Differential Equations Tutorial 7

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Contents

Recall

- Wronskin
- Homogeneous Second Order
- Inhomogeneous Second Order

The Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \text{for } s > 0.$$

- Changes a function in terms of time t to frequency s.
- The resulting expression still depends on the variable s and only on s.
- Integrate from 0 to ∞
- Require $s > 0$ (for convergence)

Find Laplace Transform

$$\mathcal{L}\{1\} \to \mathcal{L}\{e^{at}\}$$
$$\mathcal{L}\{f(t)\} \to \mathcal{L}\{e^{at} \cdot f(t)\}$$

- Frequency Shift Law
- $$\mathcal{L}\{e^{at} \cdot f(t)\} = F(s - a)$$
- When $s > a$

Find Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \cos(at)\, e^{-st}\, dt = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\{g(t)\} = G(s) = \int_0^\infty \sin(at)\, e^{-st}\, dt = \frac{a}{s^2 + a^2}$$

Method I: Integral by Parts
Method II: Complex Method

Find Laplace Transform - Piecewise Continuous

- Continuous $\rightarrow$ Piecewise Continuous
- We say that a function f is piecewise continuous on an interval $[a, b]$ if f ?
- **Unit step function** (Heaviside function).

$$u(t-a) = \begin{cases} 0, & \text{if } t < a, \\ 1, & \text{if } t \geq a. \end{cases}$$

When $a = 0$, the function becomes:

$$u(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases}$$

Transforms of Some Elementary Functions

The Laplace Transform I

Introduction to the Laplace Transform

Let's summarize transforms of some elementary functions as follows.

$f(t) =$	$\mathcal{L}\{f(t)\} =$	$f(t) =$	$\mathcal{L}\{f(t)\} =$
C	$\frac{C}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh(\omega t) = \frac{e^t - e^{-t}}{2}$	$\frac{\omega}{s^2-\omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\cosh(\omega t) = \frac{e^t + e^{-t}}{2}$	$\frac{s}{s^2-\omega^2}$

Figure: Common Transforms

Linearity, Existence and Uniqueness

- Linearity $\leftarrow$ Integral is linear
- the Laplace transform of a product is not the product of the transforms. $\leftarrow$ Convolution
- Existence: If of exponential order
  Counterexample: e^{t^2}
- Uniqueness: *Let $f(t)$ and $g(t)$ be continuous and of exponential order.*
  Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

Find Inverse Laplace Transform

- If f is a continuous function of exponential order and $\mathcal{L}(f) = F(s)$, then we call f the inverse Laplace transform of F, and write
$$f = \mathcal{L}^{-1}(F).$$
- Also, Linear
- First Shift Property
- Examples Find the inverse Laplace transform of:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\}$$

Find Inverse Laplace Transform

- $$\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{-1}{3(s+2)} + \frac{1}{3(s-1)}$$
- $$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\} = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$
- Find
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$$
- $$s^2+2s+2 = (s+1)^2+1 \Rightarrow \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$$
- $$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = e^{-t}\sin(t)$$

More Examples

- $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$
- $$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$
- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$
- $$\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin(\sqrt{7}t).$$

More Examples

EXAMPLE 2 **Termwise Division and Linearity**

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

SOLUTION We first rewrite the given function of s as two expressions by means of termwise division and then use (1):

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} \overset{\text{termwise division}}{=} \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\} \overset{\text{linearity and fixing up constants}}{=} -2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \tag{2}$$

$$= -2\cos 2t + 3\sin 2t. \quad \leftarrow \text{parts (e) and (d) of Theorem 7.2.1 with } k = 2$$

Figure: Example - Termwise

More Examples

EXAMPLE 3 Partial Fractions: Distinct Linear Factors

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$.

SOLUTION There exist unique real constants A, B, and C so that

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$
$$= \frac{A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)}{(s-1)(s-2)(s+4)}.$$

Since the denominators are identical, the numerators are identical:

$$s^2 + 6s + 9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2). \quad (3)$$

By comparing coefficients of powers of s on both sides of the equality, we know that (3) is equivalent to a system of three equations in the three unknowns A, B, and C. However, there is a shortcut for determining these unknowns. If we set $s = 1$, $s = 2$, and $s = -4$ in (3), we obtain, respectively,

$$16 = A(-1)(5), \qquad 25 = B(1)(6), \qquad \text{and} \qquad 1 = C(-5)(-6),$$

and so $A = -\frac{16}{5}$, $B = \frac{25}{6}$, and $C = \frac{1}{30}$. Hence the partial fraction decomposition is

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}, \quad (4)$$

TODO

- Properties about Derivatives. $\leftarrow$ DEFINITION
- Get familiar with the Tables.