Re-cap of L16: basics of Sep of Vars for multi-dimensional PDE problems

2-D heat eqn, cold hom BC's $u = 0$, hot IC $u = F(x, y)$ problem:

$$u_t = u_{xx} + u_{yy} \qquad 0 \le x \le L \qquad 0 \le y \le H$$

0. Problem is homogeneous, dont need to cut anything, and will be able to use shortcut (no PDE-projection or IBP part needed)
1. SV trial solution $u_k(x, y, t) = f(x)g(y)h(t)$, separate variables
2. First, got $h(t) = ce^{-\lambda t}$ (no info on λ yet)
3. Next, can separate x's from y's, get $f_k(x)$ eig-fcns and $\mu_k = (k\pi/L)^2$
4. Finally, got eig-fcns for $g_m(y)$ and $\gamma_m = (m\pi/H)^2$
5. Then combining separation constants, get $\lambda_{k,m} = \mu_k + \gamma_m$ and overall soln is

$$u(x, y, t) = \sum_k \sum_m c_{k,m} e^{-\lambda_{k,m} t} \sin(k\pi x/L) \sin(m\pi y/H)$$

6. Last step: apply IC at $t = 0$ to pick $c_{k,m}$ coefficients

$$F(x, y) = \sum_k \sum_m c_{k,m} \sin(k\pi x/L) \sin(m\pi y/H)$$

Multi-D expansion coefficients: version 1.0

1-D nested expansions: sum of sums, one variable at a time

$$\begin{aligned}
F(x,y) &= \sum_m \left(\sum_k c_{k,m} \sin(\tfrac{k\pi x}{L}) \right) \sin(\tfrac{m\pi y}{H}) \\
&= \sum_m A_m(x) \sin(\tfrac{m\pi y}{H}) \qquad A_m(x) = \sum_k c_{k,m} \sin(\tfrac{k\pi x}{L}) \\
&= \sum_m \left(\frac{2}{H} \int_0^H F(x,\tilde{y}) \sin(\tfrac{m\pi \tilde{y}}{H}) d\tilde{y} \right) \sin(\tfrac{m\pi y}{H}) \\
A_m(x) &= \sum_k c_{k,m} \sin(\tfrac{k\pi x}{L}) \qquad c_{k,m} = \frac{2}{L} \int_0^L A_m(x) \sin(\tfrac{k\pi x}{L}) dx \\
c_{k,m} &= \frac{2}{L} \int_0^L \left(\frac{2}{H} \int_0^H F(x,y) \sin(\tfrac{m\pi y}{H}) dy \right) \sin(\tfrac{k\pi x}{L}) dx
\end{aligned}$$

$$\boxed{c_{k,m} = \frac{4}{LH} \int_0^L \int_0^H F(x,y) \sin(\tfrac{m\pi y}{H}) \sin(\tfrac{k\pi x}{L}) \, dy dx}$$

For 3-D, it would be like

$$F(x,y,z) = \sum_n \left(\sum_m \sum_k c_{k,m,n} \sin(k\pi x) \sin(m\pi y) \right) \sin(n\pi z)$$

with $A_n(x,y)$ and $B_{mn}(x)$

Multi-D expansion coefficients: version 2.0 (MUCH better/shorter)

Multi-D eigenfcn expansion perspective: no need for slicing/nesting

$$\begin{aligned} F(x,y) &= \sum_m \sum_k c_{k,m} \underbrace{\sin(\tfrac{k\pi x}{L})\sin(\tfrac{m\pi y}{H})} \\ &= \sum_{km} c_{km}\phi_{k,m}(x,y) \end{aligned}$$

usual expansion, now with double integrals...

$$\begin{aligned} c_{km} &= \frac{\langle\langle F(x,y), \phi_{km}\rangle\rangle}{\langle\langle \phi_{km}, \phi_{km}\rangle\rangle} \\ &= \frac{1}{||\phi_{km}||^2}\int_0^L \int_0^H F(x,y)\phi_{km}(x,y)\, dydx \end{aligned}$$

With separation of variables, $\phi_{km}(x,y) = f_k(x)g_m(y)$ then

$$||\phi_{km}||^2 = ||f_k||^2||g_m||^2 = \frac{L}{2}\frac{H}{2}$$

Today's theme: SV can give single-variable ODE probs, but multi-var calc will give us powerful tools and a better handle on the bigger picture....

Overview: basics of Sep of Vars for multi-dimensional PDE problems

1. SV for PDE problems with $\boldsymbol{N}$-independent variables yields solns with $\boldsymbol{N-1}$ separation constants

2. CANNOT make an eigenvalue problem in time: $\boldsymbol{t}$ has only IC's not BC's **[space-time sep const (and dynamics) controlled by spatial eigenmodes]**

3. Sums will only use the indices for spatial modes $(\boldsymbol{f_n, g_m})$

4. Shortcut: If $\boldsymbol{\sigma \equiv 1}$ then Strict (one var at a time) SV $\Longrightarrow$ "simultaneous SV":

   If can write separated PDE $\quad \Longrightarrow \quad \boldsymbol{H(t) = F(x) + G(y) + E(z)}$

   Then can write each term as a sep const, $\boldsymbol{H = \lambda}$, $\boldsymbol{F = \alpha, G = \beta, \cdots}$

5. For directions/variables with homogeneous BC's, pick sep const to make the separated ODE BVP an eigenvalue problem to determine specific $\boldsymbol{\alpha_n}$ eig-values

$$\frac{\boldsymbol{L_x f(x)}}{\boldsymbol{\sigma_x(x) f(x)}} = \boldsymbol{\alpha} \qquad \Longrightarrow \qquad \boldsymbol{L_x f_n = -\lambda_n \sigma_x f_n} \qquad \text{for } \boldsymbol{\alpha = -\lambda}$$

6. $\boldsymbol{u = \sum_k c_k \phi_k}$, $\boldsymbol{c_k}$ const coeffs determined by orth projection of IC's onto $\boldsymbol{\phi_k}$'s

$$\boldsymbol{c_k = \frac{\langle\langle F(x,y), \phi_k \rangle\rangle_\sigma}{\langle\langle \phi_k, \phi_k \rangle\rangle_\sigma} = \frac{1}{||f_n||^2_{\sigma_x} ||g_m||^2_{\sigma_y}} \iint_D F f_n g_m \sigma_x \sigma_y \, dy \, dx}$$

Solving hom wave eqn on LH rectangle with hom Dirichlet BC's (H Ch 7.3)

$$u_{tt} = u_{xx} + u_{yy} \qquad \rightarrow \qquad fgh'' = f''gh + fg''h$$

Separate variables

$$\frac{h''}{h} = \frac{f''}{f} + \frac{g''}{g} \qquad \rightarrow \qquad \frac{d^2h}{dt^2} = \alpha h, \qquad \frac{d^2f}{dx^2} = -\beta f, \qquad \frac{d^2g}{dy^2} = -\gamma g.$$

Solve ODE BVP eigenvalue problems
$f_n(x) = \sin(n\pi x/L)$ with $\beta_n = (n\pi/L)^2$, then
$g_m(y) = \sin(m\pi y/H)$ with $\gamma_m = (m\pi/H)^2 = -\alpha - \beta$
Then get $\alpha_{n,m} = -\beta_n - \gamma_m < 0$. Let $\alpha = -\omega^2$ for

$$h_{nm}(t) = A_{nm}\cos(\omega_{nm}t) + B_{nm}\sin(\omega_{nm}t)$$

So re-assembled full soln is

$$u(x,y,t) = \sum_n \sum_m [A_{nm}\cos(\omega_{nm}t) + B_{nm}\sin(\omega_{nm}t)] f_n(x) g_m(y)$$

Then use IC's

IC: $u(x,y,0) = F(x,y) = \sum_{n,m} A_{nm}\phi_{nm} \qquad \rightarrow \qquad A_{nm} = \dfrac{\langle F, \phi_{nm}\rangle}{LH/4}$

IC: $\partial_t u(x,y,0) = G(x,y) = \sum_{n,m} \omega_{nm} B_{nm}\phi_{nm} \qquad \rightarrow \qquad B_{nm} = \dfrac{\langle G, \phi_{nm}\rangle}{\omega_{nm} LH/4}$

More background for multi-dimensional PDE problems

- A linear homogeneous PDE for $u(x, y, t)$ is separable if substituting

$$u_k(x, y, t) = f(x)g(y)h(t)$$

  separates the PDE into three ODE's for $f(x)$, $g(y)$, and $h(t)$ then the equation(part 1) is separable (formally separable).

- Separable BC's(part 2) depend on having **separable domains**, where each part of the boundary has one variable being constant on each edge.

- The whole problem is a **fully** separable PDE problem if the equation **and** the BC's are separable to give ODE sub-problems with simple BC's for each of the ODE problems.

Example: Solving $u_t = u_{xx} + u_{yy}$

The PDE is formally separable since (with $\alpha = -\beta - \gamma$)

$$\frac{h'}{h} = \frac{f''}{f} + \frac{g''}{g} \qquad \to \qquad \frac{dh}{dt} = \alpha h, \qquad \frac{d^2 f}{dx^2} = -\beta f, \qquad \frac{d^2 g}{dy^2} = -\gamma g.$$

But this soln also depends on the shape of the (x, y) domain of the problem...

Ex: Here we used rectangle $[0 \le x \le L] \times [0 \le y \le H]$ being separable.

Background for multi-dimensional PDE problems (cont)

- Even if full separation is not possible, **partial separation** can still work.
  Can try to separate time from space variables (Haberman Chapter 7.2):

$$u_k(x,y,t) = \phi(x,y)h(t)$$

- Example: Solving $u_t = \nabla^2 u$ becomes

$$\phi(x,y)\frac{dh}{dt} = h(t)\nabla^2\phi(x,y)$$

$$\frac{1}{h(t)}\frac{dh}{dt} = \frac{\nabla^2\phi(x,y)}{\phi(x,y)} = -\lambda$$

  so separation yields the ODE in time t **[space-time sep const λ = PDE eigval]**

$$\frac{dh}{dt} = -\lambda h \qquad \to \qquad h(t) = Ce^{-\lambda t}$$

  and PDE problem: $\boxed{\nabla^2\phi = -\lambda\phi}$ called the **Helmholtz** (eigenvalue) eqn.

Brief intro to Dynamics: If ANY u_k mode present in soln has $h(t)$ that grows in time, then system is **unstable**. If ALL $h(t)$'s decay then sys is **stable**.

If the domain is separable, then solns can be worked out in formulas, but what's always true: linear theory framework still applies even if $\phi(x,y)$ eigenfunctions have to be computed numerically...

Linear operator theory for $\boxed{\mathbf{L}u \equiv \nabla^2 u}$ (Haberman, Chapter 7.4, 7.5)

$\boxed{\text{The Helmholtz eigenvalue PDE } \nabla^2\phi = -\lambda\phi}$

Problems leading to the Helmholtz equation:

- Multi-D version of $\mathbf{L}u \equiv d^2u/dx^2$ operator and $\phi''(x) = -\lambda\phi(x)$ eigen eqn
- Separation of space/time variables $u(\vec{\mathrm{x}}, t) = \phi(\vec{\mathrm{x}})h(t)$ in the heat equation (Haberman 7.2) and wave equation (H 7.2, H 8.5)

$$\partial_t u = \nabla^2 u + S(\vec{\mathrm{x}}, t) \qquad \partial_{tt} u = \nabla^2 u + S(\vec{\mathrm{x}}, t)$$

- Poisson equation (Haberman 8.6)

$$\nabla^2 u = S(\vec{\mathrm{x}})$$

Extension of linear operator theory to 2-D (a brief review of multi-variable calculus)[a]

L^2 inner product (in 2-D) : $$\langle f(x,y), g(x,y)\rangle \equiv \iint_D f(x,y)g(x,y)\,dy\,dx$$

1-D domain: interval $a \le x \le b$ boundary $=$ endpoints: $x = a, x = b$

2-D domain: region D in xy plane boundary $=$ closed curve C (counter-clockwise)

3-D domain: region in xyz space boundary $=$ closed surface S

[a] Also see the Vector Calculus review sheet

Definition of the adjoint operator (adjoint relation) $\boxed{\langle v, \mathbf{L}u\rangle = \langle \mathbf{L}^* v, u\rangle}$

1-D via Integration by Parts and eliminating the boundary terms

$$\int_a^b vu''\,dx = (vu' - v'u)\Big|_a^b + \int_a^b v''u\,dx$$

Q: 2-D equivalent of IBP for $\mathbf{L} = \nabla^2$? A: Derived from the Divergence Theorem

Divergence Theorem: For any smooth vector fcn $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

$$\boxed{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \nabla\cdot\mathbf{F}\,dA} = \iint_D \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dy\,dx$$

where C is the boundary curve around domain D and

$\mathbf{n}$ is the unit-vector $\perp C$ pointing out of D (the unit outer normal)

1. Pick $\mathbf{F_1} = v(x, y)\nabla u(x, y)$, then $\nabla\cdot\mathbf{F_1}$ on the RHS is

$$\nabla\cdot\mathbf{F_1} = \nabla\cdot(v\nabla u) = \underline{\nabla v\cdot\nabla u} + v\nabla^2 u$$

2. Pick $\mathbf{F_2} = u(x, y)\nabla v(x, y)$, then $\nabla\cdot\mathbf{F_2}$ on the RHS is

$$\nabla\cdot\mathbf{F_2} = \nabla\cdot(u\nabla v) = \underline{\nabla u\cdot\nabla v} + u\nabla^2 v$$

3. Pick $\mathbf{F} = \mathbf{F_1} - \mathbf{F_2}$ and plug into Div Thm:

to get what is called "Green's second identity" (Haberman eqn 7.5.7, p 289):

$$\boxed{\oint_C (v\nabla u - u\nabla v)\cdot \mathbf{n}\, ds = \iint_D \left(v\nabla^2 u - u\nabla^2 v\right)\, dA}$$

Re-arrange, IBP-like and match up to adjoint relation for $\mathbf{L} \equiv \nabla^2$:

$$\underbrace{\iint_D v\nabla^2 u\, dA}_{\langle v, \mathbf{L}u\rangle_2} = \underbrace{\oint_C (v\nabla u - u\nabla v)\cdot \mathbf{n}\, ds}_{\text{boundary terms}} + \underbrace{\iint_D u\nabla^2 v\, dA}_{\langle u, \mathbf{L}^* v\rangle_2}$$

Conclusion #1: The Laplacian operator is formally self-adjoint.

Boundary terms: $\mathbf{n}\cdot\nabla f$ is the directional derivative in the normal direction, also called the "Neumann derivative": $\dfrac{\partial f}{\partial n} \equiv \mathbf{n}\cdot\nabla f$, so bdry terms $\boxed{\oint_C \left(v\dfrac{\partial u}{\partial n} - u\dfrac{\partial v}{\partial n}\right)\, ds}$.

Conclusion #2: ∇^2 with hom. Dir./Neu./Rob. BC's is fully self-adjoint on any D.

Sturm-Liouville-type results for $\nabla^2\phi = -\lambda\phi$ on any finite domain D

1. λ's are real
2. λ_k are discrete and $\lambda_k \to \infty$ (multiple roots possible (symmetry of D's shape))
3. $\phi_k(x, y)$ are a complete self-orthogonal basis in weighted inner prod with $\sigma \equiv 1$

$$F(x,y) = \sum_{\text{``}k\text{''}} c_k \phi_k(x,y) \qquad c_k = \frac{\langle F, \phi_k \rangle_\sigma}{||\phi_k||_\sigma^2} \qquad \text{``}k\text{'' multi-index} = (m,n)$$

4. To show $\lambda \geq 0$, take inner product of $\nabla^2\phi = -\lambda\phi$ with ϕ

$$\begin{aligned}
\langle \phi, \nabla^2\phi \rangle &= -\lambda\langle \phi, \phi \rangle \\
\iint_D \phi \nabla^2 \phi \, dA &= -\lambda ||\phi||^2 && \text{product rule} \\
\underbrace{\iint_D \nabla \cdot (\phi \nabla \phi)\, dA}{} - \iint_D \nabla\phi \cdot \nabla\phi \, dA &= -\lambda ||\phi||^2 && uv' = (uv)' - u'v \\
\oint_C \phi\,(\nabla\phi \cdot \mathrm{n})\; ds - ||\nabla\phi||^2 &= -\lambda ||\phi||^2 && \text{Div. Thm.} \\
\boxed{\left(||\nabla\phi||^2 - \oint_C \phi \frac{\partial \phi}{\partial n}\, ds \right) \Big/ ||\phi||^2} &= \lambda && \underline{\textbf{Rayleigh quotient}}
\end{aligned}$$

$\boxed{\lambda \geq 0}$ for Dirichlet ($\phi = 0$) or Neumann ($\partial\phi/\partial n = 0$) BC's (H, Ch 7.6)