Variation of Parameters (VoP) for Inhomogeneous forced ODE BVP

The method of variation of parameters[1] is an approach for solving forced inhomogeneous ODE problems for any choice of forcing $\boldsymbol{f(x)}$ by building from solutions of the homogeneous ODE.

Full details for **second order problems**: the general inhomogeneous equation in standard form (the coefficient of the highest derivative is normalized to be one, divide the eqn across if needed to get this)

$$\underline{1}\cdot\frac{d^2u}{dx^2}+P(x)\frac{du}{dx}+Q(x)u=f(x)\qquad a\le x\le b \tag{1a}$$

with one typical set of homogeneous BC's (other choices of homogeneous BC's similarly):

$$u'(a)=0,\qquad u(b)=0. \tag{1b}$$

Problems with inhomogeneous BC's will be solved later with an additional step.

Solution process steps:

1. Homogenize the equation and solve to get $n=2$ linearly independent solutions

$$u''+Pu'+Qu=0\qquad\to\qquad\boxed{u=\{w_1(x),w_2(x)\}} \tag{2}$$

Every solution of the homogeneous problem is a linear combination of w_1,w_2 times some constants.

2. Use linear combinations of $w_1(x),w_2(x)$ to make $\boxed{\text{two solns } u_1(x),u_2(x)}$ that satisfy the BC's: $\boxed{\text{One BC satisfied by each } u(x)}$
Pick constants A_1,A_2 to have $u_1(x)$ satisfy one BC:

$$u_1(x)=A_1w_1(x)+A_2w_2(x)\quad\to\quad u_1'(a)=A_1w_1'(a)+A_2w_2'(a)=0.$$

Pick constants B_1,B_2 to have $u_2(x)$ satisfy the other BC:

$$u_2(x)=B_1w_1(x)+B_2w_2(x)\quad\to\quad u_2(b)=B_1w_1(b)+B_2w_2(b)=0.$$

3. **The Variation of Parameters (VoP) form**

$$\boxed{u(x)=c_1(x)u_1(x)+c_2(x)u_2(x)} \tag{3}$$

If c_1,c_2 are constants then $u_h(x)=c_1u_1(x)+c_2u_2(x)$ is the homogeneous general soln.
But, via VoP, (3) gives the soln of the **inhomogeneous** problem by $\boxed{\text{finding the right fcns for } c_1(x),c_2(x) \text{ “parameters”.}}$
Warning: You need $u_1(x),u_2(x)$. Using $w_1(x),w_2(x)$ will not work!

4. Use trial solution (3) to solve (1ab)

$$u''+Pu'+Qu=f(x)\qquad u'(a)=0\qquad u(b)=0, \tag{4}$$

by using the facts about u_1,u_2:

$$u_k''+Pu_k'+Qu_k=0,\qquad u_1'(a)=0,\qquad u_2(b)=0. \tag{5}$$

Evaluate derivatives of $u(x)$ using the product rule:

$$u'(x)=c_1(x)u_1'(x)+c_2(x)u_2'(x)+\left[c_1'(x)u_1(x)+c_2'(x)u_2(x)\right]$$

$$\boxed{\text{Set } c_1'(x)u_1(x)+c_2'(x)u_2(x)=0.} \tag{6}$$

[1] This sheet is for background, we will not be using this method for problems in this course.

We have the freedom to pick this condition, and it effectively yields the same first derivative for u' as if c_1, c_2 were constants. For higher order problems (n^{th} order ODE), this step gets expanded by requiring that the sum of the c' terms in product rule expansions for higher order derivatives of u (up to $(n-1)^{\text{st}}$ order) also zero-out.
And then for the n^{th} derivative

$$\begin{aligned} u''(x) &= (c_1(x)u_1'(x) + c_2(x)u_2'(x))' \\ &= c_1u_1'' + c_2u_2'' + c_1'u_1' + c_2'u_2'. \end{aligned}$$

Substitute for u' and u'' into equation (4) and re-group terms in the original ODE:

$$\begin{aligned} u'' + Pu' + Qu &= f(x) \\ (c_1u_1'' + c_2u_2'' + c_1'u_1' + c_2'u_2') + P(c_1u_1' + c_2u_2') + Q(c_1u_1 + c_2u_2) &= \\ c_1\underbrace{(u_1'' + Pu_1' + Qu_1)}_{=0} + c_2\underbrace{(u_2'' + Pu_2' + Qu_2)}_{=0} + c_1'u_1' + c_2'u_2' &= \\ \boxed{c_1'u_1' + c_2'u_2'} &= \boxed{f(x)} \end{aligned} \tag{7}$$

In summary, we get a system of linear equations for the derivatives of the c's:

$$\begin{array}{l} u_1c_1' + u_2c_2' = 0 \\ u_1'c_1' + u_2'c_2' = f(x) \end{array} \quad \to \quad \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{8}$$

The first $(n-1)$ eqns come from u'-type conditions (6) on the derivatives of u ($\langle \mathbf{u}, \mathbf{c}' \rangle = 0$, $\langle \mathbf{u}', \mathbf{c}' \rangle = 0$, $\cdots$, $\langle \mathbf{u}^{(n-2)}, \mathbf{c}' \rangle = 0$) and the 2nd (last) eqn comes directly from the ODE (7), $\langle \mathbf{u}^{(n-1)}, \mathbf{c}' \rangle = f$).

The determinant of the matrix of the u_k solns and their derivatives is called the **Wronskian**

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$$

Write the solution of the matrix system (8) via **Cramer's rule** in terms of determinants:

$$c_1' = \frac{\begin{vmatrix} 0 & u_2 \\ f & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = -\frac{u_2 f}{W} \qquad c_2' = \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & f \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{u_1 f}{W}$$

ODE's for $c_1(x), c_2(x)$: $\quad \dfrac{dc_1}{dx} = -\dfrac{u_2(x)f(x)}{W(x)} \qquad \dfrac{dc_2}{dx} = \dfrac{u_1(x)f(x)}{W(x)}$

Use the homogeneous BC's on the u_k's to get IC's for the c_k's:

$$\begin{aligned} u'(a) &= c_1(a)\underbrace{u_1'(a)}_{=0} + c_2(a)u_2'(a) = 0 \quad &\to \quad c_2(a) = 0 \\ u(b) &= c_1(b)u_1(b) + c_2\underbrace{u_2(b)}_{=0} = 0 \quad &\to \quad c_1(b) = 0 \end{aligned}$$

Initial value problems and their solutions:

$$\frac{dc_1}{dx} = -\frac{u_2(x)f(x)}{W(x)} \qquad c_1(b) = 0 \qquad \to \qquad \boxed{c_1(x) = -\int_b^x \frac{u_2(t)f(t)}{W(t)}\,dt}$$

$$\frac{dc_2}{dx} = \frac{u_1(x)f(x)}{W(x)} \qquad c_2(a) = 0 \qquad \to \qquad \boxed{c_2(x) = \int_a^x \frac{u_1(t)f(t)}{W(t)}\,dt}$$

then the final solution is $\boxed{u(x) = c_1(x)u_1(x) + c_2(x)u_2(x)}$.

This process can be generalized to similarly get the solutions for n-th order ODE problems with forcing $f(x)$ and homogeneous BC's.