Math 551, Duke University — Review of techniques of integration

Strategy

- Identify the type of integral you have.
- Try using the simplest technique for your integral.
- If that doesn't work, try the next-level method (u-subs then IBP then Trig then ...)
- You may have to break-up the integral into several parts and use a different method for each part.

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u-Substitution

Goal: Reduce the integral to one of the 10 basic integrals in the table.

Works: If the integrand is nice enough to have exactly the chain rule form: $\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(g(x)) + C$

Ex: $\int 2x\cos(x^2)\,dx = \int \cos(u)\,du$

Integration by parts

Goal: Figure out the integral of a product of two functions (polynomials, exponentials, logs, trig, inverse trig, etc).

Works: For all the times when u-substitution fails

Notes: Needed for secant$^{\text{oddpower}}$ (Use Trig-Tricks for most other trig integrals)

Ex: $\int x(3x+4)^{777}\,dx$, $\int e^x \sin x\,dx$, $\int \ln x \tan^{-1} x\,dx$

Trig Tricks

Goal: Integrals of combinations of trig functions

Ex: $\int \cos^2(3x)\sin(4x)\,dx$, $\int \sin^2(\alpha x)\cos^2(\beta x)\,dx$

The 4 tricks

Check these tricks (in this order!) to simplify your integral as much as needed:

1: $\boxed{\alpha \neq \beta}$ If you have an integral of a product of trig functions with different angles (Ex: $3x \neq 4x$) you **MUST** use trig identities to break it up into a sum of different terms before you can make any progress. Example:
$\int \cos(3x)\sin(4x)\,dx = \frac{1}{2}\int \sin(7x) + \sin(x)\,dx$

2: $\boxed{\text{If one is odd}, u = \text{other}}$ This gives the right u-substitution for $\cos^{\text{power}_1} \times \sin^{\text{power}_2}$ integrals (with positive or zero powers). You will also need to use $\cos^2 x + \sin^2 x = 1$. Example:
$\int \cos^7 x \sin^2 x\,dx$ then $\cos^{\text{odd}}$, so $u = \sin x$ and use $\cos^2 x = 1 - u^2$ to get $\int (1-u^2)^3 u^2\,du$

3: $\boxed{\text{even} \times \text{even} \to \tfrac{1}{2}\triangleleft}$ If both powers are even, then you need to use the half-angle formulas:

$$\cos^2 x = \tfrac{1}{2}(1 + \cos(2x))$$
$$\sin^2 x = \tfrac{1}{2}(1 - \cos(2x))$$

4: $\boxed{\text{SET TOS}}$ Use $\sec^2 x = 1 + \tan^2 x$ with
- SET: **S**ec$^{\textbf{Even}}$, use $u =$ **T**an substitution
- TOS: **T**an$^{\textbf{Odd}}$, use $u =$ **S**ec substitution

Ex: $\int \sec^4 x \tan^{\sqrt{2}} x\,dx$, $\int \sec^{\pi}(6x)\tan^3(6x)\,dx$

Square roots

- If you have a $\sqrt{ax+b}$ then let $u = \sqrt{ax+b}$ so $u^2 = ax+b$ and $2u\,du = a\,dx$ and $x = (u^2 - b)/a$
- If you have a $\sqrt{a^2 - x^2}$ then $\boxed{x = a\sin\theta}$
- If you have a $\sqrt{a^2 + x^2}$ then $\boxed{x = a\tan\theta}$
- If you have a $\sqrt{x^2 - a^2}$ then $\boxed{x = a\sec\theta}$
- If you have a $\sqrt{ax^2+bx+c}$ then **First, complete the square, then let** $u = x - d$ **with** $d = b/(2a)$ **and look again**. Example:

$$\sqrt{x^2 - 4x + 7} = \sqrt{(x-2)^2 + \sqrt{3}^2} = \sqrt{u^2 + k^2}$$

- These work with positive or negative powers of the roots and (quadratic)$^{\pm k}$ too.

Rational Functions

$$\text{“Rational fcn” means} \quad \frac{\text{polynomial}}{\text{polynomial}}$$

Easy ones:

- $\int \text{poly}\,dx = \text{new poly} + C \quad (\int x^n\,dx = \frac{x^{n+1}}{n+1} + C)$
- $\displaystyle\int \frac{d\text{poly}}{\text{poly}} = \ln|\text{poly}| + C$
- $\displaystyle\int \frac{d\text{poly}}{\text{poly}^n} = -\frac{1}{n-1}(\text{poly})^{-n+1} + C \qquad n \neq 1$
- $\displaystyle\int \frac{dx}{\text{quadratic}}$: Complete the square and use square roots guide.

Everything else needs **Partial Fractions**:

1. NO Improper fractions: Convert $\frac{21}{5} \to 4 + \frac{1}{5}$. You **MUST** do this first! (Use long division or synthetic division)
2. Factor numerator, denominator, cancel stuff
3. Expand out as a sum of partial fractions (each will be one of the “easy ones”)