Math 551: Applied PDE and Complex Vars — Lecture 16

Separation of variables for elliptic PDE BVP (cont) (Haberman Ch 8.6)

The completely inhomogeneous Dirichlet problem (continued)

$$u_{xx} + u_{yy} = S(x, y) \qquad 0 \le x \le \ell \qquad 0 \le y \le h$$

$$u(x, y = 0) = C(x) \qquad u(x, y = h) = A(x) \qquad \text{Bottom/Top BC's}$$

$$u(x = 0, y) = D(y) \qquad u(x = \ell, y) = B(y) \qquad \text{Left/Right BC's}$$

- Use superposition to separate the overall solution into pieces, each dealing with one of the inhomogeneous forcing fcns (A, B, C, D, S) [using the L14 "Option A" top-solution (and its "siblings")]

$$u(x, y) = u_{\text{top}}[A(x)] + u_{\text{right}}[B(y)] + u_{\text{bot}}[C(x)] + u_{\text{left}}[D(y)] + u_p[S(x, y)]$$

- In the last term, $u_p(x, y)$, 'p' can stand for 'particular soln' (to balance the RHS inhomogeneous Source forcing), or for 'Poisson soln', as in the Poisson eqn (inhomogeneous version of Laplace eqn).
  The BVP for u_p has all-homogeneous BC's.

Homogeneous-BC Poisson problem for $u_p(x,y)$:

$$\frac{\partial^2 u_p}{\partial x^2} + \frac{\partial^2 u_p}{\partial y^2} = S(x,y) \qquad 0 \le x \le \ell \qquad 0 \le y \le h$$

$$u_p(x, y=0) = 0 \qquad u_p(x, y=h) = 0 \qquad \text{Bottom/Top BC's}$$

$$u_p(x=0, y) = 0 \qquad u_p(x=\ell, y) = 0 \qquad \text{Left/Right BC's}$$

Solution process (v1.0 via single-sum expansion)

1. Homogenize-out the forcing ($\nabla^2 u = 0$) and
   consider the SV trial soln $u_k(x,y) = \alpha_k(x)\beta_k(y)$.
   Could use either direction (x or y) for osc-eigfcns (basis fcns),
   I'll pick $\alpha_k = \phi_k(x) = \sin(k\pi x/\ell)$ for $k = 1, 2, \cdots$

2. Return to full problem and write the solution as

$$u_p(x,y) = \sum_{k=1}^{\infty} \beta_k(y)\phi_k(x) \qquad \text{Find the } \beta_k(y) \text{ coeffs!}$$

   We will get ODE BVP for each β_k, but is a bit better than L14 "Option B",
   lets call it "Solution B$^+$"

B^+ solution process for $u_p(x,y)$ (continued)

$$\frac{\partial^2 u_p}{\partial x^2} + \frac{\partial^2 u_p}{\partial y^2} = S(x,y) \qquad 0 \le x \le \ell \qquad 0 \le y \le h$$

$$u_p(x, y=0) = 0 \qquad u_p(x, y=h) = 0 \qquad \text{Bottom/Top BC's}$$

$$u_p(x=0, y) = 0 \qquad u_p(x=\ell, y) = 0 \qquad \text{Left/Right BC's}$$

2a. Expand everything using ϕ_k's:

$$S(x,y) = \sum_k s_k(y)\phi_k(x) \qquad s_k(y) = \frac{2}{\ell}\int_0^\ell S(x,y)\phi_k(x)\,dx$$

2b. **BECAUSE the BC's are ALL homogeneous: NICE SHORT-CUT**

We don't need to do projection $\langle PDE, \phi_k \rangle$, we can just plug-in the u, S expansions into the PDE to get ODE's for β_k's!

[see the usual steps...]

$\mathbf{B}^+$ solution process for $u_p(x,y)$ (continued)

2c. Solve the ODE BVP's for the $\beta_k(y)$'s on $0 \le y \le h$:

$$\underbrace{\frac{d^2\beta_k}{dy^2} - \left(\frac{k\pi}{\ell}\right)^2 \beta_k}_{\mathbf{L}_k\beta_k} = s_k(y) \qquad \beta_k(0) = 0 \qquad \beta_k(h) = 0$$

Solve each $\mathbf{L}_k\beta_k = s_k(y)$ via ODE eigen-expansion with

$$\mathbf{L}_k\Phi_m = -\Lambda_m\Phi_m \qquad \Longrightarrow \qquad \Phi_m'' = -\underbrace{\left(\Lambda_m - \frac{k^2\pi^2}{\ell^2}\right)}_{\mu_m}\Phi_m$$

Osc eqn with general soln $\Phi = a_1\cos(\sqrt{\mu}y) + a_2\sin(\sqrt{\mu}y)$

$$\Phi_m(y) = \sin\left(\frac{m\pi}{h}y\right) \qquad \mu_m = \frac{m^2\pi^2}{h^2} = \Lambda_m - \frac{k^2\pi^2}{\ell^2}$$

Expand out RHS: $s_k(y) = \sum_m c_{k,m}\Phi_m(y)$ with $c_{k,m} = \frac{2}{h}\langle s_k, \Phi_m\rangle$
and then use SHORTCUT again for $\beta_k(y)$ problem!

$$\beta_k(y) = \sum_{m=1}^{\infty}\left(\frac{-c_{k,m}}{\Lambda_{k,m}}\right)\Phi_m(y) \qquad \text{with} \qquad \Lambda_{k,m} = \pi^2\left(\frac{m^2}{h^2} + \frac{k^2}{\ell^2}\right)$$

$\mathbf{B}^+$ solution process for $u_p(x,y)$ (concluded)

$$u_p(x,y) = \sum_{k=1}^{\infty} \beta_k(y)\phi_k(x) = \sum_{k=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \left(\frac{-c_{k,m}}{\Lambda_{k,m}} \right) \Phi_m(y) \right\} \phi_k(x)$$

Ended up with a double-sum again....a sign of things to come...

There will be a better way, soon.

Separation of variables for PDE in 2^+ dimensions (Haberman Chap 7!)

- "$\mathbf{2}^+$" means 3-D (x,y,z) or 2-D plus time (x,y,t) and more...
- Notation alert:

  Haberman starts using roman letters for SV fcns, so I'll do it too:

$$\text{example}: \qquad u_k(x,y,t) = f(x)g(y)h(t)$$