Introduction to Chaos Theory

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Acknowledgements

- Learned from Prof.Jiu Ding
- International Week, in total 6 lectures.
- Reference book:
  Encounters with Chaos and Fractals, Denny Gulick, Second Edition.

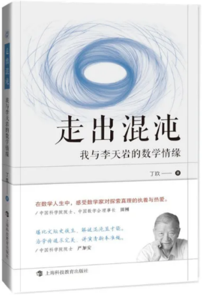

Figure: Mathmatician: Tien-yien Li.

Basics

Function $f : D \to D$ (D : Domain)

For any $x_0 \in D$, keep substituting it into f, obtaining

$$x_0,\ f(x_0),\ f(f(x_0)),\ \ldots, f(f(f...(x0))), \ldots$$

That is, continue this process until the n-th iteration: $f^{[n]}(x_0)$.

We denote this sequence as $x_0,\ x_1,\ x_2,\ \ldots,$ forming a sequence $\{x_n\}$, where $x_n = f^{[n]}(x_0)$.

We say that $\{x_n\}$ is the **orbit** of x_0.

Example: Newton's Method

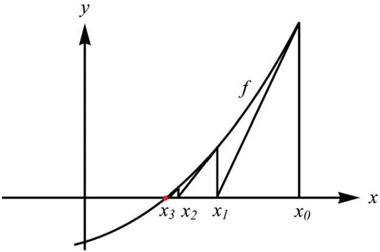

Figure: The iterations via Newton's Method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

That is,

$$x_n = g(x_{n-1}).$$

Intuition: Convergent or Divergent

- Example 1: $y = x/2$
- Graphical Analysis with Reference Line (Diagonal)

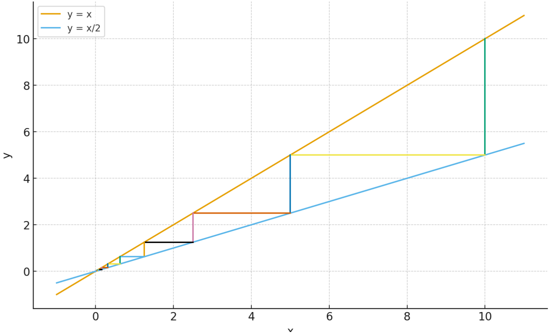

Figure: Diagram for $y = x/2$, start with $x_0 = 10.0$.

Intuition: Convergent or Divergent

- Example 2: $y = 2x$

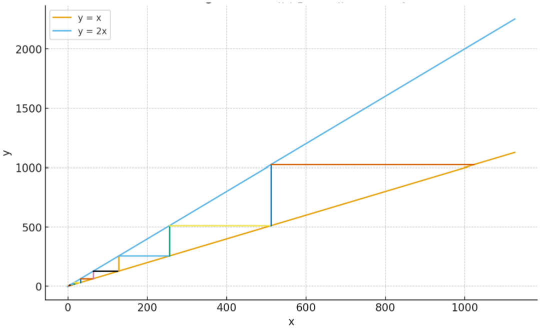

Figure: Diagram for $y = 2x$, start with $x_0 = 1.0$.

More Complicated Example

$$y = 4x(1-x), [0,1] \to [0,1]$$

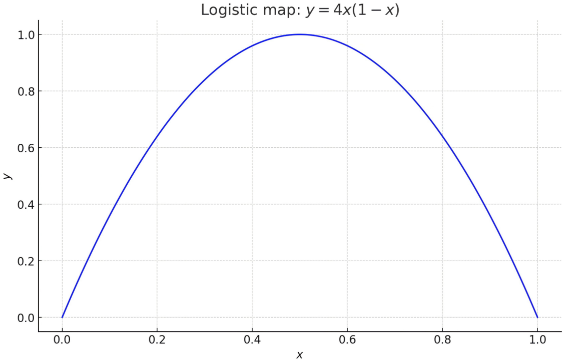

Figure: Logistic Map

Fixed Point

Definition

A point x^* is called a **fixed point** of a function f if

$$f(x^*) = x^*.$$

That is, applying the function does not change the value:

$$x_{n+1} = f(x_n) \quad \text{remains the same when } x_n = x^*.$$

How to find the fixed point(s) geometrically?

Intuition: Attractive and Repelling

Consider two functions:

$$f_1(x) = \frac{1}{2}x, \quad f_2(x) = 2x.$$

Both have a fixed point at $p = 0$.

- For $f_1(x) = \frac{1}{2}x$:

  $$x_n \text{``} \rightarrow \text{''} p$$

  Iterations $x_{n+1} = \frac{1}{2}x_n$ converge to 0.
- For $f_2(x) = 2x$:

  $$x_n \text{``} \leftarrow \text{''} p$$

  Iterations $x_{n+1} = 2x_n$ diverge away from 0.

Theorem 1

Let f be continuous on its domain. If $f^{[n]}(x) \to p$, then p must be fixed.

Proof

$$f(p) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = p.$$

Hence, p is fixed.

Example

$$f(x) = sinx$$

Show that $x_n \to 0$.

For $x > 0$, we have $\sin x < x$. Hence

$$x_0 > x_1 = \sin x_0 > x_2 = \sin x_1 > \cdots > 0.$$

The sequence $\{x_n\}$ is **decreasing** and bounded below by 0. By the monotone convergence theorem, $\{x_n\}$ converges. Let $\lim_{n\to\infty} x_n = p$.

Example: sinx

Apply Theorem 1

Since $\sin x$ is continuous,

$$p = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sin(x_n) = \sin(p).$$

Thus p must be a fixed point. The only such $p \geq 0$ is $p = 0$.

Conclusion

$$x_n \longrightarrow 0.$$

The sequence monotonically decreases to the fixed point at the origin.

Definition

Let p be a fixed point of f.

1. The point p is an *attracting fixed point* of f provided that there exists an interval $(p-\varepsilon, p+\varepsilon)$ containing p such that if x is in the domain of f and in $(p-\varepsilon, p+\varepsilon)$, then

$$f^{[n]}(x) \to p \quad \text{as } n \to \infty.$$

2. The point p is a *repelling fixed point* of f provided that there exists an interval $(p-\varepsilon, p+\varepsilon)$ containing p such that if x is in the domain of f and in $(p-\varepsilon, p+\varepsilon)$ but $x \neq p$, then

$$|f(x)-p| > |x-p|.$$

Theorem 2

Suppose that f is differentiable at a fixed point p.

a. If $|f'(p)| < 1$, then p is attracting.

b. If $|f'(p)| > 1$, then p is repelling.

c. If $|f'(p)| = 1$, then p can be attracting, repelling, or neither.

Proof of Theorem 2(a)

Proof. We notice that since $|f'(p)| < 1$, the definition of derivative implies that there is a positive constant $A < 1$ and an open interval

$$J = (p - \varepsilon,\ p + \varepsilon)$$

such that if x is in J and $x \neq p$, then

$$\left| \frac{f(x) - f(p)}{x - p} \right| \leq A.$$

Therefore

$$|f(x) - f(p)| \leq A|x - p|, \qquad \text{for all } x \in J.$$

For each such x, this means that

$$|f(x) - p| = |f(x) - f(p)| \leq A|x - p| \tag{1}$$

Proof of Theorem 2(a)

Proof.(continued) Fix x in this interval and assume $f^n(x) \neq p$ for all n. We prove by induction that

$$|f^n(x) - p| \leq A^n |x - p|. \tag{2}$$

Base case $n = 1$ follows from (1). For the inductive step, assume (2) holds for some n. Then

$$|f^{n+1}(x) - p| = |f(f^n(x)) - p| \leq A\,|f^n(x) - p| \leq A^{n+1}|x - p|.$$

Thus (2) holds for all $n \geq 1$. Since $A^n \to 0$, it follows that $f^n(x) \to p$, proving (a).
Note: The proof of (b) is similar; (c) is addressed separately.

Example

Let $\mu > 0$ be a constant, and let

$$f(x) = \mu x(1-x) = \mu x - \mu x^2, \qquad 0 \le x \le 1.$$

1. Find the values of μ for which 0 is an attracting fixed point.
2. Find the values of μ for which there is a nonzero fixed point.
3. Find the values of μ for which the nonzero fixed point is attracting.

Solution

Solution. Notice that x is a fixed point of f if and only if

$$x = \mu x - \mu x^2.$$

Thus $x = 0$ is a fixed point for every $\mu > 0$. Since $f'(0) = \mu$, it follows from Theorem 2 that 0 is attracting if $0 < \mu < 1$, and is repelling if $1 < \mu$.
Observing that if $x \neq 0$, then x is a fixed point only if

$$1 = \mu - \mu x, \qquad \text{or equivalently} \qquad x = 1 - \frac{1}{\mu}.$$

However, if $0 < \mu < 1$, then $x = 1 - \frac{1}{\mu} < 0$, so x is not in the domain of f. Therefore the nonzero fixed point $1 - \frac{1}{\mu}$ occurs only if $\mu > 1$.

Solution

Solution.(continued) We note that

$$f'\left(1-\frac{1}{\mu}\right)=\mu-2\mu\left(1-\frac{1}{\mu}\right)=2-\mu.$$

Using Theorem 2 again, we find that $1-\frac{1}{\mu}$ is attracting if $1<\mu<3$, and is repelling if $\mu>3$.
Finally, it is possible to show that $1-\frac{1}{\mu}$ is attracting if $\mu=3$.

Basins of Attraction

If a fixed point p of f is attracting, then all points near to p are attracted toward p, in the sense that their iterates converge to p. The collection of all points whose iterates converge to p is called the **basin of attraction** of p.

Definition

Suppose that p is a fixed point of f. Then the **basin of attraction** of p consists of all x such that

$$f^n(x) \to p \quad \text{as } n \text{ increases without bound,}$$

and is denoted by B_p.

Example: $f(x) = x^2, B1 = \{1, -1\}$.

Eventually Fixed Points

Definition

Let x be in the domain of f. Then x is an **eventually fixed point** of f if there is a positive integer n such that $f^{[n]}(x)$ is a fixed point of f.

A fixed point is trivially an eventually fixed point. However, if $f(x) = \sin x$, then $f(\pi) = 0$ and $f(0) = 0$, so that π is an eventually fixed point that is not a fixed point. In order not to create confusion, when we refer to x as an eventually fixed point, we will generally assume that x is *not* a fixed point.

Example: Piecewise Linear Function

Let T be defined by

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} < x \le 1. \end{cases}$$

Show that $1/8$ is an eventually fixed point.
A routine check shows that

$$T(1/8) = 1/4, \qquad T(1/4) = 1/2, \qquad T(1/2) = 1, \qquad T(1) = 0,$$

Therefore $1/8$ is an eventually fixed point.
The function T is called the *tent function* because of the shape of its graph.
One can show that if $x = k/2^n$, where k and n are positive integers and $0 < k/2^n \le 1$, then x is an eventually fixed point of T.

Periodic Points

Definition

Let x_0 be in the domain of f. Then x_0 has **period-**n (or is a **period-**n **point**) if

$$f^{[n]}(x_0) = x_0,$$

and if in addition, $x_0, f(x_0), f^{[2]}(x_0), \ldots, f^{[n-1]}(x_0)$ are distinct. If x_0 has period n, then the orbit of x_0, which is

$$\{x_0,\ f(x_0),\ f^{[2]}(x_0), \ldots,\ f^{[n-1]}(x_0)\},$$

is a *periodic orbit*, and the elements of the orbit form an *n-cycle*.

Fixed points are periodic points with period 1.

Example: $h(x) = -x^3$. Then $\{-1, 1\}$ is a 2-cycle because $h(-1) = 1$ and $h(1) = -1$.

Example

Recall: The tent function T is given by

$$T(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} < x \leq 1. \end{cases}$$

Show that $\{2/7, 4/7, 6/7\}$ is a 3-cycle for T.

Solution. A routine check yields

$$T\left(\tfrac{2}{7}\right) = \tfrac{4}{7}, \qquad T\left(\tfrac{4}{7}\right) = \tfrac{6}{7}, \qquad T\left(\tfrac{6}{7}\right) = \tfrac{2}{7},$$

confirming that $\{2/7, 4/7, 6/7\}$ is a 3-cycle for T.

Definition

Let x be a period-n point for a function f. Then x is an **attracting period-n point** if x is an attracting fixed point of $f^{[n]}$; also x is a **repelling period-n point** if x is a repelling fixed point of $f^{[n]}$.

Suppose that f is continuous at a period-n point x. If x is attracting (resp. repelling), then each point in

$$\{x,\, f(x),\, f^{[2]}(x),\, \ldots,\, f^{[n-1]}(x)\}$$

is an attracting (resp. repelling) period-n point, so we say that the n-cycle

$$\{x,\, f(x),\, f^{[2]}(x),\, \ldots,\, f^{[n-1]}(x)\}$$

is **attracting** (resp. **repelling**).

Theorem 3

Let $\{x, z\}$ be a 2-cycle of f. If $f^{[2]}$ is differentiable at x and at z, then

$$(f^{[2]})'(x) = f'(x)\, f'(z) = (f^{[2]})'(z) \tag{2}$$

Proof. Using the Chain Rule and the fact that $f(x) = z$, we find that

$$(f^{[2]})'(x) = (f \circ f)'(x) = [f'(f(x))][f'(x)] = f'(x)\, f'(z).$$

By symmetry we have $(f^{[2]})'(z) = f'(x)\, f'(z)$.

Generalized Theorem 3

Let $\{x_0, x_1, \dots, x_{n-1}\}$ be an n-cycle of f, where

$$f(x_i) = x_{i+1}, \quad f(x_{n-1}) = x_0.$$

If f is differentiable at each x_i, then the derivative of $f^{[n]}$ at any point of the cycle is given by

$$(f^{[n]})'(x_0) = f'(x_0)f'(x_1)\cdots f'(x_{n-1}) = (f^{[n]})'(x_i), \quad \forall i.$$

Proof. Using the Chain Rule and the fact that $f(x_i) = x_{i+1}$, we obtain

$$(f^{[n]})'(x_0) = (f \circ f^{[n-1]})'(x_0) = f'(f^{[n-1]}(x_0)) \cdot (f^{[n-1]})'(x_0).$$

Iterating this argument yields

$$(f^{[n]})'(x_0) = f'(x_0)\, f'(x_1)\, \cdots\, f'(x_{n-1}).$$

By cyclic permutation of indices, the same identity holds at each x_i:

$$(f^{[n]})'(x_i) = f'(x_i)f'(x_{i+1})\cdots f'(x_{i-1}).$$

Quadratic Family

Consider the *quadratic family*

$$Q_\mu(x) = \mu x(1-x), \qquad 0 < \mu \leq 4.$$

As the parameter μ increases, Q_μ exhibits the well-known *period-doubling cascade*: the fixed point loses stability and gives rise to a 2-cycle, which then gives rise to a 2^2-cycle, and so on. Let

$$\mu_0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$$

denote the sequence of bifurcation values at which a new 2^k-cycle is born. It is known that

$$\mu_0 = 3, \qquad \mu_1 = 1 + \sqrt{6}.$$

1. If $\mu_0 < \mu \leq \mu_1$, then Q_μ has two fixed points and a 2-cycle.
2. If $\mu_1 < \mu \leq \mu_2$, then Q_μ has two fixed points, a 2-cycle, and a 2^2-cycle.
3. If $\mu_2 < \mu \leq \mu_3$, then Q_μ has two fixed points, a 2-cycle, a 2^2-cycle, and a 2^3-cycle.

In general,

$$\text{if } \mu_{n-1} < \mu \leq \mu_n, \qquad \text{then } Q_\mu \text{ has a } 2^k\text{-cycle for } k = 0, 1, \ldots, n.$$

It is known, though difficult to prove, that for $k \geq 2$,

$$\mu_{k+1} \approx 1 + \sqrt{3 + \mu_k},$$

and that the sequence $\{\mu_k\}_{k=0}^{\infty}$ converges to the limit

$$\mu_\infty \approx 3.61546 \cdots ,$$

sometimes called the *Feigenbaum number* for the quadratic family. This value marks the accumulation point of the period-doubling cascade.

Feigenbaum Constant

The surprising part of the story is yet to come. Let

$$d_k = \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}, \qquad k = 2, 3, 4, \ldots$$

Since $\mu_k - \mu_{k-1}$ is the distance between successive bifurcation parameters, the ratio d_k compares the distances between two consecutive pairs of bifurcation points. Feigenbaum discovered that the sequence $\{d_k\}_{k=2}^{\infty}$ converges to a constant, which we denote by d_∞:

$$d_\infty \approx 4.669202\ldots$$

What is astonishing is that this number d_∞ is *universal*. That is, for many families of one-humped functions such as the quadratic family $\{Q_\mu\}$, the distances between successive bifurcation points shrink in a regular fashion, with the same limiting ratio d_∞. This universality is why d_∞ is called the *Feigenbaum constant*.

Bifurcation Diagram

We now study the bifurcation diagram of the quadratic family $\{Q_\mu\}$, where

$$Q_\mu(x) = \mu x(1-x), \qquad 0 < \mu \leq 4.$$

To show as much detail as possible, the bifurcation diagram is split into two parts:

$$0 < \mu \leq 1 + \sqrt{6}, \qquad 1 + \sqrt{6} \leq \mu \leq 4.$$

The diagram is generated numerically by fixing a starting point $x = \frac{1}{2}$, taking increments of $1/1000$ for μ in the interval $[0, 4]$, and plotting the points

$$(\mu,\ Q_\mu^{[n]}(x)), \qquad 201 \leq n \leq 700.$$

Bifurcation Diagram (Part I)

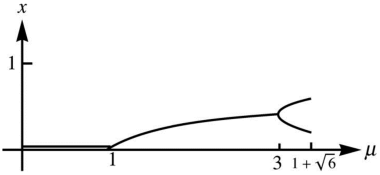

Figure: Bifurcation Diagram for $0 < \mu < 1 + \sqrt{6}$.

At each bifurcation parameter μ_k, an attracting 2^k-cycle becomes *repelling*, and simultaneously a new attracting 2^k-cycle is born. Dynamically, this means:

- for $\mu < \mu_k$, all nearby orbits are attracted to a stable 2^k-cycle;
- at $\mu = \mu_k$, the 2^k-cycle becomes unstable (repelling);
- for $\mu > \mu_k$, the system exhibits a stable 2^{k+1}-cycle.

Bifurcation Diagram (Part II)

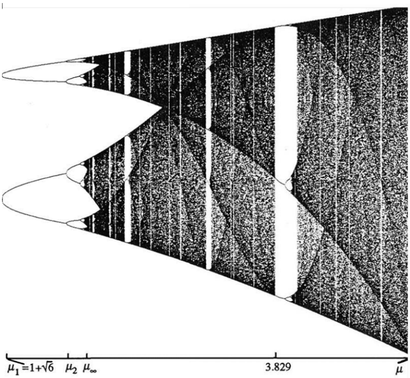

Figure: Bifurcation Diagram for $1+\sqrt{6}<\mu<4$.

Other Diagrams

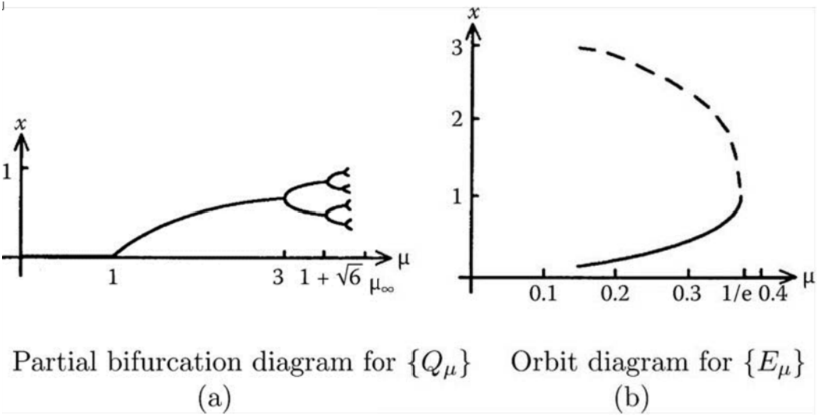

Figure: Bifurcation Diagrams.

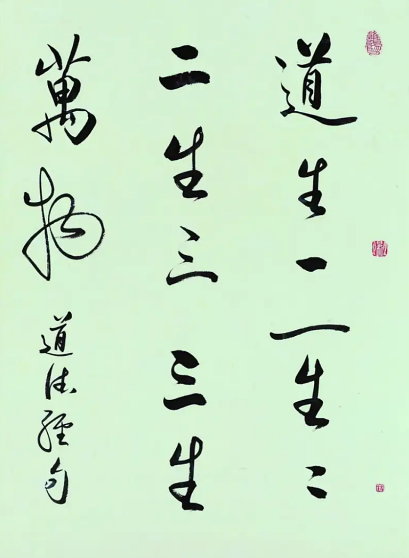

Figure: A Famous Saying from *Tao Te Ching*.

Lemmas

Lemma 1. Let f be continuous on an interval J. Let $f(J)$ denote the collection of all values $f(x)$ for $x \in J$. Then $f(J)$ is also an interval.

Proof. Suppose that $f(J)$ were not an interval. Then there would exist two numbers y and z in $f(J)$, with $y < z$, and a number $p \in (y, z)$ such that p is not in the range of f. By the Intermediate Value Theorem applied to $[y, z]$, the range of f must contain the entire interval $[y, z]$, and in particular must contain p. This contradiction implies that $f(J)$ is an interval.

Lemma 2. Let f be continuous on a closed interval J, and assume that

$$f(J) \supseteq [a, b].$$

Then there is a closed interval K such that $J \supseteq K$ and $f(K) = [a, b]$.

Lemmas

Lemma 3. Suppose that J is a closed interval, and assume that f is continuous on J and

$$f(J) \supseteq J.$$

Then f has a fixed point in J.

Lemma 4. Let f be continuous and suppose that $f(a) = b$, $f(b) = c$, and $f(c) = a$. Then f has a fixed point and a period-2 point.

Proof. The proof of lemma 2,3,4 is in the book *Encounters with Chaos and Fractals*. Read the corresponding chapter if interested.

Theorem 4

Theorem (Li–Yorke Theorem)

Suppose that f is continuous on the closed interval J, with

$$J \supseteq f(J).$$

If f has a period-3 point, then f has points of all other periods.

Later we will see the part II of this theorem.

Sensitive Dependence on Initial Conditions

Let J be an interval, and suppose that $f : J \to J$. Then f has *sensitive dependence on initial conditions at* x, or just *sensitive dependence at* x, if there is an $\varepsilon > 0$ such that for each $\delta > 0$ there is a $y \in J$ and a positive integer n such that

$$|x - y| < \delta \quad \text{and} \quad \left|f^{[n]}(x) - f^{[n]}(y)\right| > \varepsilon.$$

If f has sensitive dependence on initial conditions at each $x \in J$, we say that f has *sensitive dependence on initial conditions on* J, or that f has *sensitive dependence on* J, or simply that f has *sensitive dependence*.
The "initial conditions" in the definition refer to the given, or initial, points x and y.

Example: Baker's Function

Consider the baker's function B, given by

$$B(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} < x \leq 1. \end{cases}$$

Show that after 10 iterates, the iterates of $1/3$ and 0.333 are farther than $1/2$ apart.

Lyapunov Exponent

Definition (Lyapunov Exponent)

Let J be a bounded interval, and let $f : J \to J$ be continuously differentiable on J. Fix $x \in J$, and define $\lambda(x)$ by

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \ln \left| (f^{[n]})'(x) \right| ,$$

provided that the limit exists. In this case, $\lambda(x)$ is called the **Lyapunov exponent of f at x.**

If $\lambda(x)$ is independent of x wherever it is defined, then the common value of $\lambda(x)$ is denoted by λ, and is called the **Lyapunov exponent of f.**

Definition of Chaotic Function

Definition (Chaos)

Let f be a function defined on a bounded interval. Then f is said to be **chaotic** if it satisfies at least one of the following conditions:

1. f has a positive Lyapunov exponent at each point in its domain that is not eventually periodic;
2. f has sensitive dependence on initial conditions on its domain.

The term "chaos" in reference to functions was first used in Li and Yorke's paper *Period Three Implies Chaos* (1975). An essential part of their theorem is that the existence of period-3 points implies the existence of points of all other periods; in such cases, the function also exhibits sensitive dependence on initial conditions.

Randomness vs. Chaos

- **Randomness**: no underlying rule; outcomes are generated by a stochastic mechanism.
- **Chaos**: a completely deterministic rule produces long-term unpredictable behavior through sensitive dependence on initial conditions.

Randomness

- No determinism
- Each step governed by probability
- Unpredictable sample paths
- "No rule $\Rightarrow$ unpredictability"

Chaos

- Fully deterministic
- $x_{n+1} = f(x_n)$
- Tiny initial error grows exponentially
- "A rule $\Rightarrow$ unpredictability"

Li-Yorke Theorem

THEOREM 1. *Let J be an interval and let $F: J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfy*

$$d \leqq a < b < c \text{ (or } d \geqq a > b > c\text{)}.$$

Then

T1: *for every $k = 1, 2, \cdots$ there is a periodic point in J having period k.*

Furthermore,

T2: *there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:*

(A) *For every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0 \tag{2.1}$$

and

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0. \tag{2.2}$$

(B) *For every $p \in S$ and periodic point $q \in J$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

Figure: Period 3 implies chaos.

Interpretation

(A) Pairwise chaotic behavior

For every $p, q \in S$ with $p \neq q$,
They separate repeatedly:

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0, \tag{1}$$

They get arbitrarily close:

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0. \tag{2}$$

Together: the orbits of p and q are **neither convergent nor divergent**. They are **infinitely close and infinitely far apart**.

(B) Separation from periodic points

For every $p \in S$ and any periodic point $q \in J$,

$$\limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0. \tag{3}$$

so chaotic orbits never shadow periodic ones. This indicates **true long-term unpredictability**.

UNCOUNTABLE

S is as large as the continuum. Chaos is not exceptional: it occurs on a **massive set**.

I think the introduction should stop here.

A simple, fully deterministic function can generate long-term behavior that is fundamentally unpredictable.

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A simple, fully deterministic function can generate long-term behavior that is fundamentally unpredictable.

The universe is delightful, for everything is chaotic.


Q
&
A