Eigenfunction expansions for Inhomogeneous ODE BVP

The General Inhomogeneous Second-Order Problem

Inhomogeneous linear equation on $a \le x \le b$: $\quad \mathrm{L}u(x) = f(x) \quad$ (1a)

Inhomogeneous boundary conditions: $\quad BC_1 u(a) = c, \qquad BC_2 u(b) = d \quad$ (1b)

The Solution Process: Two steps[1]

1. Find the eigenvalues and eigenfunctions (basis fcns) $\{\,\lambda_k, \phi_k(x), \psi_k(x)\,\}$
   (a) "Homogenize" the problem: zero-out all RHS's to identify the complete linear operator:
   $$\mathrm{L}u = 0, \qquad BC_1 u(a) = 0, \qquad BC_2 u(b) = 0. \tag{2}$$
   Then write the eigenvalue problem for this operator:
   $$\mathrm{L}\phi = -\lambda\phi, \qquad BC_1\phi(a) = 0, \qquad BC_2\phi(b) = 0. \tag{3}$$
   (b) Solve the eigenvalue problem for $\{\lambda_k, \phi_k(x)\}$: Determine the general homogeneous solution, $\phi_{\text{gen}}(x)$, with λ as a constant parameter, then apply BC's to determine the condition on λ_k and finally gives the eigenfunction $\phi_k(x)$.[2]
   (c) Determine the complete adjoint operator: L^*, BC^* (use the standard L^2 inner product)
   (d) Determine the adjoint eigenfunctions $\{\psi_k(x)\}$.
   $$\mathrm{L}^*\psi = -\lambda\psi, \qquad BC_1^*\psi(a) = 0, \qquad BC_2^*\psi(b) = 0. \tag{4}$$
2. Return to original full problem and assume the solution can be written as an eigen-expansion:
   $$u(x) = \sum_{k=1}^{\infty} c_k\phi_k(x), \qquad c_k = \frac{\langle\psi_k, u\rangle}{\langle\psi_k, \phi_k\rangle}. \tag{5}$$
   (a) Start by taking the inner product of both sides of the full problem (1a) with each of the adjoint eigenfunctions
   $$\langle\psi_k, \mathrm{L}u\rangle = \langle\psi_k, f\rangle. \tag{6}$$
   (b) Use Integration By Parts (IBP) on LHS (like getting the adjoint operator)
   The IBP's will create Boundary terms ("B_k") that will depend on ψ_k and the BC values c, d from (1b)
   $$B_k + \langle\mathrm{L}^*\psi_k, u\rangle = \langle\psi_k, f\rangle. \tag{7}$$
   (c) Use the adjoint eigenvalue equation $\mathrm{L}^*\psi_k = -\lambda_k\psi_k$
   $$B_k - \lambda_k\langle\psi_k, u\rangle = \langle\psi_k, f\rangle, \tag{8}$$
   then use the formula for the expansion coefficient c_k: $\langle\psi_k, u\rangle = c_k\langle\psi_k, \phi_k\rangle$,
   $$B_k - \lambda_k c_k\langle\psi_k, \phi_k\rangle = \langle\psi_k, f\rangle. \tag{9}$$
   Solve for $\boxed{c_k}$ and then re-assemble to write the final solution for $u(x)$ as the expansion
   $$u(x) = \sum_{k=1}^{\infty} c_k\phi_k(x) \qquad c_k = \frac{\langle\psi_k, f\rangle - B_k}{-\lambda_k\langle\psi_k, \phi_k\rangle} \tag{10}$$

[1] As a process for constructing the solution of problems, this applies to all linear problems, but don't try to reduce it to a specific final "solution formula." The steps are always the same, but some problems details can enter...

[2] One scaling constant for the ϕ_k will be undetermined, pick a 'clean' convenient choice.

An example in detail

Problem: Solve for $u(x)$ on the interval $0 \le x \le 1$:

$$\frac{d^2u}{dx^2} = 9e^{4x}, \qquad u(0) = -5, \qquad u(1) = -7. \tag{11}$$

Solution process:

1. Homogenize the problem:

$$\frac{d^2u}{dx^2} = 0, \qquad u(0) = 0, \qquad u(1) = 0. \tag{12}$$

   (a) Solve the (homogeneous) eigenvalue problem:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \phi(0) = 0, \qquad \phi(1) = 0. \tag{13a}$$

   General solution (before BC's): $\phi_{\text{gen}}(x) = c_1 \sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x)$, then first BC gives $\phi_g(0) = c_2 = 0$, then second BC $\phi_g(1) = c_1 \sin(\sqrt{\lambda}) = 0$ ($c_1 \neq 0$) sets condition for λ_k's: $\sin(\sqrt{\lambda}) = 0$,

$$\lambda_k = k^2\pi^2, \qquad \phi_k(x) = \sin(k\pi x), \qquad k = 1, 2, 3, \cdots. \tag{13b}$$

   (b) Use integration by parts to determine the adjoint eigenvalue problem,

$$\frac{d^2\psi}{dx^2} = -\lambda\psi, \qquad \psi(0) = 0, \qquad \psi(1) = 0. \tag{14a}$$

   (c) The corresponding adjoint eigenfunctions are

$$\psi_k(x) = \sin(k\pi x) \qquad \text{and} \qquad \langle \psi_k, \phi_k \rangle = \int_0^1 \sin^2(k\pi x)\,dx = \frac{1}{2} \qquad k = 1, 2, 3, \cdots \tag{14b}$$

2. Returning to the full problem, we seek an expansion for the solution in the form

$$u(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x), \qquad c_k = \frac{\langle u, \psi_k \rangle}{\langle \phi_k, \psi_k \rangle} = 2\int_0^1 u(x)\sin(k\pi x)\,dx. \tag{15}$$

   (a) Take the inner product of the equation $u'' = f(x)$ with each adjoint eigenfunction, $k = 1, 2, \cdots$

$$\int_0^1 \underbrace{\psi_k(x)}_{u}\,\underbrace{u''\,dx}_{dv} = \int_0^1 f(x)\psi_k(x)\,dx. \tag{16}$$

   (b) Use IBP twice on LHS

$$\left(\psi_k u' - \psi_k' u\right)\Big|_0^1 + \int_0^1 \psi_k'' u\,dx = \int_0^1 f(x)\psi_k(x)\,dx. \tag{17}$$

   (c) Use adjoint eigenvalue equation $\psi_k'' = -\lambda_k\psi_k$ in the integral

$$\left(\psi_k u' - \psi_k' u\right)\Big|_0^1 - \lambda_k\int_0^1 \psi_k u\,dx = \int_0^1 f(x)\psi_k(x)\,dx, \tag{18a}$$

$$B_k - \lambda_k c_k \langle \psi_k, \phi_k \rangle = \langle \psi_k, f \rangle, \tag{18b}$$

   where B_k is the sum of terms due to the boundary conditions. Solving (18b) for c_k, we get

$$c_k = \frac{\langle \psi_k, f \rangle - B_k}{-\lambda_k \langle \psi_k, \phi_k \rangle}\,. \tag{19}$$

We are ready to evaluate everything; we observe a few key properties for the boundary terms,

$$\psi_k'(x) = k\pi\cos(k\pi x), \qquad \psi_k(0) = \psi_k(1) = 0, \qquad \psi_k'(0) = k\pi, \qquad \psi_k'(1) = (-1)^k k\pi,$$

and use the given information that $u(0) = -5$ and $u(1) = -7$.
Substituting-in for ϕ, ψ, λ and f and the BC's into (18b),

$$\begin{aligned}
-\psi_k'(1)u(1) + \psi_k'(0)u(0) - k^2\pi^2 c_k \int_0^1 \sin^2(k\pi x)\,dx &= \int_0^1 9e^{4x}\sin(k\pi x)\,dx, && (20a)\\
7(-1)^k k\pi - 5k\pi - \tfrac{1}{2}k^2\pi^2 c_k &= -\frac{9k\pi(-1+(-1)^k e^4)}{16+k^2\pi^2}, && (20b)
\end{aligned}$$

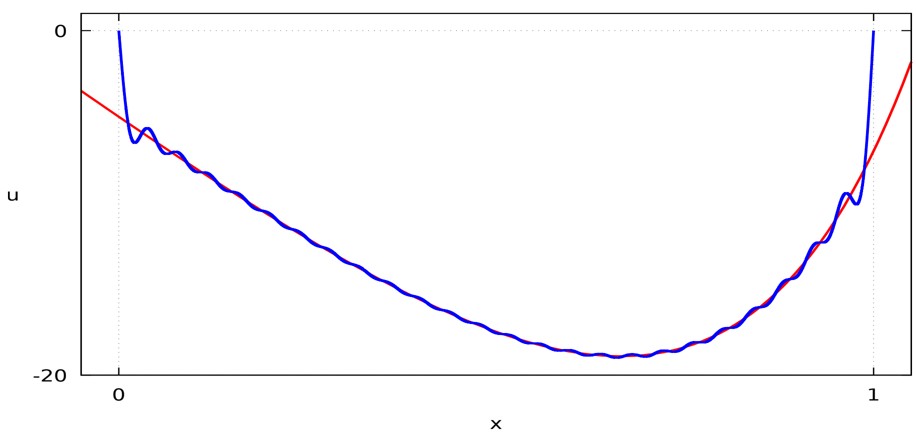
Note that the boundary terms involving $\psi_k u'$ are zero because of the BC's on ψ_k in (4), but the $\psi_k' u$ terms remain because of the BC's on u in (1).[3]

So finally,

$$u(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x) \qquad c_k = \frac{14(-1)^k - 10}{k\pi} + \frac{18(-1+(-1)^k e^4)}{k\pi(16+k^2\pi^2)} \tag{21}$$

and gives the solution $u(x)$ is as a Fourier sine series (15). See the figure for a plot of the Fourier series (first 50 c_k terms) compared against the exact solution of this problem,

$$u_{\text{exact}}(x) = \frac{1}{16}\left(9e^{4x} - [9e^4 + 23]x - 89\right). \tag{22}$$



Note the excellent agreement everywhere except at the boundaries. At $x = 0$ and $x = 1$, we have $\sin(0) = \sin(k\pi) = 0$ so the Fourier series cannot converge to the boundary conditions, but it does the best that it can; observe the Gibbs phenomenon at the jumps where the odd periodic extension cannot match the exact solution, but it converges in L^2 norm.
Also note that none of the steps involved taking derivatives applied to the expansion for $u(x)$. The use of the adjoints and inner products (sometimes called solving the "weak form" of the problem), avoided these issues.[4] If we had tried to take a short-cut by plugging the series (15) directly into (11) to find the c_k's **IT WOULD NOT WORK** – there would be no way for the boundary conditions to come into the c_k's – **DO NOT TRY** this for any problem with inhomogeneous boundary conditions!

If we had been given the exact solution (22) to start with, we'd find that (21) are its Fourier sine series coefficients from (15). Using the solution process we have been able to work out the solution by getting the c_k's without already knowing (22)!

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[3] Note that the regular and adjoint eigenfunctions are always defined with homogeneous boundary conditions.
[4] Spectral and Finite Element Methods (FEM) in numerical analysis also use this general approach.