Green's functions summary (L10 recap)

- **Final version of δ-problem for $G(x,s)$** on $a \le x \le b$

$$\boxed{\mathbf{L}_x G(x,s) = \delta(x-s) \qquad BC_1 G(a,s) = 0 \qquad BC_2 G(b,s) = 0}$$

- Piecewise defined G: $G(x,s) = \begin{cases} G_-(x,s) & \boxed{a \le x < s} \le b \\ G_+(x,s) & a \le \boxed{s < x \le b} \end{cases}$

- ($\mathbf{L}_x G_-(x) = 0$ with $x = a$ hom. BC's) and ($\mathbf{L}_x G_+(x) = 0$ with $x = b$ hom. BC's)

- Two solns (G_-, G_+) of n^{th} order ODE, but only n BC's, so there will be n constants to still pin down – Jump Conditions at $x = s$ give n eqns for those!

- If $\mathbf{L}u = A_n(x)\frac{d^n u}{dx^n} + A_{n-1}(x)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + A_2(x)\frac{d^2 u}{dx^2} + A_1(x)\frac{du}{dx} + A_0(x)u$ then Jump in $(n-1)$-th d/dx derivative of $G_\pm(x)$ evaluated at $x = s$:

$$\boxed{A_n(s)[G_+^{(n-1)}(s) - G_-^{(n-1)}(s)] = 1}$$

and all lower derivatives are continuous at $x = s$:

$$\boxed{[G_+^{(n-k)}(s) - G_-^{(n-k)}(s)] = 0 \qquad \text{for } k = 2, 3, \cdots, n}$$

For SL ($n = 2$): $p(s)[G'_+(s) - G'_-(s)] = 1$ and $G_+(s) - G_-(s) = 0$

Math 551 Part III: Separation of Variables for solving linear PDE's

Motivation: Why solve PDE's?

- Evolution-in-time and/or structure-in-space for many probs described by PDE's
- In some problems need whole solution (medical imaging, signal processing), other probs only need some key properties (max amplitude, stability, net flux, ...)
- If there were a short-cut, you'd use that, but in general need to solve the PDE and understand how to get an accurate/reliable answer (Gibbs trouble?)

The three fundamental classes of 2nd-order PDE's (see Haberman Ch 1, 2, 4)[a]

1. **Parabolic**: "diffusive spreading" $u(x,t)$
   Classic PDE: the heat equation
   $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + S(x,t)$$

2. **Hyperbolic**: "wave propagation/vibration modes"
   Classic PDE: the wave equation
   $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + S(x,t)$$

3. **Elliptic**: "equilibrium states" $u(x,y)$
   Classic PDE: Laplace's eqn/Poisson's eqn
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = S(x,y)$$

[a] with added inhomogeneous forcing terms, often called "$\boldsymbol{S}$ource terms"

Well-posed problems: PDE with the IC's and BC's needed to select a unique soln

The classes of side-conditions (see BC's in Haberman, page 156)

0. **Initial Conditions** (IC): sets the starting values of properties for u at $t = 0$

   On $0 \le x \le 1$: $$u(x, t = 0) = f(x) \qquad \text{or} \qquad u_t(x, t = 0) = v(x)$$

1. **Dirichlet** (BC): sets the value of the solution on the boundary

   $$u(x = 0, t) = A \qquad \text{for } t \ge 0$$

   More generally, the value can be a function of other variables,

   $$u(x = 0, t) = A(t) \qquad \text{or} \qquad u(x, y = 0) = B(x)$$

2. **Neumann** (BC): sets value of derivative (∂ in $\perp$ dir. thru the boundary) ("flux")

   $$\frac{\partial u}{\partial n} \equiv \hat{\mathrm{n}} \cdot \nabla u \qquad \Longrightarrow \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = B \qquad \text{or} \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0} = C$$

   and more generally $u_x(x = 0, t) = B(t)$ or $u_y(x, y = 0) = C(x)$

3. **Robin** (BC): sets a linear combination of value and derivative at boundary

   $$\text{at } x = 0: \qquad \frac{\partial u}{\partial x} + Eu = F$$

Example 1: IBVProblem for the heat equation for $u(x,t)$ on $0 \le x \le 1$ with $t \ge 0$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \underbrace{u(x=0)=0, \quad u(x=1)=0}_{\text{BC}_\ell \quad \text{and} \quad \text{BC}_r} \qquad \underbrace{u(t=0)=f(x)}_{\text{IC}}$$

$\boxed{\textbf{The Separation of Variables solution process (v1.0)}}$: for Homogeneous BC's

Goal: To write the solution of the full problem via linear superposition as

$$u(x,t) = \sum_k a_k(t)\phi_k(x)$$

where $\phi_k(x)$ are orthogonal[a] functions to be determined

$\boxed{1.}$ Divide and Conquer: Construct the solution by requiring that each product term, $u_k(x,t) = a_k(t)\phi_k(x)$, is separately a nontrivial solution of the homogenized[b] version of the boundary value problem

$$\frac{\partial u_k}{\partial t} = \frac{\partial^2 u_k}{\partial x^2} \qquad u_k(x=0)=0, \quad u_k(x=1)=0 \qquad \underbrace{u_k(t=0)=c_k\phi_k(x)}_{???(IOU)}$$

[a] Either bi-orthogonal $\langle \psi_j, \phi_k \rangle_2$ or self-orthogonal $\langle \phi_j, \phi_k \rangle_\sigma$

[b] Zero all inhomogeneous terms in the PDE and the BC's, but IC's are treated differently

Why are IC's different?

- BVP (Boundary Value Probs) for linear homogeneous eqns with zero BC's can have eigenfunctions $\phi_k(x)$ as nontrivial solutions [oscillatory solutions]
- IVP (Initial Value Probs) for linear homogeneous eqns with all zero IC's produce only the trivial soln $a_k(t) \equiv 0$ [zero at $t = 0$ stays dead for all $t > 0$] (Not useful)

$\boxed{2.}$ Substitute-in non-trivial $u_k(x,t) = a_k(t)\phi_k(x)$ into all parts of the problem

$$\text{PDE}: \qquad \frac{da_k}{dt}\phi_k = a_k \frac{d^2\phi_k}{dx^2}$$

$$\text{BC}_\ell(x = a): \qquad a_k(t)\phi_k(0) = 0$$

$$\text{BC}_r(x = b): \qquad a_k(t)\phi_k(1) = 0$$

$$\text{IC}(t = 0): \qquad a_k(0)\phi_k(x) = c_k\phi_k(x)$$

BC's must be true for all $t \geq 0 \implies \phi_k(0) = 0 \quad \phi_k(1) = 0$

IC must be true for all $0 \leq x \leq 1 \implies a_k(0) = c_k$

PDE must be true for all x and $t \implies$ $\boxed{\text{Factor/Separate the } x, t\text{-dependencies}}$

$$\underbrace{\frac{1}{a_k(t)}\frac{da_k}{dt}}_{t\text{'s only}} = \underbrace{\frac{1}{\phi_k(x)}\frac{d^2\phi_k}{dx^2}}_{x\text{'s only}} \qquad \text{“Separated form”}$$

The only way for each $\text{LHS}_k(t) = \text{RHS}_k(x)$ for all values of (x, t) is if

$$\boxed{\text{LHS}_k(t) = \text{RHS}_k(x) = s_k} = \text{constant}$$

$\boxed{s_k \text{ is a ``separation constant''}}$: unknown constant for each k, to be determined

$\boxed{3.}$ Separated t-problem: ODE IVP (start with easier part [LHS or RHS] first)

$$\frac{1}{a_k}\frac{da_k}{dt} = s_k \quad \xrightarrow[IVP]{ODE} \quad \left\{\begin{aligned} &\frac{da_k}{dt} = s_k a_k \\ &a_k(0) = c_k \end{aligned}\right\} \quad \xrightarrow{LCC} \quad \boxed{a_k(t) = c_k e^{s_k t}}$$

Soln of IVP works with any s_k, values not determined yet. Return later, move on...

$\boxed{4.}$ Separated x-problem: ODE BVP

$$\frac{1}{\phi_k}\frac{d^2\phi_k}{dx^2} = s_k \quad \xrightarrow[BVP]{ODE} \quad \left\{\begin{array}{ll} \phi_k'' = s_k\phi_k & 0 \le x \le 1 \\ \phi_k(0) = 0 & \phi_k(1) = 0 \end{array}\right\}$$

To match-up with eigenvalue BVP problems, relabel $s_k = -\lambda_k$:

$$\phi_k'' = -\lambda_k\phi_k \qquad \phi_k(0) = 0 \qquad \phi_k(1) = 0$$

$$\xrightarrow[SL]{LCC} \qquad \boxed{\phi_k(x) = \sin(k\pi x) \qquad s_k = -\lambda_k = -k^2\pi^2 \qquad k = 1, 2, 3\ldots}$$

Self-adjoint SL prob, so: completeness, orthogonality, eigenvalue results $(\lambda \ge 0)$...

$\boxed{5.}$ Everything is pinned down in $\phi_k(x), a_k(t)$ except c_k's and the problem's IC.
Final solution via linear combination of u_k solns

$$u(x,t) = \sum_k u_k(x,t) \qquad \boxed{u(x,t) = \sum_{k=1}^{\infty} c_k e^{-k^2\pi^2 t}\sin(k\pi x)}$$

Pick c_k's to match IC, $u(x,0) = f(x)$:

$$u(t=0) = \sum_{k=1}^{\infty} c_k \sin(k\pi x) = f(x) \qquad \boxed{c_k = \frac{\langle f, \phi_k\rangle_\sigma}{\langle \phi_k, \phi_k\rangle_\sigma}}$$

<u>Overall</u>: PDE solution is an eigenfunction expansion ($\phi_k(x)$ from the x-BVP) with coefficients that depend on the IC and the solution of the t-IVP ■

<u>Notes</u>:

- Separation of variables works for all basic problems for the three classes of linear PDE's (summary next)
- WARNING: There are problems where separation of variables does not work [shown later today]. These kinds of problems will not be covered in Math 551. (need numerics)

Separation of variables (SV) overviews for the fundamental classes of PDE's:

PDE solution obtained from product of separated ODE problems

1. **Parabolic**: the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad a \le x \le b \qquad t \ge 0$$

   SV problems: (x-BVP, two BC's [1 on each end])$\times$(t-IVP, one IC at $t = 0$)

2. **Hyperbolic**: the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad a \le x \le b \qquad t \ge 0$$

   SV problems: (x-BVP, two BC's [1 on each end])$\times$(t-IVP, two IC's at $t = 0$)

3. **Elliptic**: Laplace's equation (two approaches, IOU)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad a \le x \le b \qquad c \le y \le d$$

   SV problems: (x-BVP, two BC's [1 on each end])$\times$(y-BVP, two BC's [1 on each end])

From Haberman, Chapter 2, pages 50–51:

2.3.8 Summary

Let us summarize the method of separation of variables as it appears for the one example:

$$\begin{aligned}
\text{PDE:}\quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\
\text{BC:}\quad & u(0,t) = 0 \\
& u(L,t) = 0 \\
\text{IC:}\quad & u(x,0) = f(x).
\end{aligned}$$

1. Make sure that you have a linear and homogeneous PDE with linear and homogeneous BC.
2. Temporarily ignore the nonzero IC.
3. Separate variables (determine differential equations implied by the assumption of product solutions) and introduce a separation constant.
4. Determine separation constants as the eigenvalues of a boundary value problem.
5. Solve other differential equations. Record all product solutions of the PDE obtainable by this method.
6. Apply the principle of superposition (for a linear combination of all product solutions).
7. Attempt to satisfy the initial condition.
8. Determine coefficients using the orthogonality of the eigenfunctions.

These steps should be *understood*, not memorized. It is important to note that

1. The principle of superposition applies to solutions of the PDE (do not add up solutions of various different ordinary differential equations).
2. Do not apply the initial condition $u(x,0) = f(x)$ until *after* the principle of superposition.

$\boxed{\textbf{Separation of Variables (v2.0)}}$: **Solving Inhom PDE IBVP** $(+S$, inhom BC's)

1. First, determine the eigen-expansion (like v1.0): **[Like L5, L6 for ODE's]**
   (a) **Homogenize** the PDE equation and the BC's.
   (b) Plug-in the trial solution $u_k(x,t) = a_k(t)\phi_k(x)$ and separate variables.
   (c) Math 551 PDE problems $\to$ Sturm-Liouville probs: $\widetilde{\mathbf{L}}\phi_k = -\lambda_k \sigma \phi_k$.
   Identify the p, q, σ and use the general soln or known eigenfcns.
   (d) Obtain the eigenfunctions of the x-BVP's: $\boxed{\{\phi_k(x), \lambda_k\}}$ **Stop.**[a]

2. Return to full problem and use the $\phi_k(x)$'s to construct the soln in the form

$$\boxed{u(x,t) = \sum_k b_k(t)\phi_k(x)} \qquad b_k(t) = \frac{\langle u(x,t), \phi_k\rangle_\sigma}{\langle \phi_k, \phi_k\rangle_\sigma}$$

Determine eqns for the $b_k(t)$ coefficient functions (t-ODE IVP):

- For each k, do the inner product/orthogonal projection of the problem

$$\boxed{\langle \text{PDE}, \phi_k\rangle_2 \qquad \text{and} \qquad \langle \text{IC's}, \phi_k\rangle_\sigma}$$

- IC's for u will produce $b_k(0)$ IC values.
- BC's and inhomogeneous source terms in the PDE will produce inhomogeneous forcing in the the ODE for $b_k(t)$.

---

[a] The $a_k(t)$ hom solns from SV won't include needed inhomogeneous effects from $(+S$, BC's).

Some useful results for efficient solution of separation of variables problems

1. **Green's formula**: for Sturm-Liouville $\widetilde{\mathbf{L}}u \equiv \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u$, then $\langle v, \widetilde{\mathbf{L}}u\rangle = \langle u, \widetilde{\mathbf{L}}v\rangle +$ (Boundary terms) is

$$\boxed{\int_a^b v\widetilde{\mathbf{L}}u\,dx = p(x)\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\Big|_a^b + \int_a^b u\widetilde{\mathbf{L}}v\,dx}$$

See Haberman (5.5.8) page 169 or (8.4.11) page 355. (IBP^2 for SL $\widetilde{\mathbf{L}}$-operators)

2. **Integrating factor approach** for inhomogeneous ODE IVP's[a]

$$\frac{db}{dt} + rb = f(t) \qquad b(0) = b_0$$

Homogeneous solution $b_h(t) = e^{-rt}$ (soln of $db_h/dt + rb_h = 0$)

Like Variation of Parameters: plug-in $b(t) = c(t)b_h(t)$

$$\frac{dc}{dt}e^{-rt} - rce^{-rt} + rce^{-rt} = f(t) \qquad \rightarrow \qquad \frac{dc}{dt} = f(t)e^{rt}$$

$$c(t) = \int_0^t f(s)e^{rs}\,ds + C \quad \rightarrow \quad \boxed{b(t) = \left(\int_0^t f(s)e^{rs}\,ds + b_0\right)e^{-rt}}$$

[a] Eigen-expansions only work for BVP's, not IVP's!

More useful results: Eigensolutions for SL problem with $p \equiv 1, q \equiv 0, \sigma \equiv 1$
My version of Haberman's very useful table from page 65

Solutions of $\boxed{\phi_k''(x) = -\lambda_k \phi_k(x) \quad \text{on} \quad 0 \le x \le 1}$

$$\phi_{\text{gen}}(x) = A \sin(\sqrt{\lambda}\, x) + B \cos(\sqrt{\lambda}\, x)$$

Boundary Conditions	$\phi_k(0) = 0$, $\phi_k(1) = 0$, Dir./Dir.	$\phi_k'(0) = 0$, $\phi_k'(1) = 0$, Neu./Neu.	$\phi_k(0) = 0$, $\phi_k'(1) = 0$, Dir./Neu.
Indices $k =$	$1, 2, 3, \cdots$	$0, 1, 2, 3, \cdots$	$0, 1, 2, 3, \cdots$
Eig-vals $\lambda_k =$	$k^2\pi^2$	$k^2\pi^2$	$\left(\frac{2k+1}{2}\right)^2 \pi^2$
Eig-fcns $\phi_k =$	$\sin(k\pi x)$	$\cos(k\pi x)$	$\sin\left(\frac{2k+1}{2}\pi x\right)$

- Interval $0 \le \tilde{x} \le \ell$ $\quad \rightarrow \quad$ Change of var: $x = \tilde{x}/\ell$ and get $\tilde{\lambda} = \lambda/\ell^2$
- To swap BC's at $x = 0$ and $x = 1$: replace $x \rightarrow (x - 1)$ in $\phi_k(x)$.
- Haberman also gives the solutions for periodic BC's (the full Fourier Series).
- Robin boundary conditions for this equation yield problems where the λ_k's must be calculated numerically...

Math 551: linear PDE and BC's and domain must fit together for Sep of Vars to work!

Examples of non-separable (non-551) problems: where Sep of Vars is not possible

- (BC issue) A time-dependent Robin BC for a $u(x,t)$ problem at $x=0$:

$$\frac{\partial u}{\partial x} + E(t)u = F(t)$$

homogenize: $u_x + E(t)u = 0$

substitute-in $u_k(x,t) = a_k(t)\phi_k(x)$ and try to separate variables

$$a_k(t)\left[\phi_k'(0) + D(t)\phi_k(0)\right] = 0$$

- (PDE issue) A time-dependent convection-diffusion equation for $u(x,t)$:

$$u_t + C(t)u_x = u_{xx} + S(x,t)$$

homogenize: $u_t + C(t)u_x = u_{xx}$

substitute-in $u_k(x,t) = a_k(t)\phi_k(x)$ and try to separate variables

$$\frac{1}{a_k(t)}\frac{da_k}{dt} + \frac{C(t)}{\phi_k(x)}\frac{d\phi_k}{dx} = \frac{1}{\phi_k(x)}\frac{d^2\phi_k}{dx^2}$$

- (Domain) Free/Moving boundary probs: time-dependent domain $0 \leq x \leq L(t)$
- Sep of Vars wont work for nonlinear problems (in either PDE or any BC's)