Math 551, Duke University — Lecture 2

Part I: Linear Algebra ideas Part II: Orthogonality of functions and Introduction to Fourier Series

(I) Key ideas from Linear Algebra (conclusion) [Haberman Sect 5.5 App]

If you have a real matrix $\mathbf{L}_{n\times n}$ that has a complete set of n eigenvectors, and you use the definition of the inner product as $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T\mathbf{v}$ then:

- From $\langle \mathbf{v}, \mathbf{L}\mathbf{u}\rangle = \langle \mathbf{L}^*\mathbf{v}, \mathbf{u}\rangle$, the adjoint is $\mathbf{L}^* = \mathbf{L}^T$
- Find the eigenvalues from the determinant, $|\mathbf{L}-\lambda\mathbf{I}| = 0$, and $\gamma_k = \lambda_k$
- For eigenvectors ϕ_k and adjoint eigenvectors ψ_k for each λ_k, do algebra for

$$(\mathbf{L}-\lambda_k\mathbf{I})\phi_k = \vec{0} \qquad \text{and} \qquad (\mathbf{L}^T-\lambda_k\mathbf{I})\psi_k = \vec{0}$$

- The eigenvectors are bi-orthogonal: $\langle \phi_k, \psi_j\rangle = 0$ for any $k \neq j$ and $\langle \phi_k, \psi_k\rangle \neq 0$ when both come from the k-th eigenmode
- Any given $\mathbf{w} \in \mathbb{R}^n$ can be written as an "eigen-expansion" form:

$$\mathbf{w} = \sum_{k=1}^{n} c_k\phi_k \qquad \text{with} \qquad c_k = \frac{\langle \psi_k, \mathbf{w}\rangle}{\langle \psi_k, \phi_k\rangle}$$

  Need the ϕ_k's to be a complete basis set. This property guarantees that all c_k coeffs in the expansion for any $\mathbf{w}$ can be uniquely determined.
  Bi-orthogonality of ϕ, ψ's de-couples calculations of c_k's for each k.

Solving linear equations : $\boxed{\mathbf{Lu} = \mathbf{b}}$ for unknown $\mathbf{u}$

The **eigenvector expansion** method: start with the expansion formula for $\mathbf{u}$:

$$\mathbf{u} = \sum_{k=1}^{n} c_k \phi_k \qquad c_k = \frac{\langle \psi_k, \mathbf{u} \rangle}{\langle \psi_k, \phi_k \rangle}$$

But now $\mathbf{u}$ is NOT known, so can't work out numerator inner products in c_k...

So, indirect approach: Go back to the original problem $\rightarrow$

Do orthogonal projection of the **problem** onto each ψ_k for $k = 1, 2, \cdots, n$

$$\begin{aligned}
\langle \psi_k, \mathbf{Lu} \rangle &= \langle \psi_k, \mathbf{b} \rangle && \langle \psi_k, \text{Eqn} \rangle \\
\langle \mathbf{L}^* \psi_k, \mathbf{u} \rangle &= && \text{(adjoint relation)} \\
\langle \gamma_k \psi_k, \mathbf{u} \rangle &= && \text{(adjoint eig-prob, } \mathbf{L}^* \psi = \gamma \psi \text{)} \\
\lambda_k \langle \psi_k, \mathbf{u} \rangle &= && \text{(linearity and } \lambda_k = \gamma_k \text{)} \\
\lambda_k \boxed{\langle \psi_k, \mathbf{u} \rangle} &= && \text{(numerator in coeff } c_k \text{!)} \\
\lambda_k \boxed{c_k} \langle \psi_k, \phi_k \rangle &= \langle \psi_k, \mathbf{b} \rangle && \text{(using } \langle \psi_k, \mathbf{u} \rangle = c_k \langle \psi_k, \phi_k \rangle \text{)}
\end{aligned}$$

$$\boxed{c_k = \frac{\langle \psi_k, \mathbf{b} \rangle}{\lambda_k \langle \psi_k, \phi_k \rangle}} \quad \rightarrow \quad \boxed{\mathbf{u} = \sum_{k=1}^{n} \frac{\langle \psi_k, \mathbf{b} \rangle}{\lambda_k \langle \psi_k, \phi_k \rangle} \phi_k}$$

$\boxed{\textbf{Self-Adjoint problems}}$: an important special case

$\boxed{\text{If } \mathbf{L}^* = \mathbf{L}}$ (symmetric real matrices, $\mathbf{L}^T = \mathbf{L}$) then some results simplify:

- Adjoint eigenvalues $\gamma_k = \lambda_k$ (unchanged)
- Adjoint eigenvectors $\psi_k = \phi_k$ (only need to find one set of vectors!)
- The set of eigenvectors is "<u>self-orthogonal</u>": $\phi_j \perp \phi_k$ for $j \neq k$ (see HW1Q3)
- The coefficients in the expansion for a given vector $\mathbf{w}$ simplify to

$$\mathbf{w} = \sum_{k=1}^{n} c_k \phi_k \qquad c_k = \frac{\langle \phi_k, \mathbf{w} \rangle}{|\phi_k|^2}$$

- The coefficients in the expansion for the solution of $\mathbf{Lu} = \mathbf{b}$ simplify to

$$\mathbf{u} = \sum_{k=1}^{n} c_k \phi_k \qquad c_k = \frac{\langle \phi_k, \mathbf{b} \rangle}{\lambda_k |\phi_k|^2}$$

- Eigenvalues λ_k are all real numbers (see HW1Q3)

The self-adjoint version of the vector eigen-expansion carries over directly to yield Fourier series for expansions of functions...

(II) Fourier Series and Orthogonal Expansions of functions [n.e.i. Haberman Ch 3]

1. To express complicated functions as sums of simple "basis" functions, as $\boxed{\textbf{Generalized Fourier series expansions}}$

$$f(x) \text{``=''} \sum_k c_k \phi_k(x) \qquad \text{on } a \le x \le b$$

2. To express solutions of differential equations (DE) problems as Fourier series and reduce DE problems to simpler algebra for the c_k coefficients in $u(x) = \sum c_k \phi_k(x)$

Inner products for real-valued functions on an interval $a \le x \le b$

$$\text{(Definition)} \qquad \boxed{\langle f, g \rangle \equiv \int_a^b [f(x)g(x)]\sigma(x)\, dx}$$

- $\sigma(x) \ge 0$: positive **weight function** ("weighted inner product")
- Generalization for complex-valued fcns: $\langle f, g \rangle \equiv \int_a^b [f(x)\overline{g(x)}]\sigma(x)\, dx$
- Specifying $\sigma(x)$ and a, b defines the inner product for a problem.

The uniform weight case: $\sigma(x) \equiv 1$ (classic/default case)

$$\langle f, g \rangle = \int_a^b [f(x)g(x)]\, dx \qquad \text{“standard } L^2 \text{ inner product on } [a, b]\text{”}$$

- The "L^2 norm": $\quad (||f||_2)^2 = \langle f, f \rangle = \left(\sqrt{\int_a^b f^2\, dx} \right)^2 \geq 0$
- "L^2 functions": also called "square integrable fcns", have finite L^2 norm:

$$||f||_2 < \infty$$

L^2 fcns can blow-up as long as they aren't "too badly" behaved. Examples:

(a) $f(x) = x^{-1/4}$ on $0 \leq x \leq 1$: $f(0) \to \infty$ but

$$||f||_2^2 = \int_0^1 (x^{-1/4})^2\, dx = 2x^{1/2}\Big|_0^1 = 2 \qquad \text{Ok, } L^2 \text{ fcn}$$

(b) $f(x) = x^{-1/2}$ on $0 \leq x \leq 1$: $f(0) \to \infty$ and

$$||f||_2^2 = \int_0^1 (x^{-1/2})^2\, dx = \ln(x)\Big|_0^1 = \infty \qquad \text{NOT } L^2$$

Fourier series theory and eigen-expansions are guaranteed to work for L^2 fcns

__**Orthogonality of functions**__ on interval $a \le x \le b$ is defined using the σ-weighted inner product integral $\boxed{\langle f, g \rangle = 0}$

- Assume $\{\phi_k(x)\}$ is a complete set of basis functions
- General $\boxed{\textbf{orthogonal projection}}$ works as usual for the expansion of fcns:

$$f(x) \text{``}=\text{''} \sum_k c_k \phi_k(x) \qquad c_k = \frac{\langle \psi_k(x), f(x) \rangle}{\langle \psi_k(x), \phi_k(x) \rangle}$$

- Assume that the $\phi_k(x)$'s are the eigenfunctions of a self-adjoint linear operator ($L^* = L$), so $\psi_k = \phi_k$ and the set of ϕ_k's is "<u>self-orthogonal</u>"

(general case) $\quad \langle \psi_j, \phi_k \rangle = 0 \quad$ for $j \neq k$ – "bi-orthogonal"

(self-adjoint case) $\quad \langle \phi_j, \phi_k \rangle = 0 \quad$ for $j \neq k$ – "self-orthogonal"

- For the self-adjoint (Fourier series) case we have $\boxed{c_k = \dfrac{\langle \phi_k(x), f(x) \rangle}{||\phi_k(x)||^2}}$
- IOU's: (1) What does "=" mean? and
  (2) What is the self-adjoint operator L that gives the $\phi_k(x)$? (later)