Differential Equations Tutorial 1

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Week 3

Contents

Recall

- What is a **Differential Equation**?
- Classification by ?
- What is the normal form of equation of order n?

$$y^{(n)} = f\left(t, y, y', \dots, y^{(n-1)}\right)$$

- What is **Interval of Existence**?
  How is it different from "domain"?
- What is Implicit/Explicit solution?
- What is Direction Field?
  "flow of solutions" "drawing of any particular solution"

Recall

- Integral Techniuqes
- Partial Fraction

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

- Substitution

$$e.g. \int xe^{x^2}\,dx$$

Trigonometric Substitution

$$e.g. \int \frac{1}{\sqrt{1-x^2}}\,dx$$

- Integration by Parts

$$e.g. \int xe^{x}\,dx$$

Vertification of a Solution

Example

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$

$$y'' - 2y' + y = 0; \quad y = xe^x$$

Solution

From the derivatives $y' = xe^x + e^x$ *and* $y'' = xe^x + 2e^x$, *substituting which into the given equation leads to*

$$y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0.$$

Hence, the given form is indeed a solution of the given equation.

Trivial Solution

A solution of a differential equation that is identically zero on an interval I, namely, the zero solution y = 0 is said to be a trivial solution.

Example

$$y' = ty^2$$

When $y \neq 0$, solve by **separate the variables**
When $y = 0$, ?
Remark: Avoiding division by zero

Integrals as Solutions

Let us first assume that f is a function of x , hence the equation is

$$y' = f(x)$$

Integrate both sides with respect to x.

$$\int y'(x)\,dx + C_1 = \int f(x)\,dx + C_2,$$

$$y(x) = \int f(x)\,dx + C_2 - C_1.$$

$\underline{y(x) \text{ is the general solution.}}$

Integrals as Solutions

Denoting $C = C_2 - C_1$ gives us the solution in the form of

$$y(x) = \int f(x)\,dx + C. \tag{1}$$

Note that C1 and C2 are arbitrary constant, for our convenience, normally when we solve an ODE, we only add the constant on the right-hand side of an equation.

Example

Find the general solution of $y' = 3x^2$.

Solution

Integrating both sides gives

$$\int y'(x)\,dx = \int 3x^2\,dx + C,$$

that is,

$$y(x) = x^3 + C.$$

Question

Find a **particular solution** of the equation.
(Completely specified,free of arbitrary parameters)

Initial Value Problem

- IVP
- Initial Condition
- **Example**
  Suppose a car drives at a speed $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?
- The DE is given by ?
  Let y denote the distance the car traveled.
  $$y' = e^{t/2}.$$
- Initial Condition $y(0) = 0$

Example

- $y(t) = \quad ?$
- $C = \quad ?$
- $y(2) = \quad ? \qquad ,y(10) = \quad ?$
- Recall FTC

$$y(t) = \int_{t_0}^{t} f(s)ds + y_0 \tag{2}$$

$$y(t_0) = \int_{t_0}^{t_0} f(s)ds + y_0 = y_0,$$

which means y_0 is the value of the given initial condition.

Example

- $$\begin{aligned} y(t) &= \int_0^t e^{s/2}\, ds + 0 \\ &= 2e^{s/2}\Big|_0^t = 2e^{t/2} - 2, \end{aligned}$$
- *Which is exactly identical with the solution obtained using the indefinite integration.*

Separable Equations

Example

Consider the separable equation

$$y' = \frac{xy}{y^2+1}.$$

Solution

When $y \neq 0$, *we separate variables and find*

$$\frac{y^2+1}{y}dy = \left(y + \frac{1}{y}\right)dy = xdx.$$

Integrating both sides gives

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C_1$$

Separable Equations

Multiplying the equation via 2

$$y^2 + 2\ln|y| = x^2 + C,$$

where $C = 2C_1$ *.*
Note that the form is an implicit solution and we just leave the form.
Also, you can show that $y = 0$ *is also a solution of the given DE by substituting.*

Typical Models

- Exponential Growth
- Exponential Decay
- Newton's Law of Cooling
- Air Resistance in Falling Mass

Example: Exponential Equations

- $$\frac{\Delta N(t)}{\Delta t} \approx \lambda N(t).$$

- When $\Delta t \to 0$, *i.e.*, we consider the infinitesimal change, we obtain the following first order ordinary differential equation

$$\lim_{\Delta t \to 0} \frac{\Delta N(t)}{\Delta t} = \frac{dN(t)}{dt} = \lambda N(t).$$

- The equation

$$N'(t) = \lambda N(t) \tag{3}$$

is called the **exponential equation**. When $\lambda > 0$, the equation is the exponential growth equation; when $\lambda < 0$, the equation represents the exponential decay.

Example: Exponential Growth

- Population Growth: The Malthusian Model
- Introduced later in Chapter 3. Modeling and Applications
- Without limit,

$$P(t) = P_0 e^{rt}.$$

- With limit(carrying capacity)

$$\dot{P} = rP\left(1 - \frac{P}{K}\right).$$

Example: Exponential Decay

$$\frac{dP}{dt} = -\lambda P + r - rP \tag{4}$$

$$\therefore \frac{dP}{dt} = -(\lambda + r)P + r$$

$$\frac{dP}{dt} = -(\lambda + r)\left(P - \frac{r}{r+\lambda}\right)$$

Let $Q = P - \dfrac{r}{r+\lambda}$

$$\frac{dQ}{dt} = -(\lambda + r)Q$$

$$\frac{dQ}{Q} = -(\lambda + r)dt$$

$$\int \frac{dQ}{Q} = \int -(\lambda + r)dt$$

Example: Exponential Decay

$$\ln|Q| = -(\lambda + r)t + C$$

$$Q = Ae^{-(\lambda+r)t}$$

$$P(t) = \frac{r}{r+\lambda} + Ae^{-(\lambda+r)t}$$

Initial condition indicates $t = 0, P(0) = 1$

$$\therefore A = 1 - \frac{r}{r+\lambda} = \frac{\lambda}{r+\lambda}$$

$$P(t) = \frac{r}{r+\lambda} + \frac{\lambda}{r+\lambda}e^{-(\lambda+r)t}$$

Example: Air Resistance

$$\frac{d}{dt}v = g - \frac{k}{m}v^2, \quad \text{with } v(0) = 0. \tag{5}$$

Rewriting & Integrating:

$$\int \frac{1}{g - \frac{k}{m}v^2}\,dv = \int dt.$$

Let:

$$1 - \frac{k}{mg}v^2 = (1 - \alpha v)(1 + \alpha v).$$

$$\alpha^2 = \frac{k}{mg} \quad \Rightarrow \quad \alpha = \sqrt{\frac{k}{mg}}.$$

$$\int \frac{dv}{g(1 - \alpha v)(1 + \alpha v)} = \int dt.$$

By Partial Fraction

$$\frac{1}{(1-\alpha v)(1+\alpha v)} = \frac{1}{2}\left(\frac{1}{1-\alpha v} + \frac{1}{1+\alpha v}\right).$$

$$\int \frac{1}{g}\cdot\frac{1}{2}\left(\frac{1}{1-\alpha v} + \frac{1}{1+\alpha v}\right) dv = \int dt.$$

$$\frac{1}{2g}\int \left(\frac{1}{1-\alpha v} + \frac{1}{1+\alpha v}\right) dv = \int dt.$$

$$\frac{1}{2g\alpha}\left(-\ln|1-\alpha v| + \ln|1+\alpha v|\right) = t + C.$$

$$\left|\frac{1+\alpha v}{1-\alpha v}\right| = e^{2g\alpha t + C'}.$$

Setting $A = e^{C'}$, we obtain:

$$\frac{1+\alpha v}{1-\alpha v} = Ae^{2g\alpha t}.$$

Example: Air Resistance

$$v = \frac{Ae^{2g\alpha t} - 1}{\alpha(Ae^{2g\alpha t} + 1)}.$$

Substituting $\alpha = \sqrt{k/mg}$:

$$v = \sqrt{\frac{mg}{k}} \frac{Ae^{2t\sqrt{kg/m}} - 1}{Ae^{2t\sqrt{kg/m}} + 1}.$$

Apply initial condition $v(0) = 0$

$$0 = \sqrt{\frac{mg}{k}} \frac{A - 1}{A + 1}.$$

Since $A = 1$,

$$v = \sqrt{\frac{mg}{k}} \frac{e^{2t\sqrt{kg/m}} - 1}{e^{2t\sqrt{kg/m}} + 1}. \tag{6}$$

Quiz 1-a

Solve the problem

$$y' + 2y^2 = 0, \quad y(1) = y_0$$

and determine how the interval of existence depends on the initial value y_0.

Quiz 1-b

Solve the problem

$$y' + y^3 = 0, \quad y(0) = y_0$$

and determine how the interval of existence depends on the initial value y_0.