Part 1: Overview of solving ODE eigenvalue problems
Part 2: The general process for solving ODE boundary value problems
Part 3: Properties of (a) General, (b) Self-Adjoint, and (c) Sturm-Liouville linear differential operators

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Part 1: Solving ODE eigenvalue problems

1. Linear ODE boundary value problems: (i) domain, (ii) differential eqn, (iii) boundary conditions, and (iv) RHS forcing

$$a \le x \le b \qquad \mathbf{L}u = f(x) \qquad BC_1 u = c \qquad BC_2 u = d$$

   A regular n^{th} order operator $\mathbf{L}$ should have n boundary conditions.
   Finding the eig-vals/eig-fcns of $\mathbf{L}$ lets you solve the full problem for $u(x)$.

2. Write the eigen-problem for $\{\lambda, \phi(x)\}$: **(Version 1.0)**

$$a \le x \le b \qquad \mathbf{L}\phi = -\lambda\phi \qquad BC_1\phi(a) = 0 \qquad BC_2\phi(b) = 0.$$

   - LHS=Linear operator applied to eigenfcn $\phi(x)$ and RHS= $-$(eigenvalue)ϕ
   - Homogeneous version of BC's applied to ϕ
   - $\mathbf{L}\phi + \lambda\phi = 0$ is a homogeneous problem:
     it has nontrivial solutions (eigenfunctions $||\phi||^2 \neq 0$)
     only for special choices for λ values (eigenvalues)

Eigenvalue problems: Matrix vs. ODE BVP comparison

Matrix problem:

$$\mathbf{L}\vec{\phi} = \boldsymbol{\lambda}\vec{\phi}$$

vs.

ODE BVP:

$$\mathbf{L}\phi(x) = -\boldsymbol{\lambda}\phi(x) \qquad BC_1\phi(a) = 0 \qquad BC_2\phi(b) = 0$$

- Historical tradition: ODE eigenvalue equation has an extra minus sign

- n Eigenvalues for $\mathbf{L}_{n\times n}$ matrix from determinant $|\mathbf{L} - \boldsymbol{\lambda}\mathbf{I}| = 0$

  ODE BVP have an infinite number of $\boldsymbol{\lambda}$-values. (How? IOU)

Solving ODE eigenvalue problems (continued)

3. Get the general solution of the homogeneous ODE $\mathbf{L}\phi + \lambda\phi = 0$:
   - The Gen. hom. soln of n^{th} order prob is a linear combo of n independent solns:

     $$\phi_{\text{gen}}(x) = b_1\phi_1(x) + b_2\phi_2(x) + \cdots + b_n\phi_n(x)$$

   - For Linear-Const-Coeff (LCC) and Cauchy-Euler (CE) ODE's, you can find the general soln from trial solutions (LCC: $\phi = e^{mx}$, CE: $\phi = x^m$) and the roots of the characteristic polynomial $P(m) = 0$.
   - For other types of ODE's the $\phi_1, \phi_2, \cdots$ solutions must be provided.
   - In general, λ could be any complex number, and solutions could involve working out different subcases... If $\mathbf{L}$ is self-adjoint, its much easier: λ's must be real! (L04b)
4. Apply the n BC equations to the general soln to determine the condition for the eigenvalues, and the form of the eigenfunctions.
   - In general, the equations will determine $(n-1)$ of the b_k's, and one equation to be solved for the values of λ.
   - The last remaining b_{last} in $\phi_{\text{gen}}(x)$ can be set to $b_{\text{last}} = 1$ (or any $b_{\text{last}} \neq 0$)
5. From the inner product relation $\langle \mathbf{L}u, v\rangle = \langle u, \mathbf{L}^* v\rangle$, find the adjoint problem,

   $$\mathbf{L}^*\psi = -\lambda\psi, \qquad BC_1^*\psi = 0, \qquad BC_2^*\psi = 0.$$

   The λ's will be the same, solve for $\psi(x)$'s similarly via steps 3, 4.

<u>**Example 1**</u>: $\boxed{0 \leq x \leq \pi \qquad \mathbf{L}u \equiv \frac{d^2u}{dx^2} \qquad u(0) = 0 \qquad u(\pi) = 0}$

Eigenvalue problem (Dirichlet boundary condition version)

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad \phi(0) = 0 \qquad \phi(\pi) = 0$$

<u>**Note**</u>**: The problem is self-adjoint (check it...), so λ must be real!**

LCC eqn, $\phi = e^{mx}$ yields $m^2 + \lambda = 0$. Two sub-cases for general soln:

(a) $\boxed{\lambda = 0?}$: $\phi_{\mathbf{gen}}(x) = b_1 + b_2 x$ (repeated root)

Applying BC's shows that this doesn't work, no nontrivial solution.

Conclusion: "Zero is not an eigenvalue."

(b) $\boxed{\lambda \neq 0?}$: $\phi_{\mathbf{gen}}(x) = b_1 e^{\sqrt{-\lambda}\,x} + b_2 e^{-\sqrt{-\lambda}\,x}$ (distinct roots)

$\phi(0) = 0$ yields that $b_2 = -b_1$, so $\phi_{\mathbf{gen}}(x) = b_1(e^{\sqrt{-\lambda}\,x} - e^{-\sqrt{-\lambda}\,x})$

Can we have $\lambda < 0$?

BC $\phi(\pi) = 0$ can be reduced to the equation: $\boxed{\sin(\sqrt{\lambda}\,\pi) = 0}$

Eqn for the eigenvalues alone, analogous to matrix/det eqn: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

<u>Solutions</u>: $\lambda_k = k^2$ for $k = 1, 2, 3, \cdots$

Eigenmodes: $\boxed{\lambda_k = k^2 \qquad \phi_k(x) = \sin(kx)}$ $\rightarrow$ **The Fourier Sine series**

(IOU other properties of solutions....[Lecture 6])

<u>**Example 2**</u>: $\boxed{0 \le x \le \pi \qquad \mathrm{L}u \equiv \dfrac{d^2u}{dx^2} \qquad u'(0) = 0 \qquad u'(\pi) = 0}$

Eigenvalue problem (Neumann boundary condition version)

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad \phi'(0) = 0 \qquad \phi'(\pi) = 0$$

LCC eqn, $\phi = e^{mx}$ yields $m^2 + \lambda = 0$. Two sub-cases for general soln... **(DIY)**

$\boxed{\lambda_k = k^2 \quad \phi_k(x) = \cos(kx) \quad k = 0, 1, 2, \cdots}$ → **Fourier Cosine series**

$$f(x) = \sum_{k=0}^{\infty} c_k \phi_k(x) \qquad c_k = \frac{\langle f, \cos(kx)\rangle}{||\cos(kx)||^2}$$

<u>**Example 3**</u>: $\boxed{-\pi \le x \le \pi \quad \mathrm{L}u \equiv \dfrac{d^2u}{dx^2} \qquad u(-\pi) = u(\pi) \qquad u'(-\pi) = u'(\pi)}$

Eigenvalue problem (Periodic boundary condition version)

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad \phi(-\pi) = \phi(\pi) \qquad \phi'(-\pi) = \phi'(\pi)$$

LCC eqn, $\phi = e^{mx}$ yields $m^2 + \lambda = 0$. Two sub-cases for general soln... **(DIY)**

$\boxed{\lambda_k = k^2 \quad \phi_k(x) = \{\cos(kx), \sin(kx)\} \quad k = 0, 1, 2, \cdots}$ → **Full Fourier series**

$$f = \sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx) \quad a_k = \frac{\langle f, \cos(kx)\rangle}{||\cos(kx)||^2} \quad b_k = \frac{\langle f, \sin(kx)\rangle}{||\sin(kx)||^2}$$

$\boxed{\text{Part 3}}$: <u>Results on properties of Linear differential operators</u> (Re-cap)

1. <u>General $\mathbf{L}$</u> (usual process, all steps needed)
   Determine adjoint from adjoint relation $\langle \mathbf{L}u, v\rangle = \langle u, \mathbf{L}^* v\rangle$

   Eigen-problems: $\mathbf{L}\phi = -\lambda\phi$ and $\mathbf{L}^*\psi = -\lambda\psi$

   Eigenvalues λ can be complex values
   Need both sets: regular eigenfunctions $\{\phi_k\}$ and adjoint eig-fcns $\{\psi_k\}$

   L^2 inner product bi-orthogonality $\langle \phi_i, \psi_j\rangle_2 = 0$ for $i \neq j$

2. <u>Self-adjoint $\mathbf{L} = \mathbf{L}^*$</u> (symmetry reduces some work!)

   Eigen-problem: $\mathbf{L}\phi = -\lambda\phi$

   All eigenvalues λ are real-valued!

   L^2 inner product self-orthogonality $\langle \phi_i, \phi_j\rangle_2 = 0$ for $i \neq j$

   No separate calculation of ψ's (each $\psi_k(x) = \phi_k(x)$)

3. <u>**Sturm-Liouville (SL) theory**</u>: important sub-set of the self-adjoint case with even better reductions to needed calculations! $\Longrightarrow$

Results for Sturm-Liouville linear operators

SL operators, $\widetilde{\mathbf{L}}$: 2nd order differential eqns only, and must have the form:

$$\boxed{\widetilde{\mathbf{L}}u \equiv \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u} \qquad \text{on } a \le x \le b$$

- with some given real fcns for $p(x)$ and $q(x)$
- and homogeneous BC's: Dirichlet or Neumann or Robin at $x = a$ and $x = b$

Main Results for SL $\widetilde{\mathbf{L}}$'s

1. If a linear operator $\mathbf{L}$ (w/BC's) is of the SL $\widetilde{\mathbf{L}}$ form, then it is self-adjoint in the standard L^2 inner product for any $p(x), q(x)$ fcns and any of the BC's:

$$\langle v, \widetilde{\mathbf{L}}u\rangle_2 = 0 + \langle \widetilde{\mathbf{L}}v, u\rangle_2$$

   Proof: DIY HW2Q4e

   Example: $\mathbf{L}u = \dfrac{d^2u}{dx^2} \qquad \Longrightarrow \qquad$ SL with $p(x) = 1$ and $q(x) = 0$

   (i) So, $\mathbf{L}$ is self-adjoint by SL results. Dont need to do IBP to check adjoint!
   (ii) Self-adjoint, so we know that all the eigenvalues will be real.

Main Results for SL $\widetilde{\mathbf{L}}$'s

2. Define the **weighted** SL eigenvalue problem **(Version 2.0)**

$$\boxed{\widetilde{\mathbf{L}}\phi = -\lambda\sigma(x)\phi}$$

where $\sigma(x) \geq 0$ is a weight fcn (a given fcn).

Example: $\phi'' = -\lambda\phi$ has $\sigma(x) \equiv 1$

Note: Changing $\sigma(x)$ in the eigen-problem changes $\{\lambda, \phi\}$ solns!

Define the SL weighted inner product :

$$\boxed{\langle u(x), v(x)\rangle_\sigma \equiv \int_a^b u(x)v(x)\sigma(x)\,dx}$$

2.1 The eigenfcns of an SL eigenvalue problem are self-orthogonal in the σ-weighted inner product:

$$\langle \phi_i, \phi_j\rangle_\sigma = 0 \qquad \text{for } i \neq j$$