Math 551: Applied PDE and Complex Vars — Lecture 15

Solving Laplace's equation: $\nabla^2 u(x,y) = 0$ (continued)

The single-edge Dirichlet BC problem: the Lecture 14 problem (continued)

$$u_{xx} + u_{yy} = 0 \qquad 0 \le x \le \ell \qquad 0 \le y \le h$$

$$u(x=0,y) = 0 \qquad u(x=\ell,y) = 0 \qquad \text{Left/Right BC's}$$

$$u(x,y=0) = 0 \qquad u(x,y=h) = \boxed{f(x)} \qquad \text{Bottom/Top BC's}$$

- Sep of Vars trial solution $u_k(x,y) = \alpha_k(x)\beta_k(y)$ $\qquad u = \sum_k c_k \alpha_k \beta_k$
- Separated form of the PDE:

$$\frac{\alpha_k''(x)}{\alpha_k(x)} = -\frac{\beta_k''(y)}{\beta_k(y)} = s_k$$

- Separated BC's from the 3 homogeneous PDE BC's (Left, Right, Bottom):

$$\alpha(0) = 0 \qquad \alpha(\ell) = 0 \qquad \beta(0) = 0$$

- Solution process always starts from determining the oscillatory eigenfcns, ϕ_k. They could be in x-direction $\phi_k = \alpha_k(x)$ (Option A) or in the y-direction $\phi_k = \beta_k(y)$ (Option B) as the first step.

<u>Option B</u> solution process (part 1 of 2)

(a) $-\beta''(y)/\beta(y) = s_k \quad \Longrightarrow$
$\phi'' = -\lambda\phi$ eigenvalue problem on $0 \le y \le h$ with $s_k = \lambda_k$

(b) Homogenized BC's: $\phi(0) = 0$ and $\phi(h) = 0$

(c) SL Reg$^+$ Dirichlet prob, $\lambda \ge 0$ (Harmonic oscillator eqn)

(d) $\phi_k(y) = \sin\left(\frac{k\pi}{h}y\right)$ with $\lambda_k = \left(\frac{k\pi}{h}\right)^2 = s_k$ for $k = 1, 2, 3, \cdots$

(e) Looks a lot like "Option A" process so far, but differences: $\phi(h)$ has zero-ed out the top BC & if tried to solve ODE BVP for $\alpha_k(x)$, would get $\alpha_k \equiv 0$

(f) Go to "Step 2" of soln process: projection of the full soln and full prob:

$$u(x,y) = \sum_{k=1}^{\infty} \alpha_k(x)\phi_k(y) \qquad \alpha_k = \frac{\langle u, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle} = \frac{1}{h/2}\int_0^h u\phi_k\, dy$$

$$\langle u_{xx}, \phi_k\rangle + \langle u_{yy}, \phi_k\rangle = 0$$

$$\frac{h}{2}\frac{d^2\alpha_k}{dx^2} + \left(u_y\phi_k - u\phi_k'\right)\Big|_{y=0}^{y=h} + \langle u, \phi_k''\rangle = 0$$

<u>Option B</u> solution process (part 2 of 2)

(g) ODE BVP for $\alpha_k(x)$ on $0 \le x \le \ell$

$$\frac{d^2\alpha_k}{dx^2} - \frac{k^2\pi^2}{h^2}\alpha_k = \frac{2}{h}u(x,h)\phi_k'(h)$$

$$\underbrace{\frac{d^2\alpha_k}{dx^2} - \frac{k^2\pi^2}{h^2}\alpha_k}_{} = \frac{2k\pi}{h^2}(-1)^k f(x) \qquad \alpha_k(0) = 0 \qquad \alpha_k(\ell) = 0$$

(h) For each $k = 1, 2, \cdots$ problem looks like $\mathbf{L}_k\alpha_k = F_k(x)$ with hom Dir. BC's. Can solve each with eig-expansions for

$$\mathbf{L}_k\Phi_m(x) = -\Lambda_m\Phi_m \qquad \Longrightarrow \qquad \Phi_m'' = -\underbrace{\left(\Lambda_m - \frac{k^2\pi^2}{h^2}\right)}_{}\Phi_m$$

Harmonic oscillator equation with solutions for $m = 1, 2, \cdots$

$$\Phi_m(x) = \sin\left(\frac{m\pi}{\ell}x\right) \qquad \mu_m = \frac{m^2\pi^2}{\ell^2} = \Lambda_m - \frac{k^2\pi^2}{h^2}$$

(i) Solns: $\alpha_k(x) = \sum_{m=1}^{\infty}\left(-\frac{\langle F_k, \Phi_m\rangle}{\Lambda_m\langle\Phi_m, \Phi_m\rangle}\right)\Phi_m(x)$ with $\Lambda_m = \pi^2\left(\frac{m^2}{\ell^2} + \frac{k^2}{h^2}\right)$

<u>Option B</u> yields a double-sum solution for $u(x, y)$:

$$u = \sum_k \sum_m \left(\frac{(-1)^{k+1} 4\pi}{h^2 \ell \Lambda_{m,k}} \int_0^\ell f \Phi_m \, dx \right) \sin\left(\frac{m\pi}{\ell} x\right) \sin\left(\frac{k\pi}{h} y\right)$$

- Pro's: systematic form of the final soln (v 1.0)
- Con's: long derivation, Gibbs phenomena at $y = h$ BC

<u>Option A</u> version of the solution from L14:

$$u_{\text{top}} = u(x, y) = \sum_{k=1}^{\infty} c_k \sinh\left(\frac{k\pi}{\ell} y\right) \sin\left(\frac{k\pi}{\ell} x\right)$$

with $$c_k = \frac{2}{\ell \sinh(k\pi h/\ell)} \int_0^\ell f(x) \sin\left(\tfrac{k\pi x}{\ell}\right) \, dx$$

- Pro's: short derivation, single sum, no Gibbs phenomena at $y = h$
- Con's: need hyperbolic trig fcns

Useful Basic Properties of the Laplace equation: $\nabla^2 u(x,y) = 0$

- $u_{xx} + u_{yy} = 0$ LCC PDE in rectangular coordinates (x,y)
- Reflections: same PDE for

$$\tilde{u}(x,y) = u(-x,y) \qquad \text{or} \qquad \tilde{u}(x,y) = u(x,-y)$$

- Translations: same PDE for

$$\tilde{u}(x,y) = u(x-\ell,y) \qquad \text{or} \qquad \tilde{u}(x,y) = u(x,y-h)$$

- Rotations (90°): same PDE for

$$\tilde{u}(x,y) = u(-y,x)$$

- Applying these PDE "symmetries/invariants" for BVP on (x,y) rectangles...