# Math 551, Applied PDE: **The Laplacian** (continued) — Lecture 18

**Part 1**: **Re-cap of L17**: Background on the Laplacian (H, Ch. 7.4, 7.5)

- Solving problems (Poisson) via eigen-expansion or space-time separation of variables, $u(\mathbf{x},t)=\phi(\mathbf{x})h(t)$ for many problems (heat, wave) yields the **Helmholtz** PDE eigenvalue problem: $\boxed{\nabla^2\phi=-\lambda\phi}$
  These $\phi_k$ are basis fcns for the multi-D eigen-expansion approach! (i.e. SV v2.0)

- Multi-var calc and Linear operator theory for the Laplacian $\mathbf{L}u\equiv\nabla^2 u$:
  - $\nabla^2 u=\nabla\cdot\nabla u$ or in words: $\mathbf{lap}(u)=\mathbf{div}(\mathbf{grad}(u))$ ($\not\Delta$)
  - The divergence theorem $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds=\iint_D \nabla\cdot\mathbf{F}\,dA$

    can be used to derive Green's 2nd identity:

$$\underbrace{\iint_D v\nabla^2 u\,dA}_{\langle v,\mathbf{L}u\rangle_2}=\underbrace{\oint_{\partial D}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)\,ds}_{\text{boundary terms}}+\underbrace{\iint_D u\nabla^2 v\,dA}_{\langle u,\mathbf{L}^* v\rangle_2}$$

    Notation: $\partial_n f\equiv\mathbf{n}\cdot\nabla f$ Neumann derivative, and boundary of $D$: $C=\partial D$
  - The Laplacian is formally self-adjoint and with Dir/Neu BC's it is fully self-adjoint on $\boxed{\text{any } D}$ (independent of shape or coord vars!)

**Sturm-Liouville**-type results for $\nabla^2\phi = -\lambda\phi$ on any finite domain $D$

0. $\nabla^2_{xy}u$ is like a 2-D SL $\widetilde{\mathbf{L}}u$ with $p = 1, q = 0$ (and $\sigma = 1$)
1. $\lambda$'s are real
2. $\lambda_k$ are discrete and $\lambda_k \to \infty$ (multiple roots possible (symmetry of $D$'s shape))
3. $\phi_k(x, y)$ are a complete self-orthogonal basis

$$F(x, y) = \sum_{\text{“}k\text{”}} c_k\phi_k(x, y) \qquad c_k = \frac{\langle F, \phi_k\rangle_2}{||\phi_k||_2^2} \qquad \text{“}k\text{” multi-index} = (m, n)$$

4. To show $\lambda \geq 0$, take inner product of $\nabla^2\phi = -\lambda\phi$ with $\phi$:

$$\begin{aligned} \langle\phi, \nabla^2\phi\rangle &= -\lambda\langle\phi, \phi\rangle \quad \cdots \\ \boxed{\left(||\nabla\phi||^2 - \oint_C \phi\frac{\partial\phi}{\partial n}\,ds\right) \Big/ ||\phi||^2} &= \lambda \qquad \underline{\textbf{Rayleigh quotient}} \end{aligned}$$

$\boxed{\lambda \geq 0}$ for Dirichlet ($\phi = 0$) or Neumann ($\partial\phi/\partial n = 0$) BC's (H. Ch 7.6)

5. Fredholm Alternative Thm issues for Laplace problems $\mathbf{L}u = f$ and Helmholtz eigenfunctions?

**Part 2**: **The Laplacian in 2D polar coordinates** (Haberman 2.5.2)

- Convert $(r, \theta) \to (x, y)$

$$x = r\cos\theta \qquad y = r\sin\theta$$

- Convert $(x, y) \to (r, \theta)$

$$r = \sqrt{x^2 + y^2} \geq 0 \qquad \theta = \tan^{-1}(y/x)$$

- Laplacian in 2D polar coordinates. Start from usual rectangular form:

$$\nabla^2 u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Change of variables $u(x, y) = U(r, \theta)$ and chain rule for $U(r(x, y), \theta(x, y))$

$$\nabla^2_{xy} u = \nabla^2_{r\theta} U \equiv \boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}} = \nabla^2 U$$

---