Solving Fredholm integral equations (FIE): $\mathbf{L}u = f(x)$ for $u(x)$

- 1st kind (FIE$_1$): $\mathbf{L_1}u(x) \equiv \int_a^b K(x,t)u(t)\,dt$
- 2nd kind (FIE$_2$): $\mathbf{L_2}u(x) \equiv \gamma u(x) + \int_a^b K(x,t)u(t)\,dt$ (γ constant)

Adjoint operators for FIE: Use standard L^2 inner product: $\langle f,g\rangle = \int_a^b fg\,dx$

The adjoint is always defined from the inner product relation: $\langle \mathbf{L}u, v\rangle = \langle u, \mathbf{L}^* v\rangle$

$$\begin{aligned}
\langle \mathbf{L}u(x), v(x)\rangle &= \\
\int_a^b \left(\int_a^b K(\boxed{x},t)u(t)\,dt\right) v(x)\,dx &= \int_a^b\int_a^b K(x,t)u(t)v(x)\,dt\,dx = \\
\int_a^b\int_a^b K(x,t)v(x)u(t)\,dx\,dt &= \int_a^b u(t)\left(\int_a^b K(x,\boxed{t})v(x)\,dx\right)dt \\
&= \langle u(t), \mathbf{L}^* v(t)\rangle
\end{aligned}$$

Made use of interchanging the order of integration. ($\mathbf{L}^*$: ODE uses IBP, IE uses IOOI)

Adjoint eigenfunctions $\psi(x)$ solve $\mathbf{L}^*\psi = \lambda\psi$ vs. $\mathbf{L}\phi = \lambda\phi$

$$\int_a^b K(t,\boxed{x})\psi(t)\,dt = \lambda\psi(x) \qquad \text{vs.} \qquad \int_a^b K(\boxed{x},t)\phi(t)\,dt = \lambda\phi(x)$$

$\mathbf{L}^* = \mathbf{L}$ is self-adjoint if the kernel function is symmetric after swapping $x \rightleftarrows t$:

FIE$_1$ eigenvalue problems: $\mathbf{L}_1\phi = \lambda\phi \qquad \int_a^b K(x,t)\phi(t)\,dt = \lambda\phi(x)$

Simplest problems: separable ("degenerate") $K(x,t)$ kernel functions

- $K(x,t) = \sum_{j=1}^{n} \alpha_j(x)\beta_j(t)$ finite sum of separation of variables products

Can expand out the integral for $\mathbf{L}u(x)$ as a sum of terms:

$$\int_a^b K(x,t)u(t)\,dt = \int_a^b \left(\sum_j \alpha_j(x)\beta_j(t)\right)u(t)\,dt = \sum_j \int_a^b \alpha_j(x)\beta_j(t)u(t)\,dt = \sum_{j=1}^{n}\left(\int_a^b \beta_j(t)u(t)\,dt\right)\alpha_j(x)$$

$\Longrightarrow \quad \alpha(x)$'s factor out of separate integral terms

Degenerate FIE problems reduce to $n \times n$ matrix problems $+ (\cdots$ [IOU])

Solve $\mathbf{L}_1\phi = \lambda\phi$, 2nd-order kernel ($n = 2$) example:

$$\left(\int_a^b \beta_1(t)\phi(t)\,dt\right)\alpha_1(x) + \left(\int_a^b \beta_2(t)\phi(t)\,dt\right)\alpha_2(x) = \lambda\phi(x)$$

(a) If $\lambda \neq 0$ then LHS(x) determines $\phi(x)$ trial soln form on $a \le x \le b$

(b) If $\lambda = 0$ then what is $\phi(x)$?