Math 551, Duke University — The Method of Undetermined Coefficients

This is a method for solving homogeneous/inhom. Linear Constant Coefficient (LCC) DE's.[1]

$\boxed{\textbf{(1)}}$ Put the problem in standard form:[2]

$$y'' + py' + qy = f_0(x) + f_1(x) + \dots$$

For a 2nd order LCCDE, the **general** solution has **two** constants of integration.
For a solution of a specific **initial (or boundary) value problem** for a 2nd order LCCDE, you will be given **two** conditions that you can use to solve for the **two** unknown constants of integration at the end.
If RHS $= 0$ then the DE is homogeneous (unforced) and you only need the homogeneous solution (step 2),

$$y(x) = y_h(x)$$

If RHS $\neq 0$ then the DE is called inhomogeneous and the general solution is the sum of solutions from step 2 and step 3,

$$\boxed{y(x) = y_h(x) + y_p(x)}$$

where $y_p(x)$ is the "particular" (forced) solution, and it includes contributions from each of the RHS $f(x)$ terms. For this method to work, each of the $f(x)$ terms on the RHS must look like

$$\boxed{f(x) = P(x)e^{Rx}} = (\text{polynomial}) \cdot (\text{exponential})$$

Constant R can be complex and the exponential can be a complex exponential with sine and/or cosine terms. If the RHS is not of this form then UC will not work and you must use the variation of parameters method for $y_p(x)$.

$\boxed{\textbf{(2)}}$ FIRST, **ignore the RHS**, solve the homogeneous DE (RHS=0):

$$y'' + py' + qy = 0$$

Use the trial solution form:

$$y(x) = e^{rx}$$

You can use the quick substitution rule:[3]

$$y \to r^0 = 1 \qquad y' \to r^1 = r \qquad y'' \to r^2 \qquad \dots$$

to get the characteristic polynomial equation

$$r^2 + pr + q = 0$$

Get the roots r_1 and r_2.[4]
The DE will have one solution[5] for each root. The first solution is always

$$y_1(x) = e^{r_1 x}$$

If $r_2 \neq r_1$, then the other solution of the DE is

$$y_2(x) = e^{r_2 x}$$

If $r_2 = r_1$ (a double root), then the second solution for r_1 is $y_1(x)$ multiplied by x,

$$y_2(x) = xy_1(x) = xe^{r_1 x}$$

If r_1 is complex, $r_1 = a + ib$, then

$$y_1(x) = e^{ax}e^{ibx}$$

The general solution of the homogeneous DE is the sum of these solutions, each multiplied by a different constant

$$\boxed{y_h(x) = c_1 y_1(x) + c_2 y_2(x)}$$

For the complex case, by playing with the constants, all solutions can be written in terms of purely real functions:

$$y_h(x) = c_3 e^{ax}\cos bx + c_4 e^{ax}\sin bx$$

or

$$y_h(x) = Ae^{ax}\cos(bx - \phi)$$

$\boxed{\textbf{(3)}}$ Finally, focus on the RHS of the ODE to guess the "particular forcing" solutions. There will be one $y_p(x)$ solution from each $f(x)$ term on the RHS and the final particular solution will be the sum

$$y_p(x) = y_{p0}(x) + y_{p1}(x) + \dots$$

The final $y_p(x)$ solution will not have any unknown constants. To figure out the proper form of $y_p(x)$, look at each $f(x)$ term and let

$$\boxed{y_p(x) = Q(x)x^m e^{Rx}}$$

R in $y_p(x)$ is the same as R in $f(x)$

$$Q(x) = Ax^n + Bx^{n-1} + Cx^{n-2} + \dots + Z$$

is a polynomial of the same order as $P(x)$ in $f(x)$, but you will have to figure out all of the unknown coefficients, $A, B, \cdots$.
What's $\mathbf{x^m}$? If R from $f(x)$ is the same number as one of the roots of the characteristic polynomial from the LHS, then you need to multiply the "usual" $y_p(x)$ up by x (just like with a double root). If R matches m roots then multiply up by x^m. Example $R = r_1 = r_2$, then $m = 2$. If R is not r_1 or r_2 then $m = 0$. If R is complex, then guess

$$y_p(x) = Q_1(x)x^m e^{ax}\cos(bx) + Q_2(x)x^m e^{ax}\sin(bx)$$

After that, plug $y_p(x)$ into the original LCCDE and grunge through the algebra until you determine all of the coefficients in $Q(x)$.

Notes

[1] This trick works for 1st, 2nd, ... any-th order LCCDE's, but 2nd order is most common. This also works similarly for Cauchy-Euler ODEs

[2] Standard form is (a sum of constants times derivatives of $y(x)$ on the LHS) equals (a sum of functions $f(x)$ on the RHS).

[3] Namely, derivatives of $y \to r^{(\text{to the order of the derivative of } y(x))}$

[4] Factor the polynomial or use the quadratic formula.

[5] one distinct "linearly independent" solution for each root