Recap of linear differential operators

- Standard (unweighted) L^2 inner product $\langle f, g\rangle_2 \equiv \int_a^b f(x)g(x)\,dx$
- The adjoint relation $\langle v, \mathbf{L}u\rangle_2 = \langle \mathbf{L}^* v, u\rangle_2$ calculation defines $\mathbf{L}^*, BC^*$
- **The (unweighted) eigenvalue problem** **version 1.0**

$$\mathbf{L}\phi = -\lambda\phi \qquad \text{and} \qquad \mathbf{L}^*\psi = -\lambda\psi$$

yields L^2 bi-orthogonality

$$\langle \phi_j, \psi_k\rangle_2 = 0 \qquad \text{if } j \neq k$$

If $\mathbf{L}$ is self-adjoint, then $\psi_k = \phi_k$ and ϕ's are a self-orthogonal set

$$\langle \phi_j, \phi_k\rangle_2 = 0 \qquad \text{if } j \neq k$$

Solution of $\mathbf{L}u = f$ with BC's is

$$u(x) = \sum_{k=1}^{\infty} c_k\phi_k \qquad c_k = \frac{\langle \psi_k, f\rangle_2 - B_k}{-\lambda_k\langle \psi_k, \phi_k\rangle_2}$$

Sturm-Liouville operators (2nd order ODE) **[Haberman, Chap 5.3]**

$$\widetilde{\mathrm{L}}u \equiv \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u \qquad a \le x \le b$$

$$\widetilde{BC}_1 u(a) \equiv \alpha_1 u(a) + \alpha_2 u'(a) \qquad \widetilde{BC}_2 u(b) \equiv \beta_1 u(b) + \beta_2 u'(b)$$

General separated ('un-mixed') boundary conditions (1st, 2nd, or 3rd kind BC's), with coeffs $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$ (i.e. not everything zeroed out).

SL operators are self-adjoint in the standard L^2 inner product

$$\langle v, \widetilde{\mathrm{L}}u\rangle_2 = \langle \widetilde{\mathrm{L}}v, u\rangle_2 \qquad \leftrightarrow \qquad \widetilde{\mathrm{L}}^* = \widetilde{\mathrm{L}}, \quad \widetilde{BC}^* = \widetilde{BC}$$

The SL weighted eigenvalue problem with a given $\sigma(x) \ge 0$ weight fcn:

version 2.0 $\quad \boxed{\widetilde{\mathrm{L}}\phi = -\lambda\sigma(x)\phi} \quad \widetilde{BC}_1\phi(a) = 0 \quad \widetilde{BC}_2\phi(b) = 0$

The eigenfunctions $\{\phi_k(x)\}$ are self-orthogonal in the σ-weighted product ($\perp_\sigma$):

$$\langle f, g\rangle_\sigma \equiv \int_a^b f(x)g(x)\sigma(x)\,dx \qquad \rightarrow \qquad \boxed{\langle \phi_j, \phi_k\rangle_\sigma = 0 \quad \text{for } j \ne k}$$

and the eigvals λ_k are all real. Each $\sigma(x)$ fcn creates a different set of $\{\lambda_k, \phi_k\}$

Important basic Sturm-Liouville results (I)

1. The σ-weighted eigen-expansion for any given L^2 function $g(x)$

$$g(x) = \sum_{k=1}^{\infty} c_k \phi_k(x) \qquad c_k = \frac{\langle \phi_k, g \rangle_\sigma}{\langle \phi_k, \phi_k \rangle_\sigma} = \frac{1}{||\phi_k||_\sigma^2} \int_a^b g \phi_k \sigma \, dx$$

2. **Solving inhomogeneous Boundary Value Probs (BVP) with SL operators**:

$$\widetilde{\mathbf{L}} u = f(x) \qquad \widetilde{BC}_1 u(a) = c \qquad \widetilde{BC}_2 u(b) = d$$

Project the ODE $\widetilde{\mathbf{L}} u = f$ onto ϕ_k using the standard L^2 inner product:

$$\begin{aligned}
\langle \phi_k, \widetilde{\mathbf{L}} u \rangle_2 &= \langle \phi_k, f \rangle_2 \\
\tilde{B}_k + \langle \widetilde{\mathbf{L}}^* \phi_k, u \rangle_2 &= \\
\tilde{B}_k + \langle \boxed{-\lambda_k \sigma \phi_k}, u \rangle_2 &= \\
\tilde{B}_k - \lambda_k \langle \phi_k, u \rangle_{\boxed{\sigma}} &= \\
\tilde{B}_k - \lambda_k \underline{c_k \langle \phi_k, \phi_k \rangle_\sigma} &= \langle \phi_k, f \rangle_2 \\
\boxed{p(x)\left(\phi u' - \phi' u\right)\Big|_a^b - \lambda_k c_k \int_a^b \phi_k^2(x) \sigma(x) \, dx} &= \boxed{\int_a^b \phi_k(x) f(x) \, dx}
\end{aligned}$$

Important basic Sturm-Liouville results (II)

3. **Second-order Linear Differential Operators**: General form

$$\mathbf{L}u \equiv A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u \qquad \text{and separated BC's}$$

In general, $\mathbf{L}$ is not self-adjoint in L^2 prod: $\mathbf{L}^* \neq \mathbf{L}$ for $\langle v, \mathbf{L}u\rangle_2 = \langle \mathbf{L}^* v, u\rangle_2$

BUT.... $\boxed{\textbf{Any 2nd-order } \mathbf{L} \textbf{ can be converted into a } L^2\textbf{-self-adjoint SL } \widetilde{\mathbf{L}} \textbf{ !!}}$

$$\mathbf{L} = \frac{1}{\sigma(x)}\widetilde{\mathbf{L}}$$

Need to find the right σ, p, q from $\mathbf{L}$'s A, B, C (HW3)

then the eigenvalue problems are related by

$$\mathbf{L}\phi = -\lambda\phi \qquad \leftrightarrow \qquad \widetilde{\mathbf{L}}\phi = -\lambda\sigma\phi \qquad \text{(same } \lambda\text{'s, same } \phi(x)\text{'s)}$$

and the inner products are related by

$$\langle v, \mathbf{L}u\rangle_\sigma = \int_a^b v(\mathbf{L}u)\sigma\, dx = \int_a^b \frac{v}{\sigma}(\widetilde{\mathbf{L}}u)\sigma\, dx = \cdots$$

So, $\mathbf{L}$ is σ-self-adjoint: $\langle v, \mathbf{L}u\rangle_\sigma = \langle \mathbf{L}v, u\rangle_\sigma$ for the right choice of $\sigma(x)$!

Important basic Sturm-Liouville results (III)

4. **General inhomogeneous second order BVP's**:

$$\mathbf{L}u = f(x) \qquad BC_1 u(a) = c \qquad BC_2 u(b) = d$$

Solution = Eigenfunction expansion $\qquad u(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$

Can either:

(a) Find c_k's using usual bi-orthogonality (from projections on $\psi_j, j = 1, 2, \cdots$)

$$B_k - \lambda_k c_k \langle \psi_k, \phi_k \rangle_2 = \langle \psi_k, f \rangle_2 \qquad \text{(Lecture 5)}$$

OR

(b) Convert to SL form, then find c_k's (same values)

$$\mathbf{L}u = f \quad \to \quad \frac{1}{\sigma}\widetilde{\mathbf{L}}u = f \quad \to \quad \widetilde{\mathbf{L}}u = \sigma f$$

then

$$\tilde{B}_k - \lambda_k c_k \langle \phi_k, \phi_k \rangle_\sigma = \langle \phi_k, f \rangle_\sigma \qquad \text{(Compare w/Slide \#2!?)}$$

Work trade-offs: (a) Need $\mathbf{L}^*, BC^*, \psi_k$, vs. (b) Need SL $\widetilde{\mathbf{L}}$ form (p, q, σ)

Important specialized results: **Sturm-Liouville Theory** $\widetilde{\mathbf{L}}\phi = -\lambda\sigma\phi$

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi = -\lambda\sigma(x)\phi$$

0. Reading: Haberman, sections 5.3, 5.5, 5.6.
1. **"Regular" SL problems** are SL with the additional condition that

$$p(x) > 0 \qquad \text{on } a \leq x \leq b$$

then the problem gains the following properties (results of SL theorems):

(a) There is a smallest eigenvalue, λ_1 (a finite real number).

(b) The eigenvalues are discrete, distinct, ordered real numbers:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \tag{1}$$

(i.e. there are no double, triple, etc. roots).

(c) The sequence of eigenvalues has $\lambda_k \to \infty$ as $k \to \infty$.

(d) There is a unique eigenfunction $\phi_k(x)$ for each λ_k.

(e) The $\phi_k(x)$ are oscillatory fcns with $(k-1)$ zeros on $a < x < b$ for $k = 1, 2, \cdots$ (this result is called "the oscillation theorem")

Regular SL problems: results of the SL theorems (continued)

(f) The $\{\phi_k(x)\}$ eig-fcns are a **complete basis** for expansion of L^2 fcns

$$g(x) = \sum_{k=1}^{\infty} c_k \phi_k(x) \qquad c_k = \frac{\langle \phi_k, g \rangle_\sigma}{\langle \phi_k, \phi_k \rangle_\sigma}$$

(g) For Regular SL problem ($p > 0$) with two additional conditions:

$$q(x) \leq 0 \qquad \text{for } a \leq x \leq b \tag{1}$$

and boundary conditions with coefficients:

$$\alpha_1 \alpha_2 \leq 0 \qquad \text{and} \qquad \beta_1 \beta_2 \geq 0 \tag{2}$$

then we get the important result: $\boxed{\text{all } \lambda_k \geq 0}$ Proof: Start with

$$\begin{aligned}
\langle \widetilde{\mathbf{L}} \phi_k,\ \phi_k \rangle_2 &= \langle -\lambda_k \sigma \phi_k,\ \phi_k \rangle_2 \\
\int_a^b \underbrace{\phi_k}\ \underbrace{(p\phi_k')'\, dx} + \int_a^b q\phi_k^2\, dx &= -\lambda_k \int_a^b \phi_k^2 \sigma\, dx \\
p\phi_k \phi_k' \Big|_a^b - \int_a^b p\left(\phi_k'\right)^2\, dx + \int_a^b q\phi_k^2\, dx &= -\lambda_k \int_a^b \phi_k^2 \sigma\, dx
\end{aligned}$$

Regular SL problems: Proof of (g) on getting $\lambda \geq 0$ (concluded)

Use boundary conditions: $\alpha_1\phi(a) + \alpha_2\phi'(a) = 0$ with $\alpha_1\alpha_2 \leq 0$

$$\phi'(a) = -\tfrac{\alpha_1}{\alpha_2}\phi(a) \;\text{ if } \alpha_2 \neq 0 \qquad \text{else } \phi(a) = 0 \;\text{ if } \alpha_2 = 0$$

So boundary terms become

$$p(a)\phi(a)\phi'(a) = \underbrace{-\frac{\alpha_1}{\alpha_2}}_{\geq 0}\underbrace{p(a)}_{>0}\underbrace{\phi^2(a)}_{\geq 0} \qquad \geq 0$$

similarly, using $\beta_1\beta_2 \geq 0$

$$p(b)\phi(b)\phi'(b) = -\frac{\beta_1}{\beta_2}p(b)\phi^2(b) \qquad \leq 0$$

and use $q(x) \leq 0$ so $\int q\phi^2\,dx \leq 0$

$$\lambda_k = \frac{1}{||\phi_k||_\sigma^2}\left(p\phi\phi'\Big|_{x=a} - p\phi\phi'\Big|_{x=b} - \int_a^b q\phi^2\,dx + \int_a^b p(\phi')^2\,dx\right) \quad \geq 0$$

The "Rayleigh quotient" $\dfrac{\langle u, Lu\rangle}{\langle u, u\rangle}$ is used in many theory proofs and numerical computations of eigenvalues...

Important examples of SL probs (I): $\{a, b, \widetilde{\mathrm{L}}[p, q, \sigma], \widetilde{BC}[\alpha_1, \alpha_2, \beta_1, \beta_2]\}$

1. $0 \le x \le 1$ with $p(x) \equiv 1$ and $q(x) \equiv 0$ and $\sigma(x) \equiv 1$

$$\widetilde{\mathrm{L}}u = \frac{d^2u}{dx^2} \qquad : \qquad \widetilde{\mathrm{L}}\phi = -\lambda\sigma\phi \quad \Longrightarrow \quad \frac{d^2\phi}{dx^2} = -\lambda\phi$$

Pick boundary conditions to have $\alpha_1\alpha_2 = 0$ and $\beta_1\beta_2 = 0$ ((g) Slide #6)
then $\boxed{\lambda \ge 0}$, and general soln:

$$\phi_{\text{gen}}(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

Subcases

1. Dirichlet BC's ($\alpha_2 = \beta_2 = 0$): $\phi(0) = 0, \phi(1) = 0$
   This is the **Fourier Sine Series** – Regular SL problem.
   SL theory: $\{\phi_k(x), k = 1, 2, \cdots\}$ is **complete** for L^2 expansions
2. Neumann BC's ($\alpha_1 = \beta_1 = 0$): $\phi'(0) = 0, \phi'(1) = 0$
   This is the **Fourier Cosine Series** – Regular SL problem.
   SL theory: $\{\phi_k(x), k = 0, 1, 2, \cdots\}$ is **complete** for L^2 expansions
3. But...

Singular SL problems

If any of the assumptions/conditions for regular SL problems are *violated* then the problem is called "singular" and some of the results (on λ's or $\phi(x)$'s) may work differently.

"Violations" usually involve one of:

(a) Forms of the $p(x), q(x), \sigma(x)$ functions.
Singular endpoints: If $p(x) = 0$ at a boundary $(x = a$ or $x = b)$ then one of the two linearly independent homogeneous solutions in the general solution may be "singular" ($\phi_1 \to \infty$ or $\phi_1' \to \infty$ or ...) there and hence would be un-usable for building the eigenfunction.
Then no boundary condition is needed at the singular endpoint and bounded, well-behaved eigenfcns $\phi_k(x)$ will be determined by the other BC.

(b) Boundary conditions
Violating the assumptions of unmixed BCs on finite domains $a \leq x \leq b$ will change results for λ_k.

(c) Singular domain: $a \leq x \leq b$ with $a = -\infty$ or $b = \infty$

Singular SL examples (II): $\{a, b, \widetilde{\mathrm{L}}[p, q, \sigma], \widetilde{BC}[\alpha_1, \alpha_2, \beta_1, \beta_2]\}$

3b. $0 \leq x \leq 1$ with $p(x) \equiv 1$ and $q(x) \equiv 0$ and $\sigma(x) \equiv 1$

$$\widetilde{\mathrm{L}}u = \frac{d^2u}{dx^2} \quad : \quad \widetilde{\mathrm{L}}\phi = -\lambda\sigma\phi \quad \Longrightarrow \quad \frac{d^2\phi}{dx^2} = -\lambda\phi$$

with periodic boundary conditions

$$\phi(0) = \phi(1) \qquad \phi'(0) = \phi'(1)$$

- Mixed BC's (info from $x = 0, x = 1$ together in same eqns
- Violates Regular SL un-mixed BC's condition, so it is a Singular problem!
- This is the **Full Fourier Series** – Singular SL problem.
- But most SL results still hold:
  - $\widetilde{\mathrm{L}}$ self-adjoint
  - λ real and $\lambda \geq 0$ too!
  - $\{\phi_k(x)\}$ **complete** for L^2 expansions
- The only thing the is violated is now there are **TWO** eigenfunctions for each $k = 1, 2, \cdots$ (and $\phi_0(x) \equiv 1$ for $\lambda_0 = 0$)

$$\lambda_k = (2\pi k)^2 \qquad \phi_k(x) = \cos(2\pi kx) \qquad \phi_k(x) = \sin(2\pi kx)$$

<u>**Singular SL examples**</u> (III): $\{a, b, \widetilde{\mathrm{L}}[p, q, \sigma], \widetilde{BC}[\alpha_1, \alpha_2, \beta_1, \beta_2]\}$

3a. A singular endpoint problem for formal SL operator:

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) - \frac{1}{4x}\phi = -\lambda x\phi \qquad 0 \le x \le 1$$

Can read off SL coeff fcns: $\quad p(x) = x \qquad q(x) = -\dfrac{1}{4x} \qquad \sigma(x) = x$

- It is a singular problem because $p(0) = 0$ violates $p > 0$ condition at $x = 0$
- This problem has general solution

$$\phi_{\mathrm{gen}}(x) = c_1\frac{\sin(\sqrt{\lambda}\,x)}{\sqrt{x}} + c_2\frac{\cos(\sqrt{\lambda}\,x)}{\sqrt{x}}$$

- $\displaystyle\lim_{x\to 0}\frac{\sin(\sqrt{\lambda}\,x)}{\sqrt{x}} = 0$ (OK) $\qquad \displaystyle\lim_{x\to 0}\frac{\cos(\sqrt{\lambda}\,x)}{\sqrt{x}} = \infty$ (Blows-up, **BAD!**)
- So $\phi(x) = \sin(\sqrt{\lambda}\,x)/\sqrt{x}$ is the only acceptable bounded soln.
  No need for a BC at $x = 0$, just kill the "bad" soln ($c_2 = 0$).
- In this problem only need one BC to pick λ's, like $\phi(1) = 0$.