- **The simplest problems**: single eqn/scalar problems for unknown u

  Statics : $au = b$ Dynamics : $\frac{du}{dt} = au - b, \quad u(0) = c$

  Solutions : $u = b/a$ $u(t) = (c - b/a)e^{at} + b/a$

- **The next level**: matrix/vector systems

  Statics : $\mathbf{A}\mathbf{u} = \mathbf{b}$ Solution : $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$

  but, what if you couldn't use the inverse matrix "$\mathbf{A}^{-1}$"?
  There's a different way to write the soln if you know the eigenvalues ($\boldsymbol{\lambda}$) and eigenvectors ($\boldsymbol{\phi}$) of matrix $\mathbf{A}$ (from $\mathbf{A}\boldsymbol{\phi} = \boldsymbol{\lambda}\boldsymbol{\phi}$) then

  Eigen-expansion Solution : $\mathbf{u} = \sum_{k=1}^{n} c_k \phi_k$ [IOU: how to calculate c_k's]

  Also works for Dynamics problems: $\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u} - \mathbf{b}, \quad \mathbf{u}(0) = \mathbf{c}$

(For many problems $\mathbf{A}^{-1}$ is not defined, but there are ways in linear algebra to work around this...)

- **The general level (551)**: linear operator problems
  Writing solns as eigen-expansions works for a large array of types of problems:
  - ODE's, example: $$A\frac{d^2u}{dx^2} + B\frac{du}{dx} + Cu = f(x)$$
  - Integral equations
  - PDE's in 1-D, 2-D, 3-D
  - Other classes of linear problems, numerical methods (FFT, Galerkin FEM), and linear stability analysis (comparing experiments and model predictions)

- Key issues will be determining the $\{\lambda, \phi\}$'s, how to write the expansion and how to calculate the coefficients....
- The eigen-expansion approach also connects to: Green's fcn integrals, spectral theory, the Fredholm alternative theorem, Bessel fcn expansions, Fourier transforms, Laplace transforms, complex contour integrals....
- 551 is not about specific formulas or solns, it's about a **solution process** for getting solns and properties (is the soln unique? does a soln exist at all? and results you need to calculate from the soln) in the most direct way.

Math 551, Duke University $\boxed{\text{Part I}}$ Lecture 1

Extending basic ideas from Linear Algebra [Haberman Sect 5.5 Appendix]

$\boxed{\textbf{Inner products } \langle \mathbf{u}, \mathbf{v}\rangle}$ are generalizations of the vector dot product.

For real vectors, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, define the inner product as $\langle \mathbf{u}, \mathbf{v}\rangle \equiv \mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v}$.

Other definitions for the inner product are made for other classes of problems (ODEs, PDEs, ...). All inner products share common properties:

- Inner products are used to define norms, orthogonality, adjoints.
- The **linearity property** of inner products: for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

$$\langle \underline{a\mathbf{u}} + \underline{b\mathbf{v}}, \mathbf{w}\rangle = \langle \underline{a\mathbf{u}}, \mathbf{w}\rangle + \langle \underline{b\mathbf{v}}, \mathbf{w}\rangle = \underline{a}\langle \underline{\mathbf{u}}, \mathbf{w}\rangle + \underline{b}\langle \underline{\mathbf{v}}, \mathbf{w}\rangle$$

  inner prod. of linear-combination-of-vectors = lin. combo. of inner-prods.
  (i.e. can expand-out sums, and factor-out constant scalar multiples)

- The **norm property** of inner products: for all vectors $\mathbf{u}$'s,

$$|\mathbf{u}|^2 \equiv \langle \mathbf{u}, \mathbf{u}\rangle \geq 0 \qquad \text{and } |\mathbf{u}| = 0 \text{ if and only if } \mathbf{u} = \vec{\mathbf{0}}$$

- **Orthogonality** of vectors is defined by the inner product,

$$\mathbf{u} \perp \mathbf{v} \qquad \leftrightarrow \qquad \boxed{\langle \mathbf{u}, \mathbf{v}\rangle = 0}$$

$\boxed{\textbf{Linear Operators, } \mathbf{L}}$ are a generalization of matrices ($\mathbf{u}$ vector, $\mathbf{L}$ matrix)

Calculating $\mathbf{Lu}$ produces another vector, $\mathbf{v} = \mathbf{Lu}$ ($\mathbf{L}$ for $\underline{\mathbf{L}}$inear operator)

- Eigenvalues $\boldsymbol{\lambda}$ and eigenvectors $\boldsymbol{\phi}$ of $\mathbf{L}$ are defined by:

$$\boxed{\mathbf{L}\boldsymbol{\phi} = \boldsymbol{\lambda}\boldsymbol{\phi}} \quad \text{with} \quad |\boldsymbol{\phi}| \neq \mathbf{0}$$

Eigen-vals/vecs calculated using: $\det(\mathbf{L} - \boldsymbol{\lambda}\mathbf{I}) = \mathbf{0}$ then $(\mathbf{L} - \boldsymbol{\lambda}_k\mathbf{I})\boldsymbol{\phi}_k = \vec{\mathbf{0}}$

- The <u>**adjoint operator**</u> $\mathbf{L}^*$ is defined by the inner product <u>adjoint relation</u>:

$$\boxed{\langle \mathbf{v}, \mathbf{Lu} \rangle = \langle \mathbf{L}^*\mathbf{v}, \mathbf{u} \rangle} \qquad \text{for any } \mathbf{u}, \mathbf{v}$$

The adjoint eigenvalues $\boldsymbol{\gamma}$ and adjoint eigenvectors $\boldsymbol{\psi}$ are defined by:

$$\mathbf{L}^*\boldsymbol{\psi} = \boldsymbol{\gamma}\boldsymbol{\psi} \;\; \text{with} \;\; |\boldsymbol{\psi}| \neq \mathbf{0} \implies \det(\mathbf{L}^* - \boldsymbol{\gamma}\mathbf{I}) = \mathbf{0} \;\to\; (\mathbf{L}^* - \boldsymbol{\gamma}_j\mathbf{I})\boldsymbol{\psi}_j = \vec{\mathbf{0}}$$

<u>For real vectors</u>, with $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$, for real matrix $\mathbf{L}$, the adjoint relation:

$$\langle \mathbf{v}, \mathbf{Lu} \rangle = \mathbf{v}^T(\mathbf{Lu}) = (\mathbf{v}^T\mathbf{Lu})^T = \mathbf{u}^T\mathbf{L}^T\mathbf{v} = \mathbf{u}^T(\mathbf{L}^T\mathbf{v}) =$$

$$(\mathbf{u}^T(\mathbf{L}^T\mathbf{v}))^T = (\mathbf{L}^T\mathbf{v}))^T\mathbf{u} = \langle \mathbf{L}^T\mathbf{v}, \mathbf{u} \rangle$$

Using $c = c^T$ for scalars, and $(ABC)^T = C^TB^TA^T \implies$ $\mathbf{L}^* = \mathbf{L}^T$ (transpose).

Further results for Linear Operators on real vectors:

- Relations between adjoint eigenvalues and regular eigenvalues:

$$\det(\mathbf{L}^T - \gamma \mathbf{I}) = \mathbf{0} \qquad \text{vs.} \qquad \det(\mathbf{L} - \boldsymbol{\lambda}\mathbf{I}) = \mathbf{0}$$

$$\text{Start:} \quad \det(\mathbf{L}^T - \gamma \mathbf{I}) = \det((\mathbf{L}^T - \gamma \mathbf{I})^T) = \det(\mathbf{L} - \gamma \mathbf{I}) = \mathbf{0}$$

Using properties: $\det(\mathbf{M}) = \det(\mathbf{M}^T)$, $(\mathbf{M}+\mathbf{N})^T = \mathbf{M}^T + \mathbf{N}^T$, $(\mathbf{L}^T)^T = \mathbf{L}$, $\mathbf{I}^T = \mathbf{I}$

So the adjoint eigenvalues satisfy the eig-val eqn, and are the same: $\boxed{\gamma = \boldsymbol{\lambda}}$

But the eig-vecs $\boldsymbol{\phi_k}$ and adj-eig-vecs $\boldsymbol{\psi_j}$ are generally different.

- Relations between eigenvectors and adjoint eigenvectors:

$\boxed{\text{If } \boldsymbol{\lambda_k} \neq \boldsymbol{\lambda_j} \text{ then } \boldsymbol{\phi_k} \perp \boldsymbol{\psi_j}}$ $\boxed{\textbf{The bi-orthogonality of the sets of eigenvectors}}$

$$\begin{aligned}
\text{Start with} \quad \boldsymbol{\lambda_k}\langle\boldsymbol{\psi_j}, \boldsymbol{\phi_k}\rangle &= \langle\boldsymbol{\psi_j}, \boldsymbol{\lambda_k}\boldsymbol{\phi_k}\rangle && \text{[via linearity]}\\
&= \langle\boldsymbol{\psi_j}, \mathbf{L}\boldsymbol{\phi_k}\rangle && [\text{via } \boldsymbol{L}\boldsymbol{\phi_k} = \boldsymbol{\lambda_k}\boldsymbol{\phi_k}]\\
&= \langle\mathbf{L}^T\boldsymbol{\psi_j}, \boldsymbol{\phi_k}\rangle && \text{[via adjoint relation]}\\
&= \langle\boldsymbol{\lambda_j}\boldsymbol{\psi_j}, \boldsymbol{\phi_k}\rangle && [\text{via } \boldsymbol{L}^*\boldsymbol{\psi_j} = \boldsymbol{\lambda_j}\boldsymbol{\psi_j}]\\
&= \boldsymbol{\lambda_j}\langle\boldsymbol{\psi_j}, \boldsymbol{\phi_k}\rangle && \text{[via linearity, End]}
\end{aligned}$$

$$\text{(Start)}-\text{(End)}=0 \qquad \to \qquad (\boldsymbol{\lambda_k}-\boldsymbol{\lambda_j})\langle\boldsymbol{\psi_j}, \boldsymbol{\phi_k}\rangle = \mathbf{0} \implies \boxed{\langle\boldsymbol{\psi_j}, \boldsymbol{\phi_k}\rangle = \mathbf{0}}$$

<u>**Completeness**</u>: The set of vec's $\{\phi_k\}$ is called a <u>**complete basis set**</u> if <u>every</u> given vec $\mathbf{w} \in \mathbb{R}^n$ can be written as a linear combination of the ϕ_k basis vec's:

$$\boxed{\mathbf{w} = \sum_{k=1}^{n} c_k \phi_k}$$

How do we determine the c_k coefficients for $\mathbf{w}$?

<u>Key step</u>: $\boxed{\textbf{Orthogonal projection}}$ = The universal problem-solving approach. Always starts with taking the inner product of both sides of the equation with each of the adjoint eigenvectors ψ_j for $j = 1, 2, 3, \cdots, n$: $\boxed{\langle \psi_j, \text{Eqn} \rangle}$

$$\begin{aligned}
\langle \psi_j, \mathbf{w} \rangle &= \left\langle \psi_j, \sum c_k \phi_k \right\rangle && [\text{via } \langle \psi_j, \text{LHS} \rangle = \langle \psi_j, \text{RHS} \rangle] \\
&= \sum_k c_k \langle \psi_j, \phi_k \rangle && [\text{via linearity}] \\
&= c_j \langle \psi_j, \phi_j \rangle && [\text{via bi-orthogonality for } j \neq k]
\end{aligned}$$

and finally, swap index letters, j and k, to get a formula for each coefficient:

$$\Longrightarrow \quad \boxed{c_k = \frac{\langle \psi_k, \mathbf{w} \rangle}{\langle \psi_k, \phi_k \rangle}} \quad \rightarrow \quad \boxed{\mathbf{w} = \sum_{k=1}^{n} \frac{\langle \psi_k, \mathbf{w} \rangle}{\langle \psi_k, \phi_k \rangle} \phi_k}$$