Math 551: Applied PDE and Complex Vars — Lecture 9

Part 1: Fredholm Integral equations: Solving $\mathbf{L}u = f(x)$ for FIE (conclusion)

Part 2: The Green's function for solving ODE BVP

$\boxed{\textbf{Part 1}}$: $\mathbf{L_1}u(x) = \int_a^b K(x,t)u(t)\,dt$ Degenerate FIE$_1$, n-term separable kernel fcn: $K(x,t) = \sum_{j=1}^n \alpha_j(x)\beta_j(t)$

1. Finite-multiplicity eigenmodes ($\mathbf{L}\phi = \lambda\phi$ with $\lambda \neq 0$): $n \times n$ matrix eigenvalue problem to get $\{\lambda_k, \phi_k, \psi_k\}_{k=1\cdots n}$ with each $\phi(x) = \sum_j c_j\alpha_j(x)$ and separable FIE$_1$ always has infinite-multiplicity zero eigenvalue $\lambda_0^\infty = 0$
2. FAT: $\mathbf{L_1}$ having a zero eigenvalue means soln of $\mathbf{L_1}u = f$ is never unique.
   $u(x) = \sum_{k=1}^n c_k\phi_k(x) + \sum_m c_m^\infty \phi_{0,m}^\infty(x)$ (ϕ^∞s contrib to solns) but
   $\mathbf{L_1}u = \sum_{k=1}^n c_k\lambda_k\phi_k(x) + 0$ (but ϕ^∞s dont help balance LHS vs RHS)
   No soln possible if $f \neq \sum_{j=1}^n f_j\alpha_j(x)$
3. Practical approach: construct $\boxed{\textbf{A}}$ soln of $\mathbf{L_1}u = f$, either:
   - Use $u = \sum_k c_k\phi_k$, the equation for each c_k is decoupled.
     (but you need to work out the $\lambda_k, \phi_k(x), \psi_k(x)$'s first) (eigenfunctions)
   - Use $u = \sum_j d_j\alpha_j$, then must solve coupled algebra eqns for d_j's.
     (no λ, ϕ, ψ's needed!) (un-determined coefficients)
4. Solving $\mathbf{L_2}u = f$ with $L_2u = \gamma u + L_1u$
   Use undetermined coefficients $u = \frac{1}{\gamma}f(x) + \sum_j d_j\alpha_j$ will work with ANY $f(x)$

$\boxed{\text{Part 2}}$: Introduction to Green's functions

Return to ODE BVP problems for $u(x)$

$$\mathbf{L}u = f(x) \qquad a \le x \le b$$

with homogeneous BC's:

$$BC_a u = 0 \qquad BC_b u = 0$$

Definition: The **Green's function**[a] for an ODE BVP with homogeneous BC's is the kernel function $G(x,t)$ that gives the solution of the ODE as the inner product of the Green's fcn with the forcing:

$$\boxed{u(x) = \langle G(x,t), f(t) \rangle} \qquad \leftrightarrow \qquad u(x) = \int_a^b G(x,t) f(t)\, dt$$

- This is called an "integral representation" of the solution – it requires working out an integral to get u at any x.
- How messy this might be depends a lot on the form of the Green's function $G(x,t)$ – there are a few ways to write it...

[a] Named after George Green (UK 1800s, Green's theorem...), NOT the color green

Recall, solving ODE BVP

1. Non-self-adjoint inhomogeneous BVP's: $\mathbf{L}u = f,\ BC_a u = c,\ BC_b u = d$

$$u(x) = \sum_k c_k \phi_k(x) \qquad c_k = \frac{B_k(c,d) - \langle \psi_k, f\rangle_2}{\lambda_k \langle \psi_k, \phi_k\rangle_2}$$

2. ... with homogeneous BC's: $BC_a u = 0, BC_b u = 0$

$$u(x) = \sum_k c_k \phi_k(x) \qquad c_k = -\frac{\langle \psi_k, f\rangle_2}{\lambda_k \langle \psi_k, \phi_k\rangle_2}$$

Then re-group/re-interpret:

$$\begin{aligned} u(x) &= \sum_k \left(-\frac{1}{\lambda_k \langle \psi_k, \phi_k\rangle} \int_a^b \psi_k(t) f(t)\, dt \right) \phi_k(x) \\ &= \int_a^b \left(-\sum_k \frac{\psi_k(t)\phi_k(x)}{\lambda_k \langle \psi_k, \phi_k\rangle} \right) f(t)\, dt \\ u(x) &= \int_a^b G(x,t) f(t)\, dt \end{aligned}$$

Green's fcn (v1.0) $\boxed{G(x,t) = -\sum_{k=1}^{\infty} \frac{\psi_k(t)\phi_k(x)}{\lambda_k \langle \psi_k, \phi_k\rangle}}$ (bi-linear eigen-expansion)

Green's function (v1.0) (continued)

- If problem is self-adjoint ($\mathbf{L}^* = \mathbf{L}$) then $\psi_k(x) = \phi_k(x)$ and $G(x,t) = G(t,x)$ symmetric kernel fcn:

$$G(x,t) = -\sum_{k=1}^{\infty} \frac{\phi_k(t)\phi_k(x)}{\lambda_k ||\phi_k||^2}$$

- ODE BVP: given $f(x)$, solve $\mathbf{L}u = f$ for $u(x)$ (forward problem)

  $$\updownarrow$$

  FIE prob: given $u(x)$ solve $u = \int G f\, dt$ for $f(x)$ (inverse problem)

  and

  If FIE is self-adjoint then the ODE BVP is also self-adjoint

  (a 2nd way to justify that $G(x,t) = G(t,x)$)

- Integral soln $u(x) = \int G f\, dt$ of ODE BVP sometimes "conceptually" written as $u = \mathbf{L}^{-1} f$ with $\mathbf{L}^{-1} v \equiv \int G v\, dt$

  $$(\mathbf{L}u = \mathbf{L}(\mathbf{L}^{-1} f) = f)$$

- Infinite series for $G(x,t)$, not a degenerate (separable) kernel, there is no λ^{∞}.

Options for solving ODE BVP $\mathbf{L}u = f$

- (Math 551, v1.0) Eigen-expansion: use $\{\boldsymbol{\lambda}, \boldsymbol{\phi}, \boldsymbol{\psi}\}$ of $\mathbf{L}$: $\boldsymbol{u = \sum c_k \phi_k(x)}$
  Long but full-proof, and extends to PDE
  Green's fcn v1.0 is part of this approach.

- (ODE class) $\boldsymbol{u(x) = u_{\mathbf{hom}}(x) + u_{\mathbf{par}}(x)}$ Particular solution?
  (a) Un-determined coefficients (see `SummarySheets/Undetcoeff.pdf`)
  Trial soln for $\boldsymbol{u_{\mathbf{par}}(x)}$ and matching ODE LHS/RHS terms
  Quick, easy, but only works for "simple $\boldsymbol{f(x)}$" (very limited)
  (b) Variation of Parameters: $\boldsymbol{u_1, u_2}$ fund. hom. solns (see `L09a.pdf` derivation)

$$\begin{aligned} u(x) &= c_1(x)u_1(x) + c_2(x)u_2(x) \\ &= \left(-\int_b^x \frac{u_2(t)f(t)}{W(t)}\,dt\right)u_1(x) + \left(\int_a^x \frac{u_1(t)f(t)}{W(t)}\,dt\right)u_2(x) \end{aligned}$$

  Works for all $\boldsymbol{f(x)}$'s but longer derivation of the integrals....
  Do not use in Math 551.

- (Math 551, v2.0) **Green's fcn v2.0**: Distribution theory, a better version without $\{\boldsymbol{\lambda}, \boldsymbol{\phi}, \boldsymbol{\psi}\}$! First, use Vari. of Params to get v1.9 version of this

Background/theory for Green's fcns

Distribution theory: a better way to get the piecewise-defined Green's function

History: the theory of "generalized fcns and distributions"

- Oliver Heaviside (British EE), Heaviside step function
- Paul Dirac (British Quantum Physics), Dirac delta function

The Heaviside step function: $H(x - x_*)$ has unit jump at $x = x_*$

$$H(x - x_*) = \begin{cases} 0 & x < x_* \\ 1 & x > x_* \end{cases}$$

It "switches" other functions off/on:

$$f(x)H(x - x_*) = \begin{cases} 0 & x < x_* \\ f(x) & x > x_* \end{cases}$$

and chops off integrals of fcns at x_* in $a < x_* < b$

$$\langle f(x), H(x - x_*) \rangle = \int_a^b fH\,dx = \int_{x_*}^b f(x)\,dx$$

<u>The Dirac delta function</u>: $\delta(x - x_*)$ has an infinite "spike" at $x = x_*$

fcn "values" (sort of)

area= "∞" $\cdot\, 0 = 1$

$$\delta(x - x_*) = \begin{cases} \text{“}\infty\text{”} & x = x_* \\ 0 & x \neq x_* \end{cases}$$

The real working definition: $\boxed{\delta(x) = H'(x)}$

What does the delta function do?

$\boxed{\textbf{The sifting property}}$ for any nice $f(x)$: δ pulls out a single value of f from inside an integral:

$$\langle f, \delta(x - x_*)\rangle = \boxed{\int_a^b f(x)\delta(x - x_*)\, dx = f(x_*)} \qquad a < x_* < b$$

Proof of the sifting property via IBP

$$\begin{aligned} \int_a^b f(x)H'(x - x_*)\, dx &= f(x)H(x - x_*)\Big|_a^b - \int_a^b f'(x)H(x - x_*)\, dx \\ &= (f(b) - 0) - \int_{x_*}^b f'(x)\, dx \\ &= (f(b) - 0) - (f(b) - f(x_*)) \\ &= f(x_*) \end{aligned}$$

Green's fcn (v2.0) for solving ODE BVP: $\mathbf{L}u = f(x)$ on $a \le x \le b$

Warning: Will need to switch around independent variables x or t or s ... to make final solution look right, $u = u(x)$, so keep track and follow along step by step!
(Haberman uses x, x_0 – dont use these, easy to mix them up by accident)

(Other books use x and ζ or ξ – dont use these squiggly Greek letters, also easily confused and hard to write neatly)

Switch ODE BVP to be in terms of t (instead of x) on $a \le t \le b$:

$$\mathbf{L}_t u(t) = f(t) \qquad BC_a u = 0 \qquad BC_b u = 0$$