Math 551: Applied PDE and Complex Vars — Lecture 10

Part 1: Deriving the δ problem for the piecewise $G(x,s)$
Part 2: Solving ODE BVP with inhomogeneous BC's via the Green's fcn
Part 3: constructing the piecewise G from the δ problem

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(0) Recap from Lecture 9 **Green's functions** **[Haberman, Chap 9.3]**

To solve the ODE BVP with homogeneous version of BC's for $u(x)$:

$$a \le x \le b: \quad \mathrm{L}_x u(x) = f(x) \quad BC_1 u(x=a) = 0 \quad BC_2 u(x=b) = 0$$

Soln: $$\boxed{u(x) = \int_a^b G(x,s) f(s)\, ds \quad G(x,s) = \begin{cases} G_-(x,s) & \boxed{a \le x < s} \le b \\ G_+(x,s) & a \le \boxed{s < x \le b} \end{cases}}$$

Idea: $G(x,s)$ combines the forcing from all source positions, $a < s < b$, to get the soln $u(x)$ at any field position, $a < x < b$

- Heaviside step fcn $H(x-s) = \begin{cases} 0 & x < s \\ 1 & x > s \end{cases}$
- Dirac delta fcn $\delta(x-s) = H'(x-s)$ and sifting property:

$$\int_a^b f(s)\delta(s-x)\, ds = f(x) \qquad \delta(\text{[integration variable] - [spike position]})$$

$\boxed{\textbf{Part 1}}$: Re-write hom-BC version of problem with variable x swapped to be s:

$$a \le s \le b: \quad \mathbf{L}_s u(s) = f(s) \qquad BC_1 u(a) = 0 \qquad BC_2 u(b) = 0$$

Project the problem onto the Green's fcn: $\langle \mathbf{L}_s u, G \rangle_2 = \langle f, G \rangle_2$:

$$\int_a^b \mathbf{L}_s u(s) G(x,s)\, ds = \underline{\int_a^b u(s) \mathbf{L}_s^* G(x,s)\, ds = u(x)} = \int_a^b f(s) G(x,s)\, ds$$

LHS via adjoint relation, RHS via definition of $G(x,s)$,
and then apply sifting property to get version 1.0 problem:

$$a \le s \le b: \qquad \mathbf{L}_s^* G(x,s) = \delta(s-x) \qquad BC_1^* G(x,a) = 0 \qquad BC_2^* G(x,b) = 0$$

Can do better: The <u>**Reciprocity theorem**</u> – create a fcn $F(y,s)$ that satisfies
$\mathbf{L}_s F = \delta(s-y)$ with hom. BC's on F at $s=a$ and $s=b$

$$\langle \mathbf{L}_s F, G \rangle_2 = \int_a^b \mathbf{L}_s F(y,s) G(x,s)\, ds = \int_a^b \delta(s-y) G(x,s)\, ds = G(x,y)$$

$$\langle \mathbf{L}_s F, G \rangle_2 = \langle F, \mathbf{L}_s^* G \rangle_2 = \int_a^b F \mathbf{L}_s^* G\, ds = \int_a^b F(y,s) \delta(s-x)\, ds = F(y,x)$$

Conclusion: $F(y,x) = G(x,y)$ for any $a \le x, y \le b$.

The "adjoint Green's fcn" would be a good name for F. *(continued)*

Part 1: (concluded)

δ problem for $F(y,s)$ on $a \le s \le b$:

$$\mathrm{L}_s F(y,s) = \delta(s-y) \qquad BC_1 F(y,a) = 0 \qquad BC_2 F(y,b) = 0$$

Use Reciprocity, $F(y,s) = G(s,y)$

$$\mathrm{L}_s G(s,y) = \delta(s-y) \qquad BC_1 G(a,y) = 0 \qquad BC_2 G(b,y) = 0$$

Relabel variables: change $s \to x$ and then $y \to s$ everywhere to get:

Final version of δ-problem for $G(x,s)$ on $a \le x \le b$:

$$\boxed{\mathrm{L}_x G(x,s) = \delta(x-s) \qquad BC_1 G(a,s) = 0 \qquad BC_2 G(b,s) = 0}$$

- Same Linear operator L_x on $a \le x \le b$ as original problem for $u(x)$
- Same-type **homogeneous** BC's at $x = a$ and $x = b$
- RHS delta-fcn forcing, spiking at $x = s$ (at some $a < s < b$)

Example: $G(x,s)$ on $0 \le x \le 1$

$$\frac{d^2G}{dx^2} = \delta(x-s) \qquad G(x=0) = 0 \qquad G(x=1) = 0$$

Part 2: **Solving inhomogeneous ODE BVP $\mathbf{L}u = f$ via $G(x,s)$**

Re-write full problem (Dir BC example) with variable x swapped to be s:

$$a \le s \le b \qquad \mathbf{L}_s u(s) = f(s) \qquad u(a) = c \qquad u(b) = d$$

Project the problem onto the Green's fcn: $\langle \mathbf{L}u, G\rangle_2 = \langle f, G\rangle_2$:

$$\begin{aligned}
\int_a^b \mathbf{L}u(s) G(x,s)\, ds &= \int_a^b f(s) G(x,s)\, ds \\
(\text{Bdry Terms})\Big|_{s=a}^{s=b} + \int_a^b u(s) \mathbf{L}_s^* G(x,s)\, ds &= \\
(\text{Bdry Terms})\Big|_{s=a}^{s=b} + \int_a^b u(s) \delta(s-x)\, ds &= \\
(\text{Bdry Terms } [G(x,s), c, d])\Big|_{s=a}^{s=b} + u(x) &= \int_a^b f(s) G(x,s)\, ds
\end{aligned}$$

$$\boxed{u(x) = -(\text{Bdry Terms})(x,s)\Big|_{s=a}^{s=b} + \int_a^b f(s) G(x,s)\, ds}$$

Soln = usual $f(x)$-forcing integral (w/hom BC's) [the particular soln "$u_F(x)$"] + Bdry-terms [soln of hom eqn $Lu(x) = 0$ with inhom BC-forcing "$u_B(x)$"]

$\boxed{\text{Part 3}}$: Steps in constructing the piecewise-defined $G(x, s)$

$$a \le x \le b \qquad L_x G(x,s) = \delta(x-s) \qquad G(a,s) = 0 \qquad G(b,s) = 0$$

1. (away from the spike) For $x \neq s$, the delta fcn does not spike, it is zero. Separate into cases:

   - For $a \le x < s$: left part of the Green's fcn: $G_-(x,s)$ with left BC

   $$a \le x < s \qquad L_x G_-(x,s) = 0 \qquad G_-(a,s) = 0$$

   - For $s < x \le b$: right part of the Green's fcn: $G_+(x,s)$ with right BC

   $$s < x \le b \qquad L_x G_+(x,s) = 0 \qquad G_+(b,s) = 0$$

2. (near the spike) Use ODE to connect G_- and G_+ together at the $x = s$ spike (the "Jump eqns")

3. This gives the piecewise $G(x,s) = \begin{cases} G_-(x,s) & a \le x < s \\ G_+(x,s) & s < x \le b \end{cases}$

$\boxed{\textbf{Part 3}}$: **The Jump conditions** (conclusion)

$$a \le x \le b \qquad L_x G(x,s) = \delta(x-s) \qquad G(a,s) = 0 \qquad G(b,s) = 0$$

Integrate the δ-ODE near the spike

$$\int_{s-\epsilon}^{s+\epsilon} \mathbf{L}_x G \, dx = \int_{s-\epsilon}^{s+\epsilon} \delta(x-s)\, dx = 1$$

Take limit $\epsilon \to 0$, and consider example $\mathbf{L}_x u(x) = A(x)u'' + B(x)u' + C(x)u$

$$LHS = \int_{s-\epsilon}^{s+\epsilon} A(x)\frac{d^2G}{dx^2}\, dx + \int_{s-\epsilon}^{s+\epsilon} B(x)\frac{dG}{dx}\, dx + \int_{s-\epsilon}^{s+\epsilon} C(x)G\, dx$$

- Shortcut notation: $G = G(x)$ (hide the s parameter)
- Note: integrals over very narrow range $|x - s| \le \epsilon \to 0$
- Can assume coeff fcns A, B, C are smooth, so can use Taylor series and approx $A \approx A(s), B \approx B(s), C \approx C(s)$ const...
- Derivatives, Distribution theory, $\int_{-\epsilon}^{\epsilon} \delta\, dx = 1$, and $\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} f(x)\, dx = 0$ for bounded functions $|f(x)| < M$
- For n^{th} order ODE, yields set of n equations connecting G_- and G_+ (and derivatives) at $x = s$ spike position.