Lecture 8: FIE background and details

1st kind Fredholm integral operators: $\quad Lu(x) = \int_a^b K(x,t)u(t)\,dt$

n-term separable kernel fcn: $K(x,t) = \sum_{j=1}^n \alpha_j(x)\beta_j(t)$

$\boxed{\textbf{Eigen-problem for FIE: } L\phi = \lambda\phi}$ (Note: RHS has $+\lambda$ like matrix not $-\lambda$ like ODE)

(i) Finite multiplicity eigenmodes, $\lambda_{1,2,\cdots,n}$: $\phi_k(x) =$ linear combo's of $\alpha(x)$'s for each $k = 1, 2, \cdots, n$:

$$\phi_k(x) = \sum_{\ell=1}^n c_\ell \alpha_\ell(x) \quad \to$$

Match LHS/RHS coeffs of $\alpha_j(x)'s$:
$n \times n$ matrix eigen-problem for $\lambda, \vec{\mathbf{c}}$

(ii) Infinite multiplicity eigenmodes, all FIE_1 have the eig-val $\boxed{\lambda_0^\infty = 0}$ with infinite number of eig-fcns $\phi_{0,m}^\infty(x)$, $m = 1, 2, \cdots, \infty$.
Each $\phi_{0,m}^\infty$ satisfies n orthogonality conditions:

$$\langle \beta_j, \phi_m^\infty \rangle = \int_a^b \beta_j(t)\phi_{0,m}^\infty(t)\,dt = 0 \qquad j = 1, 2, \cdots, n$$

$\phi_{0,m}^\infty \neq$ linear combo's of α's. Many options on writing them

Example: FIE$_1$ $\quad \int_0^1 (4xt - 5x^2t^2)\phi(t)\,dt = \lambda\phi(x)$ $\quad \begin{cases} \alpha_1 = x & \beta_1 = 4t \\ \alpha_2 = x^2 & \beta_2 = -5t^2 \end{cases}$

- $\lambda_1 = \frac{1}{2} \quad \phi_1(x) = -6x + 5x^2$
- $\lambda_2 = -\frac{1}{6} \quad \phi_2(x) = 2x - 3x^2$
- $\lambda_0^\infty = 0 \quad \phi_{0,1}^\infty(x) = 1 - 4x + \frac{10}{3}x^2, \quad \phi_{0,2}^\infty(x) = x - \frac{10}{3}x^2 + \frac{5}{2}x^3, \ \cdots$

2nd kind Fredholm integral operators: $Lu(x) = \gamma u(x) + \int_a^b K(x,t)u(t)\,dt$

$$\text{“FIE}_2 = \text{FIE}_1 + \gamma\text{”} \qquad \to \qquad L_2 u = L_1 u + \gamma u$$

Shifting results: FIE$_1$ $\to$ FIE$_2$ (Or matrix problem (i) for FIE$_2$ directly via $\phi = \sum_\ell c_\ell \alpha_\ell$)

- All eigenvalues shift: $\lambda_{k,2} = \lambda_{k,1} + \gamma$ and $\lambda_{0,2}^\infty = \lambda_{0,1}^\infty + \gamma$
- All eigenfunctions are unchanged: $\phi_{k,2}(x) = \phi_{k,1}(x)$

Example: (continued) $\quad \frac{1}{2}\phi(x) + \int_0^1 (4xt - 5x^2t^2)\phi(t)\,dt = \lambda\phi(x)$

FIE$_1$:	$\lambda_1 = \frac{1}{2}$	$\lambda_2 = -\frac{1}{6}$	$\lambda_0^\infty = 0$
FIE$_2$:	$\lambda_1 = 1$	$\lambda_2 = \frac{1}{3}$	$\lambda_0^\infty = \frac{1}{2}$

(Not clear how to get FIE$_2$ λ_0^∞ without shifting from FIE$_1$ $\lambda_0^\infty = 0$)

Solving $Lu = f(x)$ via eigen-fcn expansion (Method 1)

Expansion of solution $u(x) = \sum_{k=1}^{n} d_k \phi_k(x) + \sum_{m=1}^{\infty} d_m^{\infty} \phi_{0,m}^{\infty}(x)$

Group them all together

$$u(x) = \sum_{\ell} d_\ell \phi_\ell(x) \qquad d_\ell = \frac{\langle u, \psi_\ell \rangle}{\langle \phi_\ell, \psi_\ell \rangle}$$

The usual orthogonal projection soln process for $Lu = f$:

$$\begin{aligned}
\langle Lu, \psi_\ell \rangle &= \langle f, \psi_\ell \rangle \\
\langle u, L^* \psi_\ell \rangle &= \\
\langle u, \lambda_\ell \psi_\ell \rangle &= \\
\lambda_\ell \langle u, \psi_\ell \rangle &= \\
\lambda_\ell d_\ell \langle \phi_\ell, \psi_\ell \rangle &=
\end{aligned}$$

So $d_\ell = \int_a^b f(x) \psi_\ell(x)\, dx \Big/ \left(\lambda_\ell \int_a^b \phi_\ell(x) \psi_\ell(x)\, dx \right)$

This works (for $\lambda_\ell \neq 0$), but needs ψ_ℓ's.

For those, we need the adjoint operator L^* (with $L^* \psi = \lambda \psi$).

Solving $Lu = f(x)$ for FIE$_1$ via a direct approach (Method 2)

$\boxed{\text{The method of un-determined coefficients}}$

1. Expand out LHS: Lu acting on $u(x) = \sum_{j=1}^{n} d_j \phi_j(x) + \sum_{m=1}^{\infty} d_m^{\infty} \phi_{0,m}^{\infty}(x)$

$$Lu = \sum_{j=1}^{n} d_j (L\phi_j) + \sum_{m=1}^{\infty} d_m^{\infty} (L\phi_{0,m}^{\infty}) = \sum_{j=1}^{n} d_j (\lambda_j \phi_j) + \underbrace{\sum_{m=1}^{\infty} d_m^{\infty} (\lambda_0^{\infty} \phi_{0,m}^{\infty})}_{0}$$

2. Expand each $\phi_j(x) = \sum_\ell c_{\ell,j} \alpha_\ell(x)$ as linear combo of α's and re-group

$$Lu = \sum_{j=1}^{n} d_j \lambda_j \left(\sum_{\ell=1}^{n} c_{\ell,j} \alpha_\ell(x) \right) = \sum_{j=1}^{n} F_j \alpha_j(x) = f(x)$$

$\boxed{\text{Could have just started with } u(x) = \sum_j c_j \alpha_j(x)}$ and plug into $Lu = f(x)$!

- If you use $u = \sum_j d_j \phi_j$, the equation for each d_j comes out decoupled.
  (but you need to work out the $\lambda_j, \phi_j(x), \psi_j(x)$'s first) (eigenfunctions)

- If you use $u = \sum_j c_j \alpha_j$, then must solve coupled algebra eqns for c_j's.
  (no λ, ϕ, ψ's needed!) (un-determined coefficients)

$\boxed{\textbf{Consequences of the FAT for solving FIE}_1\ Lu = f(x)}$

$\lambda_0^\infty = 0$ is always a FIE$_1$ eigenvalue, so FIE$_1$ problems are always FAT case A.

There is <u>never</u> a unique solution of $Lu = f$, either $\boxed{\textbf{no soln}}$ or $\boxed{\infty \textbf{ solns}}$.

$\boxed{\text{If } f(x) = \sum_j f_j \alpha_j(x)}$ then pick $F_j(c_1, c_2, \cdots, c_n) = f_j$ for a soln.

$$Lu = f \qquad \to \qquad \sum_j F_j \alpha_j = \sum_j f_j \alpha_j \qquad \to \qquad F_j = f_j \xrightarrow{\text{solve}} c_k$$

<u>Example</u>: (continued)

$$Lu = 8x - 7x^2 \qquad \text{with trial soln: } u(x) = c_1 x + c_2 x^2$$

Expand out LHS: $(\frac{4}{3}c_1 + c_2)x + (-\frac{5}{4}c_1 - c_2)x^2 = 8x + (-7)x^2$

$$\alpha_1(x): \qquad \frac{4}{3}c_1 + c_2 = 8$$

$$\alpha_2(x): \qquad -\frac{5}{4}c_1 - c_2 = -7$$

Solve $\quad \to \quad \{c_1 = 12 \;,\; c_2 = -8\} \quad \to \quad u(x) = 12x - 8x^2$

$u_1(x) = 12 - 8x^2$ is <u>**A**</u> soln, but other solns too (FAT case A$_2$ with ∞^∞ solns)

$$u_{2,1} = 12x - 8x^2 + c_1^\infty \phi_{0,1}^\infty(x), \quad u_{2,\ldots} = 12x - 8x^2 + \sum_m c_m^\infty \phi_{0,m}^\infty(x)$$

Consequences of the FAT for solving FIE$_1$ $Lu = f(x)$ (conclusion)

If $f(x) \neq \sum_j f_j \alpha_j(x)$ then can't match $Lu = f$ LHS/RHS terms $\forall x$, so the problem would be FAT case A$_1$ with **no solution**. Example: (continued)

$$Lu = x^3 \qquad \to \qquad (\tfrac{4}{3}c_1 + c_2)x + (-\tfrac{5}{4}c_1 - c_2)x^2 \neq x^3$$

Solving FIE$_2$ $Lu = f(x)$: if $\lambda \neq 0$ then FAT says problem has unique soln!

Can plug in expansion into LHS: $u = \displaystyle\sum_j d_j \phi_j + \sum_m d_m^\infty \phi_{0,m}^\infty$ (Method 1)

$$L\left(\sum_j d_j\phi_j + \sum_m d_m^\infty \phi_{0,m}^\infty\right) = \sum_j d_j \lambda_j \phi_j + \sum_m d_m^\infty \lambda_0^\infty \phi_{0,m}^\infty = f(x)$$

- Any $f(x)$ can be balanced by $\sum \phi_j$ and $\sum \phi_{0,m}^\infty$ to give a soln!
- But this is the long way (need λ, ϕ, ψ's) and calculating all d_j and d_m^∞ coeffs
- **A much better (shorter) way** is by expanding ϕ_j's as $\sum \alpha$'s and using un-determined coefficients $\quad \Longrightarrow$

Solving FIE$_2$ $Lu = f(x)$: Un-determined coefficients ($\{c_j\}, p$) (Method 2)

Assume a trial solution of the form: $\boxed{u(x) = \sum_{j=1}^{n} c_j \alpha_j(x) + \frac{1}{p} f(x)}$

Plug into LHS: $Lu \equiv \gamma u + \underline{\underline{\int K u\, dt}}$

$$\begin{aligned}
Lu &= \sum_{j=1} \gamma c_j \alpha_j(x) + \frac{\gamma}{p} f(x) + \underline{\underline{\int_a^b K(x,t) \sum_j c_j \alpha_j(t)\, dt + \int_a^b K(x,t) \frac{f(t)}{p}\, dt}} \\
&= \frac{\gamma}{p} f(x) + \underline{\sum_{j=1}^{n} \gamma c_j \alpha_j(x) + \underline{\underline{\sum_{j=1}^{n} F_j \alpha_j(x)}}} \\
Lu &= f(x) \quad + \qquad \underline{0}
\end{aligned}$$

Match LHS/RHS terms: match $f(x) \implies p = \gamma$ and match $\alpha_j(x)$'s to get c_j's

Example: (continued) (FIE$_2$ with $\lambda = \{1, \frac{1}{3}, \frac{1}{2}\}$)

$$\tfrac{1}{2} u + L_1 u = e^x$$

$$u(x) = c_1 x + c_2 x^2 + \frac{1}{p} e^x \quad \to \quad u(x) = (72 - 30e)x + (55e - 140)x^2 + 2e^x$$

Since $\lambda \neq 0$, this must be THE unique solution! (FAT case B)

Fredholm Integral Linear Operator Theory: general theory

For non-degenerate/non-separable kernel functions $K(x,t) = \sum_{j=1}^{\infty} \alpha_j(x)\beta_j(t)$

Need an infinite series: like Fourier or Taylor, $K(x,t) = \left\{e^{xt},\ \frac{1}{x-t}, e^{-ixt}, \cdots\right\}$

Hilbert-Schmidt condition: for "compact operator" L

If $\displaystyle\int_a^b \int_a^b K(x,t)^2\, dx\, dt < \infty$ then good properties (K is L^2 fcn)

Hilbert-Schmidt (HS) theory for FIE (like Sturm-Liouville for ODE BVP)

If Lu is a self-adjoint, compact Fredholm integral operator with a non-degenerate kernel and the HS condition holds, then nice properties:

1. All λ's real
2. Discrete eigenvalues, each with finite multiplicity!

$$|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \cdots \qquad \boxed{\text{NO } \lambda_0^\infty \text{ !}}$$

3. $\{\phi_j(x)\}$ self-orthogonal and complete basis set, $u(x) = \sum_{j=1}^{\infty} c_j\phi_j(x)$
4. FAT and eigen-expansions for solving $Lu = f(x)$ still hold

References: If you want to read more background/introductory material on integral equations...

- J. D. Logan, Applied Mathematics, 4th ed, Chapter 5
- K. F. Riley, M. P. Hobson, S. J. Bence, Mathematical Methods for Physics and Engineering, 3rd ed, Chapter 23
- R. B. Guenther and J. W. Lee, Partial Differential Equations of Mathematical Physics and Integral Equations, Chapter 7
- G. B. Arfken and H. J. Weber, Mathematical Methods for Physicists, 4th ed, Chapter 16
