Math 551: Applied PDE and Complex Vars — Lecture 13

Separation of Variables method for inhom PDE problems (H Chap 8)

1-D version for $u(x,t)$ PDE initial-boundary value problems (IBVP) (continued)

1. SV trial solution: $u_k(x,t) = a_k(t)\phi_k(x)$, get eigen-modes: $\{\phi_k(x), \lambda_k\}$
2. Orthogonal projection of full problem: $\langle \text{PDE}, \phi_k\rangle$ and $\langle u, \phi_k\rangle = c_k(t)||\phi_k||^2$

$$u(x,t) = \sum_k c_k(t)\phi_k(x) \qquad c_k = \frac{\langle u, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle}$$

Get forced ODE for each $c_k(t)$ and $c_k(0)$ from PDE-IC

Last time (Lecture 12): Solving a heat eqn (parabolic PDE IBVP) (H Chap 8.4)

$$u_t = u_{xx} - 3u + S(x,t) \quad u(0,t) = A(t) \quad u_x(1,t) = B(t) \quad u(x,0) = f(x)$$

Steps: 1. $\{\phi_k(x), \lambda_k\}(\mathbf{L}, Dir, Neu)$, and 2. $I_k(A,B,S)$ and f_k

$$\frac{dc_k}{dt} + \lambda_k c_k = I_k \quad c_k(0) = f_k \quad \xrightarrow{*} \quad c_k(t) = \left(f_k - \frac{I_k}{\lambda_k}\right)e^{-\lambda_k t} + \frac{I_k}{\lambda_k}$$

What's been learned: Solution looks like exp decay from IC ($t\to 0$) to

steady state based on inhom forcing only ($t \to \infty$)

Note: Inhom BC's will produce Gibbs ripples at boundaries. If IC doesnt match BC's then more Gibbs at 'corners' too...

Solving a wave equation Dirichlet IBVP: (Hyperbolic PDE) (H Chap 8.5)

PDE Problem on domain $0 \le x \le 1$ for $t \ge 0$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + S(x,t)$$

Boundary conditions

$$u(x=0) = A(t) \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = B(t)$$

TWO Initial conditions

$$u(t=0) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

A, B, S, f, g are given functions

Part 1 of Solving Elliptic PDE: "1-D Expansions" (H Chap 8.6, pp. 366-9)

Laplacian operator: notation $\nabla^2 u \equiv \partial_{xx} u + \partial_{yy} u$ (for $u(x,y)$ in 2-D)

THE basic "single edge" Dirichlet-Laplace problem on a rectangle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{on } [0 \le x \le \ell] \times [0 \le y \le h]$$

Left/Right BC's: $u(x=0) = 0$ and $u(x=\ell) = 0$

Bottom/Top BC's: $u(y=0) = 0$ and $u(y=h) = \boxed{f(x)}$

SV trial solution: $u_k(x,y) = \alpha_k(x)\beta_k(y)$ Which one is the ϕ_k?