Math 551: Sep of Vars for PDE's (conclusion) Lecture 26

$\boxed{1}$ **Problems in spherical coordinates**: 3 examples
– Legendre polynomials for spherical problems (H 7.10)
– **Closing summary of separation of variables for PDEs (H 8.6)**

$\boxed{2}$ **Stability theory: analyzing PDE time-dynamics & predicting behaviors**

(3) Introduction to Green's functions for PDE's (H 9.5)

$\boxed{1}$ **The heat eqn in spherical coords**: $\partial_t u = \nabla^2 u$ on ball $\boxed{0 \le \rho \le b}$

$$\frac{\partial u}{\partial t} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right)$$

Dirichlet problem BC's: $u(\rho = b) = 0$ and
(periodic in $0 \le \theta \le 2\pi$) and (bounded at $\rho = 0$ and $\phi = 0, \pi$)
Different types of IC's:

(a) $u(t=0) = F(\rho)$ "spherically symmetric" (No θ, ϕ's)

(b) $u(t=0) = F(\rho, \phi)$ "axially symmetric" (No θ's)[a]

(c) $u(t=0) = F(\rho, \phi, \theta)$ general case $u = u(\rho, \theta, \phi, t)$

[a] A solution $u(\rho, \theta, t)$ or $u(\theta, t)$ (with no ϕ's) is not possible (see the $u_{\theta\theta}$ term)!

(a) Spherically symmetric: $\partial_t u = \rho^{-2}\partial_\rho(\rho^2\partial_\rho u)$ $\qquad u_k(\rho,t) = f(\rho)h(t)$

$$\frac{h'(t)}{h(t)} = \frac{(\rho^2 f'(\rho))'}{\rho^2 f(\rho)} = -\lambda \qquad \to \qquad h(t) = e^{-\lambda t} \qquad (\lambda\,?)$$

The basic spherical Bessel equation (order zero) $\qquad \boxed{\textbf{SL form with } \sigma = \rho^2}$

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) = -\lambda\rho^2 f$$

Convert to standard form via $\boxed{f(\rho) = y(z)/\sqrt{z}}$ with $z = \sqrt{\lambda}\,\rho$

$$\tfrac{d}{dz}\left(z\tfrac{dy}{dz}\right) - \tfrac{1}{4z}y = -zy \qquad \to \qquad y(z) = c_1 J_{1/2}(z) + c_2 Y_{1/2}(z)$$

$$f(0) \text{ bdd}, \quad f(b) = 0 \quad \to \quad \left\{ f_k(\rho) = \frac{J_{1/2}(\sqrt{\lambda_k}\,\rho)}{\sqrt{\rho}} \quad \lambda_k = \left(\frac{k\pi}{b}\right)^2 \right\}$$

can also be written as $f_k(\rho) = \sin(\sqrt{\lambda_k}\,\rho)/\rho$ (see H p.335)

$$u(\rho,t) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} f_k(\rho) \qquad c_k = \frac{\langle F, f_k\rangle_\sigma}{||f_k||_\sigma^2} = \frac{1}{||f_k||_\sigma^2}\int_0^b F(\rho) f_k(\rho)\rho^2\,d\rho$$

"**Chain of command**": $\rho \to t$: BC's on $f(\rho)$ sets λ's for $h(t)$'s ■

(b) Axisymmetric: $u_k(\rho, \phi, t) = f(\rho)g(\phi)h(t)$ (No θ)

First: $$\frac{h'}{h} = \frac{(\rho^2 f')'}{\rho^2 f} + \frac{(\sin\phi g')'}{\rho^2 \sin\phi g} = -\lambda \quad \rightarrow \quad h(t) = e^{-\lambda t} \qquad (\lambda\,?)$$

$$\frac{(\rho^2 f')'}{f} + \lambda\rho^2 = -\frac{(\sin\phi g')'}{\sin\phi g} = \mu \quad \rightarrow \quad f(\rho) \text{ or } g(\phi) \text{ next?} \qquad (\mu\,?)$$

Try $f(\rho)$ first: SL form with $p = \rho^2, q = -\mu$ and $\boxed{\sigma = \rho^2}$

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda\rho^2 - \mu)f = 0$$ General spherical Bessel eqn $(\lambda\,?)$

Convert to standard form via $\boxed{f(\rho) = y(z)/\sqrt{z}}$ with $z = \sqrt{\lambda}\,\rho$

$$\frac{d}{dz}\left(z\frac{dy}{dz}\right) - \frac{\mu + \frac{1}{4}}{z}y = -zy \quad \rightarrow \quad y(z) = c_1 J_m(z) + c_2 Y_m(z)$$

$$f(\rho) = c_1 \frac{J_m(\sqrt{\lambda}\,\rho)}{\sqrt{\rho}} + c_2 \frac{Y_m(\sqrt{\lambda}\,\rho)}{\sqrt{\rho}} \qquad f(0) \text{ bdd} \rightarrow \; c_2 = 0$$

But what is μ, m? ....stuck in dead-end!?

Try $g(\phi)$ first: $$\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \mu \sin\phi g = 0 \qquad \Longrightarrow$$

General ODE for $g(\phi)$: $$\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\mu\sin\phi-\frac{\gamma}{\sin\phi}\right)g=0$$

Let $\boxed{z=\cos\phi}$ then $g(\phi)=y(z)$ with $y(z)$ solving $\boxed{\textbf{Legendre's equation}}$:

$$\frac{d}{dz}\left((1-z^2)\frac{dy}{dz}\right)+\left(\mu-\frac{\gamma}{1-z^2}\right)y=0 \qquad -1\le z\le 1$$

- Singular Sturm-Liouville, singular at $z=\pm 1$: No BC's needed there ($p=1-z^2, q=-\gamma/(1-z^2), \sigma=1$, eigenvalue μ)
- General soln depends on parameters μ,γ: $\boxed{y(z)=c_1P^{\gamma}_{\mu}(z)+c_2Q^{\gamma}_{\mu}(z)}$
- All $Q(z)$'s blow up at $z=\pm 1$
- Most $P(z)$'s also blow up, EXCEPT for special choices of μ

Legendre eigenvalues : $\boxed{\mu_n=n(n+1)}$ $\quad n=0,1,2,3,\cdots$

- Param γ is the coupling parameter to trig$(m\theta)$: $\boxed{\gamma_m=m^2}$ (order m)
- $\boxed{m=0 \text{ axisymmetric case}}$: Legendre polynomials $P_n(z)$ (even/odd)

$$P_0(z)=1 \qquad P_1(z)=z \qquad P_2(z)=\tfrac{1}{2}(3z^2-1) \qquad \cdots$$ HW#4 Q1

- General case $m=1,2,3,\cdots$: Associated Legendre functions "$P^m_n(z)$"

$$g_{n,m}(\phi)=P^m_n(\cos\phi)$$ (Haberman Sec 7.10.19)

(b) Axisymmetric solution (concluded): $m = 0$ mode only
Got $h(t) = e^{-\lambda t}$ then

$$g_n(\phi) = P_n(\cos\phi) \qquad \mu_n = n(n+1) \qquad n = 0, 1, 2, \cdots$$

then SL q-term $\mu_n + \frac{1}{4} = n^2 + n + \frac{1}{4} = m^2 \implies$ Bessel order $m = n + \frac{1}{2}$

$$f(\rho) = \frac{J_{n+1/2}(\sqrt{\lambda}\,\rho)}{\sqrt{\rho}}$$

then BC $f(b) = 0$ picks λ_k's: $J_{n+1/2}(\sqrt{\lambda_k}\,b) = 0$ for $k = 1, 2, \cdots$

$$u(\rho, \phi, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} c_{n,k} P_n(\cos\phi) \frac{J_{n+1/2}(\sqrt{\lambda_{n,k}}\,\rho)}{\sqrt{\rho}} e^{-\lambda_{n,k} t}$$

Finally, IC at $t = 0$ on $u(\rho, \phi) = F(\rho, \phi)$ sets $c_{n,k}$ coefficients

"**Chain of command**": $\phi \to \rho \to t$
ϕ-problem selects P_n modes and μ_n, $n = 0, 1, 2, \cdots$
which sets order of $J_{n+1/2}$ in ρ direction,
and BC at $\rho = b$ selects $\lambda_{n,k}$ eigenvalues for $e^{-\lambda t}$ decay rates ∎

(c) Spherical coordinate probs for Poisson, heat, and wave eqns

$$\nabla^2 u = S(\rho,\theta,\phi) \qquad u_t = \nabla^2 u + S \qquad u_{tt} = \nabla^2 u + S$$

Separating time and spatial-dependence, are all solved in terms of the spatial eigenfunctions of the Helmholtz problem:

$$\nabla^2 \Phi = -\lambda \Phi \qquad \text{(with homogenized BC's)}$$

$$\Phi_{m,n,k}(\rho,\theta,\phi) = (\text{Trig } h_m(\theta)) \cdot (\text{Legendre } g_n(\phi)) \cdot (\text{spherical Bessel } f_k(\rho))$$

$\boxed{\text{General spherical probs for } u(\rho,\phi,\theta,t)}$: "spherical harmonic fcns" $\Phi(\rho,\theta,\phi)$

$$u(\rho,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{k=1}^{\infty}\left[c_{m,n,k}\cos(m\theta) + d_{m,n,k}\sin(m\theta)\right] \cdot P_n^m(\cos\phi) \cdot \frac{J_{n+1/2}(\sqrt{\lambda_{n,k}}\,\rho)}{\sqrt{\rho}} \cdot e^{-\lambda_{n,k}t}$$

"**Chain of command**": $\theta \to \phi \to \rho \to t$

θ problem 2π-periodic modes $m = 0, 1, 2, \cdots$ which selects the order for generalized ϕ-problem selects P_n^m Legendre modes, $n = 0, 1, 2, \cdots$

which sets order of $J_{n+1/2}$ in ρ direction, and

the BC at $\rho = b$ selects the $\lambda_{n,k}$ eigenvalues for $e^{-\lambda t}$ decay rates ■

Summary of solns via 2D (N-D) Helmholtz eigenfcns (Haberman 8.6)

Problems for Poisson, heat, and wave equations,

$$\nabla^2 u = S(x,y) \qquad u_t = \nabla^2 u + S(x,y) \qquad u_{tt} = \nabla^2 u + S(x,y)$$

can be solved in terms of eigensolns of the corresponding Helmholtz problem:

$$\nabla^2 \Phi = -\lambda \Phi \qquad \text{(with homogenized BC's)}$$

"multi-index" notation: "$\mathbf{k}$" $= (n,m)$

$$u(x,y) = \sum_{n,m} c_{n,m} \Phi_{n,m}(x,y) \qquad \text{and} \qquad u = \sum_{\text{“}\mathbf{k}\text{”}} c_{\mathbf{k}}(t) \Phi_{\mathbf{k}}(x,y)$$

Solve Helmholtz $\{\lambda_k, \Phi_k\}$ then project onto PDE, $\langle \Phi, \text{PDE} \rangle$, via Green's 2nd:

$$\langle \Phi, \mathbf{L}u \rangle = \iint_D \Phi \nabla^2 u \, dA = \oint_C \left(\Phi \frac{\partial u}{\partial n} - u \frac{\partial \Phi}{\partial n} \right) ds + \iint_D u \nabla^2 \Phi \, dA.$$

$$\iint_D \Phi_k S \, dA = \oint_C \left(\Phi_k \frac{\partial u}{\partial n} - u \frac{\partial \Phi_k}{\partial n} \right) ds - \lambda_k c_k ||\Phi_k||^2$$

This "2-D" expansion approach (HW#7 Q2) can be overall faster than the "1-D" approach (HW#7 Q1): $\left[u(x,y) = \sum_n b_n(y) f_n(x) \right.$ then split up each $b_n(y) = \sum_m c_{m,n} g_m(y)$ for same final $\left. u = \sum_n \sum_m c_{mn} f_n g_m \right]$

$\boxed{2}$ Linear Stability theory: describing dynamics (time evolution) of PDE solns

$$\frac{\partial u}{\partial t} = \mathbf{L}u \qquad \underset{\textbf{Problem}}{\overset{\textbf{Solution}}{\rightleftharpoons}} \qquad u(x,t) = \sum_{k=0}^{\infty} c_k e^{\Lambda_k t} \Phi_k(x)$$

k called the "wavenumber" of spatial oscillations in Φ_k

Λ_k exponential growth rate of k-th eigenmode (Set of Λ_k's called "the spectrum")

(eigenvalues of $L\Phi = \Lambda\Phi$ (+sign!) : space-time separation constants)

How Λ depends on k is called the "dispersion relation"

Linear Stability results:

- If <u>all</u> $\Lambda_k < 0$ then all modes decay $u_k = e^{\Lambda_k t}\Phi_k(x) \to 0$ as $t \to \infty$ (asymptotically stable) $\implies$ $u \to 0$ (the solution is **<u>stable</u>**)
- If <u>any</u> $\Lambda_k > 0$ then that mode grows $u_k \to \infty$ as $t \to \infty$ (asymptotically unstable) $\implies$ $u \to \infty$ (the solution is **<u>unstable</u>**)
- In general (for non-self adjoint $\mathbf{L}$) Λ_k can be complex, $\Lambda = \sigma + i\omega$, with σ (modal growth rate) and ω (modal oscillation frequency)
  Then stability results apply to $\sigma_k = \text{Re}(\Lambda_k) \lessgtr 0$[a]
- Includes Λ_k from 2-D/N-D expansions....

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[a] $\text{Re}(\Lambda_k) = 0$ borderline "marginal cases" needs further checks... and $\Lambda_0 = 0 \implies$ FAT

Stability theory: extension to inhom-forced PDE and nonlinear PDE

$$\frac{\partial u}{\partial t} = N(u,x) \qquad \text{like } N = \mathbf{L}u + S(x) \text{ or } N = (u^2)_{xx} \text{ or } \cdots$$

- Find a steady state solution $u = \bar{u}$ with $N(\bar{u}, x) = 0$ (check FAT!)
- To determine the stability of $\bar{u}$ to small (infinitesimally small) perturbations, let

$$u(x,t) = \bar{u} + \epsilon \widetilde{u}(x,t) + \epsilon^2(\cdots) \qquad \text{with } \epsilon \to 0$$

- Plug into the full PDE

$$\epsilon \frac{\partial \widetilde{u}}{\partial t} = N(\bar{u} + \epsilon \widetilde{u})$$

- Use Taylor series to expand RHS for $\epsilon \to 0$

$$N(\bar{u} + \epsilon \widetilde{u}) = N(\bar{u}) + \epsilon \left. \frac{\delta N}{\delta u} \right|_{\bar{u}} \widetilde{u} + \epsilon^2(\cdots)$$

- Linearized stability equation – collect ϵ^1 terms on LHS, RHS:

$$\frac{\partial \widetilde{u}}{\partial t} = \bar{\mathbf{L}} \widetilde{u} \qquad \text{where} \quad \bar{\mathbf{L}} \widetilde{u} = \left. \frac{\delta N}{\delta u} \right|_{\bar{u}} \widetilde{u}$$

  Use separation of variables to determine the spectrum/stability of $\bar{\mathbf{L}}$

(3) Introduction to Green's functions for PDE's

Recall Green's fcns for self-adjoint ODE BVP, $\mathbf{L}u = f(x)$, with hom. BC's (H 9.3)

Solve via eig-fcn expansion: $\mathbf{L}\Phi = -\lambda\Phi$

$$u(x) = \sum_k \left(\frac{-1}{\lambda_k ||\Phi_k||^2} \int_a^b f\Phi_k \, dx \right) \Phi_k(x) = \int_a^b \left(\sum_k \frac{\Phi_k(x)\Phi_k(\widetilde{x})}{-\lambda_k ||\Phi_k||^2} \right) f(\widetilde{x}) \, d\widetilde{x}$$

Self-adjoint bilinear form of the Green's function:

$$G(x, x_0) = \sum_k \frac{\Phi_k(x)\Phi_k(\widetilde{x})}{-\lambda_k ||\Phi_k||^2} \qquad \to \qquad u(x) = \int_a^b G(x, \widetilde{x}) f(\widetilde{x}) \, d\widetilde{x}$$

Constructing the piecewise Green's function:

$$\mathbf{L}G = \delta(x - \widetilde{x}) \qquad \text{with hom. BC's on } G(x)$$

Problems with inhomogeneous BC's: start by projecting ODE onto G,

$$\langle \mathbf{L}u, G \rangle = \langle f, G \rangle$$

leads to the solution in the form

$$u(x) = \int_a^b G(x, \widetilde{x}) f(\widetilde{x}) \, d\widetilde{x} + (\text{"IBP" boundary terms})$$

The Green's function for Poisson's equation (Haberman 9.5)

$$\nabla^2 u = f(x,y) \qquad \text{with homogeneous BC's}$$

Eigenfunction expansion approach (9.5.3)

$$u = \sum_n \sum_m c_{n,m} \Phi_{n,m}(x,y) \qquad c_{n,m} = \frac{\langle f, \Phi_{n,m} \rangle}{-\lambda_{n,m} ||\Phi_{n,m}||^2}$$

Green's function (self-adjoint bilinear form)

$$G(x,y,\tilde{x},\tilde{y}) = \sum_n \sum_m \frac{\Phi_{n,m}(x,y)\Phi_{n,m}(\tilde{x},\tilde{y})}{-\lambda_{n,m} ||\Phi_{n,m}||^2}$$

$$u(x,y) = \int_0^L \int_0^H G(x,y,\tilde{x},\tilde{y}) f(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}$$

Delta function approach for determining the Green's function (9.5.5)

$$\nabla^2 G = \delta(\mathrm{x} - \tilde{\mathrm{x}})$$

Further issues:

1. Inhomogeneous boundary conditions
2. G formulas for infinite domain problems
3. Finite domain problems – brief mention (more difficult...) (H 9.5.7-9)

Green's functions for Poisson's equation: Inhomogeneous BC's

$$\nabla^2 u = f(x,y) \qquad \text{with BC's: } u = h(x,y) \text{ (Dir)} \quad \text{or} \quad \frac{\partial u}{\partial n} = h(x,y) \text{ (Neu)}$$

1. First: get Green's function, $\nabla^2 G = \delta(\mathrm{x} - \widetilde{\mathrm{x}})$ (IOU, next slide)
2. Then project the Poisson problem onto G

$$\int_0^L \int_0^H G(x,y,\widetilde{x},\widetilde{y}) \widetilde{\nabla}^2 \widetilde{u}\, d\widetilde{x}\, d\widetilde{y} = \int_0^L \int_0^H G(x,y,\widetilde{x},\widetilde{y}) f(\widetilde{x},\widetilde{y})\, d\widetilde{x}\, d\widetilde{y}$$

Use Green's second identity on the LHS

$$\text{LHS} = \iint \left(\widetilde{\nabla}^2 G\right) \widetilde{u}\, d\widetilde{x}\, d\widetilde{y} + \oint G \frac{\partial \widetilde{u}}{\partial \widetilde{n}}\, d\widetilde{s} - \oint \widetilde{u} \frac{\partial G}{\partial \widetilde{n}}\, d\widetilde{s}$$

$$= \iint \delta(\widetilde{\mathrm{x}} - \mathrm{x}) \widetilde{u}\, d\widetilde{x}\, d\widetilde{y} +$$

$$= \quad u(x,y) \quad + \oint G \frac{\partial \widetilde{u}}{\partial \widetilde{n}}\, d\widetilde{s} - \oint \widetilde{u} \frac{\partial G}{\partial \widetilde{n}}\, d\widetilde{s}$$

Solution:

$$u(x,y) = \iint G \widetilde{f}\, d\widetilde{x}\, d\widetilde{y} + \begin{cases} \oint h(x(\widetilde{s}), y(\widetilde{s})) \frac{\partial G}{\partial \widetilde{n}}\, d\widetilde{s} & \text{Dir BC} \\ -\oint h(x(\widetilde{s}), y(\widetilde{s})) G\, d\widetilde{s} & \text{Neu BC} \end{cases}$$

Solution by "boundary integrals"

Green's functions for Poisson's equation: infinite domain problems

$$\nabla^2 G = \delta(\mathbf{x} - \widetilde{\mathbf{x}})$$

Shift coordinates so the spike is at the origin: let $\hat{\mathbf{x}} = \mathbf{x} - \widetilde{\mathbf{x}}$

Use "piecewise" description of G: $$\nabla^2 G = \begin{cases} \text{“}\infty\text{”} & \hat{\mathbf{x}} = 0 \\ 0 & \text{else} \end{cases}$$

Find 3D spherically symmetric solution for $\rho > 0$: $\rho = |\mathbf{x} - \widetilde{\mathbf{x}}|$

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{dG}{d\rho}\right) = 0 \qquad \to \qquad \rho^2 \frac{dG}{d\rho} = C_1 \qquad \to \qquad G(\rho) = -\frac{C_1}{\rho}$$

Jump condition: Integrate over volume w/origin: $\iiint \nabla^2 G \, dV = \iiint \delta \, dV = 1$

Use divergence theorem on LHS on a spherical volume at origin:

$$\iint \frac{\partial G}{\partial n} \, dS = 1 \qquad\qquad \frac{\partial G}{\partial n} = \frac{dG}{d\rho} = \frac{C_1}{\rho^2}$$

$$\int_0^{2\pi} \int_0^{\pi} \left(\frac{C_1}{\rho^2}\right) \rho^2 \sin\phi \, d\phi d\theta = 4\pi C_1 = 1 \qquad \to \qquad G(\rho) = -\frac{1}{4\pi\rho}$$

$$\nabla^2 u = f \qquad \to \qquad \boxed{u(x,y,z) = -\iiint \frac{f(\widetilde{\mathbf{x}})}{4\pi|\mathbf{x} - \widetilde{\mathbf{x}}|} \, d\widetilde{V}}$$

Electromagnetic fields, gravity, other problems....