Math 553, Duke University

1 Review: Integrating First Order Ordinary Differential Equations

1. Integration: If you are solving a specific initial value problem, use definite integrals. If you are finding a general solution of an ODE, use indefinite integrals. When calculating an integrating factor, use an indefinite integral and drop the "$+C$." If you get confused by using too many different letters for dummy variables of integration, use CAPITALs:

$$\frac{dy}{dx} = g(x) \qquad y(x_0) = y_0 \qquad \to \qquad y(x) = \int_{x_0}^{x} g(X)\,dX + y_0$$

Review the various methods of integration: **partial fractions**, **trigonometric substitutions**, **integration by parts**, **etc**.

2. Deriving ODE's: Use the given assumptions appropriate for the science in the problem – chemistry, physics, etc. Remember that "rate of change of $y(t)$" means dy/dt For geometry, remember that the derivative is the slope, $dy/dx = \tan\theta$.

3. Determine the appropriate way to solve the problem by identifying the type of the ODE:

First, put the ODE in **standard form**:[1] [2]

$$\frac{dy}{dx} = H(x,y)$$

then check the type of ODE:

a) Is it **simple**?

$$\frac{dy}{dx} = f(x)$$

Just integrate.

b) Is it **separable**?

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Separate x's and y's and integrate each, $g(y)dy = f(x)dx$.[3]

c) Is it **linear**?[4]

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Calculate integrating factor $\boxed{\rho(x) = e^{\int P(x)\,dx}}$ then multiply the ODE across by $\rho(x)$, let $\boxed{w = \rho(x)y}$ and integrate the resulting **simple** ODE for $w(x)$:

$$\frac{dw}{dx} = \rho(x)Q(x)$$

If these don't work, try:

d) Is it **Bernoulli**?

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Let $\boxed{w = y^{1-n}}$ and multiply the ODE to get a **linear** ODE for $w(x)$.[5]

e) Is it **equi-dimensional**?[6] [7]

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Let $\boxed{y = vx}$ to get a **separable** ODE for $v(x)$.

If these don't work, try:

f) Is it **upside-down**?

$$\frac{dy}{dx} = \text{MESS} \qquad \text{but} \qquad \frac{dx}{dy} = \text{NICE} = \frac{1}{\text{MESS}}$$

Integrate the easier ODE for $x(y)$ to get an implicit solution. For **equi-dim** ODE, let $\boxed{x = uy}$.

g) Does the ODE look like it came from a **product rule** or **chain rule** for a derivative? – Try to undo it!

h) **Fancy substitutions**: If you see some "interesting" combination of y's and x's inside a $\exp$, $\sin$, $\cos$, or etc, try to let $w =$"that combination of variables" and use (g)

i) **Existence and Uniqueness**: Does a theorem say that no solution exists? or more than one solution exists? Does $\frac{\partial H}{\partial y}$ blow up?

Some ODE's fit into several types, they can be solved in several different ways. Try these:

1) Use (b), (e) for

$$\frac{dy}{dx} = \frac{y}{x}$$

2) Use (d), (e) for

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^3}{x^3}$$

3) Use (e), (f) for

$$\frac{dy}{dx} = -\frac{2x^3y}{x^4+y^4}$$

4) Use (h) for

$$\frac{dy}{dx} - \frac{4}{x}y = 2x^5e^{y/x^4}$$

Notes

[1] Standard form means: no factors multiplying the derivative, put everything else on RHS.

[2] These techniques are arranged in order of (usually) increasing difficulty to integrate, so try to use the first method that applies to your ODE.

[3] Formally true, but recall full details: reduce (b) to a **simple** ODE (a) by multiplying across by $g(y)$ and using the chain rule to get the ODE for $w(x)$:

$$w = \int g(y)\,dy = G(y) \qquad \to \qquad \frac{dw}{dx} = f(x)$$

[4] linear in $y(x)$: no terms with more than one y or y'

[5] The **Bernoulli** approach doesn't work for $n = 1$ (**separable** (b)) or $n = 0$ (**linear** (c)), but you already know how to solve these cases.

[6] Confusingly, sometimes also called "homogeneous"

[7] A good way to test if the ODE is **equi-dim** is to plug in $y = vx$ and see if the right-side is $F(v)$ (no y's or x's alone remaining).