The Fredholm Alternative Theorem (FAT)

Either

(A) The homogeneous adjoint problem

$$\mathrm{L}^*\psi_0 = 0 \qquad BC_1^*\psi_0(a) = 0 \qquad BC_2^*\psi_0(b) = 0 \qquad (1)$$

has a **nontrivial** solution.

Xor (eXclusive OR - one case or the other exclusively, never both together, never neither)

(B) The inhomogeneous problem

$$\boxed{\mathrm{L}u = f(x) \qquad BC_1u(a) = c \qquad BC_2u(d) = d} \qquad (2)$$

has a **unique** solution $u(x)$ for any choices of $f(x), c, d$.

(a) The homogeneous adjoint problem $\mathrm{L}^*\psi = 0$ in **(A)** corresponds to the adjoint eigenvalue problem $\mathrm{L}^*\psi = -\lambda\psi$ for a zero eigenvalue $\boxed{\lambda_0 = 0}$.

(b) The alternative, case **(B)**, is automatically guaranteed, giving existence and uniqueness of the solution of (2), **if** case **(A)** is not true.

The Fredholm alternative is usually used in practice as a quick way, via **(A)**, to determine what to expect from solving an inhomogeneous problem with the same linear operator (2) in **(B)** before working through the whole expansion. It is used in this way:

- If L (and hence L*) has no zero eigenvalue, then **Case (B)** holds and the orthogonal expansion/eigenfcn expansion process will work and will produce a unique solution with no difficulties:

$$B_k - c_k\lambda_k\langle\phi_k, \psi_k\rangle = \langle f, \psi_k\rangle \quad \xrightarrow{\forall k} \quad \{c_k \text{ values}\} \quad \to \quad u(x) = \sum c_k\phi_k(x) \qquad (3)$$

- If **Case (A)** holds, then there are two sub-cases for what happens to the soln of (2) based on the $\lambda_0 = 0$ version of eqn (3): $\boxed{B_0 - 0 = \langle f, \psi_0\rangle}$. This equation is called the $\boxed{\textbf{solvability condition}}$.

(A₁) **No solution of (2) exists** if the solvability equation is a contradiction, $B_0 \neq \langle f, \psi_0\rangle$.

(A₂) **The solution of (2) is non-unique** if the solvability equation is consistent, $B_0 = \langle f, \psi_0\rangle$, then solutions exist. Since the value of the c_0 coefficient does not get pinned down by this condition, any value may be selected for this coefficient. Meanwhile, the values for the c_k for $k = 1, 2, 3, \cdots$ are uniquely determined, as usual by (3) for all non-zero λ_k. So:

$$\text{All solns of (2) are} \qquad u(x) = c_0\phi_0(x) + \sum_{k=1}^{\infty} c_k\phi_k(x) \qquad \text{with \textbf{any} value for } c_0. \qquad (4)$$

- To determine between cases **(A)** vs. **(B)** check for a zero eigenvalue: $\boxed{\lambda = 0?}$
- To determine between cases $(\mathbf{A_1})$ vs. $(\mathbf{A_2})$, you must find the $\lambda_0 = 0$ adjoint eigenfunction: $\boxed{\psi_0(x)}$ $\psi_0(x)$ is needed to calculate both the forcing $\langle f, \psi_0\rangle$ and the boundary-conditions term B_0.

The Fredholm alternative is universal – it works the same way for matrices, linear differential operators for ODE boundary value problems (Sturm-Liouville or General), integral equations, linear algebra, $\cdots$, everything*
(Also see Haberman, sections 9.4.1, 9.4.2 for more on the FAT)