Shared feature: Linear constant coefficient (LCC) equations (general N-th order and 2nd order example)

$$\sum_{n=0}^{N} a_n \frac{d^n\phi}{dx^n} = 0 \qquad \xrightarrow{N=2} \qquad a_2\frac{d^2\phi}{dx^2} + a_1\frac{d\phi}{dx} + a_0\phi = 0 \tag{1}$$

and Cauchy-Euler (CE) equations

$$\sum_{n=0}^{N} b_n x^n \frac{d^n\phi}{dx^n} = 0 \qquad \xrightarrow{N=2} \qquad b_2x^2\frac{d^2\phi}{dx^2} + b_1x\frac{d\phi}{dx} + b_0\phi = 0 \tag{2}$$

These are homogeneous equations, whose general solutions can be written as linear combinations of N linearly independent fundamental solutions:

$$\phi_{\text{gen}}(x) = \sum_{\ell=1}^{N} c_\ell\phi_\ell(x) \qquad \xrightarrow{N=2} \qquad \phi_{\text{gen}}(x) = c_1\phi_1(x) + c_2\phi_2(x) \tag{3}$$

where each $\phi_\ell(x)$ can be determined using appropriate trial solutions that reduce the ODE to algebra for finding roots of a characteristic polynomial. The c_ℓ coeffs get selected by additional BC or IC conditions.

1 LCC solutions

For LCC equations, the trial solution is $\boxed{\phi(x) = e^{mx}}$. The values for the exponents m are roots of the characteristic polynomial equation:

$$a_n m^n + a_{n_1}m^{n-1} + \cdots + a_2m^2 + a_1m + a_0 = 0 \qquad (a_2m^2 + a_1m + a_0 = 0) \tag{4}$$

• If m_* is a multiple root (order k) then it yields k distinct $\phi_\ell(x)$ solutions in the form:

$$e^{m_*x} \qquad xe^{m_*x} \qquad x^2e^{m_*x} \qquad \cdots \qquad x^{k-1}e^{m_*x} \tag{5}$$

• If all the coefficients in the ODE are real, then if $m_* = \alpha+i\beta$ is a complex root then the complex-conjugate $\overline{m_*} = \alpha - i\beta$ is always also a root. Therefore, two ϕ_ℓ solutions are

$$c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} = c_1e^{\alpha x}e^{i\beta x} + c_2e^{\alpha x}e^{-i\beta x} \tag{6}$$

using $e^{c+d} = e^ce^d$ and Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, the solutions can always be re-written in terms of only real functions:

$$= c_1e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) + c_2e^{\alpha x}(\cos(\beta x) - i\sin(\beta x)) = c_3\boxed{e^{\alpha x}\cos(\beta x)} + c_4\boxed{e^{\alpha x}\sin(\beta x)} \tag{7}$$

For real-valued problems, always use form (7), not (6)!

2 CE solutions

For CE equations, the trial solution is $\boxed{\phi(x) = x^m}$. The powers m are roots of the characteristic polynomial equation (for an example 4th order CE ODE):

$$b_4m(m-1)(m-2)(m-3) + b_3m(m-1)(m-2) + b_2m(m-1) + b_1m + b_0 = 0 \tag{8}$$

• If m_* is a k^{th} order multiple root, then solutions are $x^{m_*}, x^{m_*}\ln(x), x^{m_*}(\ln(x))^2, \cdots x^{m_*}(\ln(x))^{k-1}$.
• If $m_* = \alpha + i\beta$ (and $\overline{m_*} = \alpha - i\beta$) are complex roots, then using $x^r = \left(e^{\ln(x)}\right)^r = e^{r\ln(x)}$ and $e^{i\theta} = \cos\theta + i\sin\theta$:

$$\begin{aligned} c_1x^{\alpha+i\beta} + c_2x^{\alpha-i\beta} &= c_1e^{\alpha\ln(x)}(\cos(\beta\ln(x)) + i\sin(\beta\ln(x)) + c_2e^{\alpha\ln(x)}(\cos(\beta\ln(x)) - i\sin(\beta\ln(x)) \\ &= c_3\boxed{x^\alpha\cos(\beta\ln(x))} + c_4\boxed{x^\alpha\sin(\beta\ln(x))} \qquad \text{(use this form!)} \end{aligned}$$

Inhomogeneous LCC or CE equations $Ly = f$ can be solved in terms of sums of homogeneous and particular solutions (see summary sheet on the method of Un-determined Coefficients) or using variation or parameters or eigenfunction expansions.