Differential Equations Tutorial 9

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Week 14

Contents

- Recall
- Heaviside
- Dirac Delta
- Convolutions

Recall

- State free solution, Input free solution
- Shift properties and their inverses
- $$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = sY(s) - y(0)$$

  $$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$$
- **Note:** The initial values $y(0)$ and $y'(0)$ are already embedded in the Laplace transform formulas for derivatives.

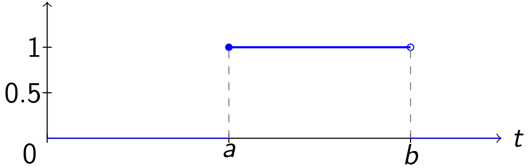
Heaviside Function

Consider:

$$H_{ab}(t) = H_a(t) - H_b(t) = \begin{cases} 0, & \text{for } t < a, \\ 1, & \text{for } a \leq t < b, \\ 0, & \text{for } t \geq b. \end{cases}$$

where we assume that $a < b$. We call $H_{ab}(t)$ the **interval Heaviside function**.

$$H_a(t) - H_b(t) = H(t-a) - H(t-b)$$



Find Laplace Transform with H(t)

Example

Find the Laplace transform of the function

$$f(t) = H\left(t - \frac{\pi}{6}\right)\sin(2t)$$

$$\mathcal{L}\{H(t-a)\cdot f(t-a)\} = e^{-as}\cdot\mathcal{L}\{f(t)\}$$

Which is EQUIVALENT to,

$$\mathcal{L}\{H(t-a)\cdot f(t)\} = e^{-as}\cdot\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}\left\{H\left(t-\frac{\pi}{6}\right)\sin(2t)\right\} = e^{-\frac{\pi}{6}s}\cdot\frac{1+\frac{\sqrt{3}}{2}s}{s^2+4}$$

Inverse Transform - Second Shift Property

Example

Find the inverse Laplace transform of the function

$$\frac{e^{-2s}}{s(s^2+9)}.$$

Solution

$$F(s) = \frac{1}{s(s^2+9)}.$$

$$F(s) = \frac{1}{9}\left(\frac{1}{s} - \frac{s}{s^2+9}\right).$$

$$f(t) = \mathcal{L}^{-1}\{F\} = \frac{1}{9}\left(1 - \cos 3t\right).$$

$$\to \qquad ?$$

Example

Solve $y' + y = f(t), \quad y(0) = 5$, where

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

SOLUTION The function f can be written as

$$f(t) = 3\cos t \cdot H(t - \pi)$$

We have

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 3\mathcal{L}\{\cos t \cdot H(t - \pi)\}$$

$$sY(s) - y(0) + Y(s) = -3 \cdot \frac{s}{s^2+1} e^{-\pi s}$$

$$(s+1)Y(s) = 5 - \frac{3s}{s^2+1} e^{-\pi s}$$

Piecewise Forcing Term & IVP

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} = e^{-(t-\pi)}H(t-\pi),$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} = \sin(t-\pi)H(t-\pi),$$

and

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\} = \cos(t-\pi)H(t-\pi).$$

Piecewise Forcing Term & IVP

Thus the inverse is:

$$
\begin{aligned}
y(t) = 5e^{-t} &+ \frac{3}{2}e^{-(t-\pi)}H(t-\pi) \\
&- \frac{3}{2}\sin(t-\pi)H(t-\pi) - \frac{3}{2}\cos(t-\pi)H(t-\pi)
\end{aligned}
$$

$$
= \begin{cases} 5e^{-t}, & 0 \leq t < \pi \\ 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\sin t + \frac{3}{2}\cos t, & t \geq \pi \end{cases}
$$

Derivatives of a Transform

Example

Evaluate $\mathcal{L}\{t \sin kt\}$.

Solution With $f(t) = \sin kt$, $F(s) = \frac{k}{s^2+k^2}$, and $n = 1$, Theorem gives

$$\mathcal{L}\{t \sin kt\} = -\frac{d}{ds}\mathcal{L}\{\sin kt\} = -\frac{d}{ds}\left(\frac{k}{s^2+k^2}\right) = \frac{2ks}{(s^2+k^2)^2}.$$

Dirac Delta

- $$\int_{-\infty}^{\infty} \delta(t-p)f(t)\,dt = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \varphi^{\epsilon}(t-p)f(t)\,dt$$
- $$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{p}^{p+\epsilon} f(t)\,dt \overset{t=p+\epsilon\tau}{=}$$
- $$\lim_{\epsilon\to 0}\int_{0}^{1} f(p+\epsilon\tau)\,d\tau = f(p)\int_{0}^{1} d\tau = f(p)$$

  as ϵ is small.

Convolutions

- $$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\} = F(s)G(s),$$
- $$\int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau;$$
- Example:

  Solve $f(t) = 3t^2 - e^{-t} - \displaystyle\int_0^t f(\tau)e^{t-\tau}\, d\tau$ for $f(t)$.
- $\displaystyle\int_0^t f(\tau)e^{t-\tau}\, d\tau \quad f = ?, \quad g = ?$

Conclusion

Review – 2nd Order ODE System $ay'' + by' + cy = f(t)$

Method 1: The Undetermined Coefficient Method

$$y = y_p + y_h = y_p + C_1 y_1 + C_2 y_2$$

Structure of Solutions for an Inhomogeneous Linear DE

Method 2: The Method of Variation of Parameters

$$y_p = v_1 y_1 + v_2 y_2$$

$$y = y_h + y_p = C_1 y_1 + C_2 y_2 + v_1 y_1 + v_2 y_2$$

Method 3: (IVP) Laplace Transform with $y(0) = y_0$, and $y'(0) = y_1$

$$Y(s) = Y_s(s) + Y_i(s) = \frac{F(s)}{as^2 + bs + c} + \frac{(as+b)y_0 + ay_1}{as^2 + bs + c}$$

Method 4: (IVP) Convolution with $y(0) = y_0$, and $y'(0) = y_1$

$$y_s(t) = \boxed{e} * f(t)$$

$$y_i(t) = ay_0 e'(t) + (ay_1 + by_0)e(t)$$

Figure: Solving 2nd DEs

Quiz 9-a

$$\text{Evaluate } \mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}.$$

Hint: Using the property of convolutions.

Quiz 9-b

Find the Laplace transform of the periodic function shown in the figure.

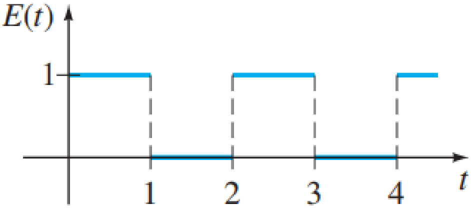


Figure: Periodic Function $E(t)$

Hint:

$$\sum_{n=0}^{\infty} e^{-snT} = \frac{1}{1 - e^{-sT}}$$