Calculating complete adjoint differential operators

An example

$$\mathrm{L}u(x) \equiv \frac{d^2u}{dx^2} + p(x)\frac{du}{dx} \qquad \text{formal linear differential operator} \tag{1a}$$

$$0 \le x \le 1 \qquad \text{on domain} \tag{1b}$$

$$u(0) = 0 \qquad u'(1) = 0 \qquad \text{with boundary conditions} \tag{1c}$$

Use the real inner product on the domain (standard L^2 inner product),

$$\langle u, v\rangle \equiv \int_0^1 uv\,dx \tag{2}$$

The inner product relation defines the adjoint operator:

$$\langle v, \mathrm{L}u\rangle = \langle \mathrm{L}^* v, u\rangle \tag{3}$$

which must hold for all acceptable functions $u(x)$ that satisfy the homogeneous boundary conditions (1c), and functions $v(x)$ (which will need to satisfy some other homogeneous boundary conditions that we will determine along the way).

Using integration by parts, we get

$$\begin{aligned}
\langle v, \mathrm{L}u\rangle = \int_0^1 v(u'' + pu')\,dx &= \int_0^1 vu''\,dx + \int_0^1 (vp)u'\,dx \\
&= (vu' + vpu)\Big|_0^1 - \int_0^1 v'u'\,dx - \int_0^1 (vp)'u\,dx \\
&= (vu' + pvu)\Big|_0^1 - \left(v'u\Big|_0^1 - \int_0^1 v''u\,dx\right) - \int_0^1 (vp)'u\,dx \\
&= (vu' + pvu - v'u)\Big|_0^1 + \int_0^1 [v'' - (vp)']\,u\,dx \\
&= \qquad 0 \qquad + \qquad \langle \mathrm{L}^* v, u\rangle
\end{aligned}$$

To match to the inner product relation, the final form of the integrand in the integral on the right must be the formal adjoint operator:

$$\mathrm{L}^* v \equiv \frac{d^2v}{dx^2} - \frac{d}{dx}\Big(p(x)v\Big) \tag{4}$$

To match the inner product relation, the boundary terms produced by the integration by parts steps must vanish for all acceptable choices of $u(x), v(x)$ functions:

$$(vu' + pvu - v'u)\Big|_0^1 = \begin{array}{r} v(1)u'(1) + p(1)v(1)u(1) - v'(1)u(1) \\ -v(0)u'(0) - p(0)v(0)u(0) + v'(0)u(0) \end{array} = 0 \tag{5}$$

Some terms can be eliminated using the boundary conditions on $u(x)$; for this problem using $u(0) = 0$ kills the last two terms, leaving

$$v(1)u'(1) + p(1)v(1)u(1) - v'(1)u(1) - v(0)u'(0) = 0 \tag{6}$$

Other boundary conditions on $u(x)$ can be used to re-write remaining terms; for this problem using $u'(1) = 0$ kills off the first term, leaving

$$p(1)v(1)u(1) - v'(1)u(1) - v(0)u'(0) = 0 \qquad \to \qquad \Big(p(1)v(1) - v'(1)\Big)u(1) - v(0)u'(0) = 0 \tag{7}$$

We have no more information about $u(x)$ to simplify this further. The remaining values of $u(1)$ and $u'(0)$ in the solution can be ANY numbers, these depend on the details[1] of the problem $Lu = f$ being solved for $u(x)$, Despite this, we still need the remaining boundary terms to always zero out; the only way to guarantee that this will happen is by imposing conditions on acceptable $v(x)$ functions so that each of the terms zero-out independently:

$$\Big(p(1)v(1) - v'(1)\Big)u(1) = 0 \qquad \xrightarrow{\forall u(1)} \qquad p(1)v(1) - v'(1) = 0 \tag{8a}$$

$$-v(0)u'(0) = 0 \qquad \xrightarrow{\forall u'(0)} \qquad v(0) = 0 \tag{8b}$$

these are the adjoint boundary conditions!

To see that this is the only guaranteed way to eliminate the boundary terms, consider two possible cases for the values of $u(1), u'(0)$ for example: (a) $\{u(1) = 3, u'(0) = 0\}$ and (b) $\{u(1) = 0, u'(0) = 2\}$. The two adjoint BC's on $v(x)$ cover these and all other possible cases of BC's from $u(x)$ functions.

So, in summary, we have used integration by parts to find the formal adjoint operator:

$$\mathrm{L}^* v \equiv \frac{d^2 v}{dx^2} - \frac{d}{dx}\Big(p(x)v\Big) \tag{9}$$

and the complete adjoint operator (L^* along with the adjoint BC's)

$$\mathrm{L}^* v \equiv \frac{d^2 v}{dx^2} - \frac{d}{dx}\Big(p(x)v\Big) \qquad v(0) = 0 \qquad p(1)v(1) - v'(1) = 0 \tag{10}$$

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[1] namely, what are $p(x), f(x)$ and any inhomogeneous BC values