Math 551: Applied PDE and Complex Vars — Lecture 12

Separation of Variables method for inhomogeneous PDE problems

1-D version for $u(x,t)$ PDE initial-boundary value problems (IBVP)

The generic two-step solution process:

- Step 1 (short)
  Use Separation of Variables on the **homogenized** version of the problem.
  - SV trial solution: $u(x,t) = a(t)\phi(x)$
  - Find the spatial-expansion basis fcns as eigensolutions: $\{\phi_k(x), \lambda_k\}$[a]
- Step 2 (longer)
  Use orthogonal projection on the full problem.
  - If ϕ_k's are self-orthogonal then write $\langle \text{PDE}, \phi_k\rangle$ for all k's and reduce to ODE problems for the $c_k(t)$ coefficients in the expansion of the solution:

$$u(x,t) = \sum_k c_k(t)\phi_k(x) \qquad c_k = \frac{\langle u, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle}$$

    Goal is to reduce PDE problem to solvable ODE IVP for each $c_k(t)$.

    WARNING: If you have inhom BC's you CANNOT plug the expansion directly into PDE: BC's wont work!

References: Lecture 5 solution process for ODE BVP and Haberman Chap 8.4

[a] And $\psi_k(x)$ or p, q, σ if not in self-adjoint form.

Solving a heat equation problem: an example

Problem statement:

PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 3u + S(x,t)$$

Domain

$$t \geq 0 \qquad 0 \leq x \leq 1$$

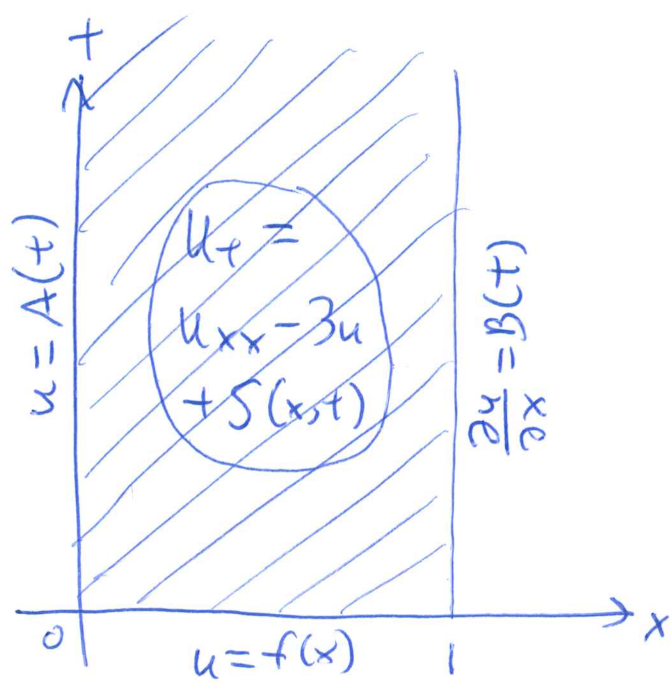
Boundary conditions

$$u(x=0) = A(t) \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = B(t)$$

Initial conditions

$$u(t=0) = f(x)$$

A, B, S, f are given functions



Some useful results

1. From multi-var calc: can interchange order of integrals/derivatives on different independent vars

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \qquad \frac{d}{dx}\left(\int_a^b F(x,y)\,dy\right) = \int_a^b \frac{\partial F}{\partial x}\,dy \qquad \cdots$$

2. From Math 551: For SL $\tilde{\mathbf{L}}w = \frac{d}{dx}\left(p(x)\frac{dw}{dx}\right) + q(x)w$ then for any u, v

$$\int_a^b v\tilde{\mathbf{L}}u\,dx = p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_a^b + \int_a^b u\tilde{\mathbf{L}}v\,dx$$

Important details to follow:

1. When/how are values for separation constants s_k justified and determined?
2. How is each term in the PDE handled?
3. When do the inhom BC's get used?
4. How is the inhom source $S(x,t)$ used?
5. When do the IC's get used?
