}^n$$

- Phase portraits can be studied geometrically without reference to t. $\leftarrow$ "Autonomous"
- Equilibrium points: solutions where $\mathbf{f}(\mathbf{x}) = 0$.

Example (2D)

$$\frac{dx}{dt} = x(1-y), \qquad \frac{dy}{dt} = y(x-1)$$

This is an autonomous system since the right-hand sides depend only on x and y.

Autonomous System

- Draw Phase Line & Draw solution curves
- Nonequilibrium solutions
- A point is **stable** if nearby solutions flow toward it.
- A point is **unstable** if nearby solutions flow away from it.
- **Stable:** • (filled dot), with arrows pointing inward.→ Converge
- **Unstable:** ∘ (hollow dot), with arrows pointing outward.→ Diverge away
- Revisit:

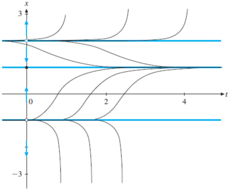


Figure: Equilibrium and Nonequilibrium Solutions

Review

Given a DE

- Directly Solve...
- e.g.
$$x\,dx + y\,dy = 0$$
- Question:
  Integral Curve vs Solution Curve
- BTW, "tricks"
  e.g.
$$y\cos(x+y)\,dx + \left(y\cos(x+y) + \sin(x+y)\right)\,dy = 0$$

Continuing

- Check Exactness $\Rightarrow$ Exact
- But we could apply

$$d\left(y\sin(x+y)\right)=0 \quad \Rightarrow \quad y\sin(x+y)=C$$

- For some problems we do ask you to solve in a particular method.
- Separable...
- Some are quite obvious, some need to simplify
- Pay attention to trivial solutions and singular points

Continuing

- First Order & Linear ...($\star$)
- Integrating factor or variation of parameters.
- Bernoulli ... $\Rightarrow$ ($\star$)
- Homogeneous ... $\Rightarrow$ ($\star$)

Continuing

- Exact ...
- Non-Exact Made Exact ...
- How to Derive ??
- Example

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

Not exact. With the identifications $P = xy$, $Q = 2x^2 + 3y^2 - 20$, we find the partial derivatives $Q_x = 4x$ and $P_y = x$. Since

$$\frac{Q_x - P_y}{P} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}$$

depends only on y, suggesting an integrating factor $\mu(y)$.

Continuing

$$\mu(y) = e^{\int \frac{3}{y}\,dy} = e^{3\ln y} = y^3.$$

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0.$$

This equation is now exact. To solve it, we find a potential function $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = xy^4, \quad \frac{\partial F}{\partial y} = 2x^2y^3 + 3y^5 - 20y^3.$$

Integrating $\frac{\partial F}{\partial x} = xy^4$ with respect to x, we get:

$$F(x, y) = \frac{1}{2}x^2y^4 + h(y).$$

Differentiate with respect to y:

$$\frac{\partial F}{\partial y} = 2x^2y^3 + h'(y).$$

Non-Exact Made Exact

Matching with given $\frac{\partial F}{\partial y}$, we get:

$$h'(y) = 3y^5 - 20y^3 \Rightarrow h(y) = \frac{1}{2}y^6 - 5y^4.$$

Thus, the solution is:

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C.$$

- Not Above ...
- Observing
  e.g. term $e^{-y/x}$
- Modeling – Set DE (system/piece-wise)

Schedule

- Midterm1
- Time Apr.20
- Coverage Week 1-7
- Type of Questions similar to assignment
- No Lecture & Tutorial Next Week

Notes

- Show steps if you want partial credit.
- Discuss case by case.
- Details are significant.

Quiz 5-a

Consider the autonomous differential equation

$$\frac{dx}{dt} = (x-1)(x-2)(x-3)(x-4).$$

- Sketch the phase line.
- Sketch all representative solution curves of the equation.
- State which of the equilibrium points are stable.

Quiz 5-b

Consider the autonomous differential equation

$$\frac{dx}{dt} = (x - r)(x - 2)(x - 3),$$

where $r \in \mathbb{R}$ is a real parameter.

(a) For which values of r is the equilibrium point $x = r$ stable?

(b) For such a value of r, sketch the all typical solution curves $x(t)$.