Math 551, Duke University — Lecture 7

Part 1: Sturm-Liouville problems (conclusion)

Part 2: The Big Questions: Existence and Uniqueness of Solutions, and the Fredholm Alternative Theorem (FAT)

Part 3: Integral equations: another class of linear operator problems

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Part 1: Linear operators and ODE BVP eigenvalue problems: SL in context

1. n^{th} order non-self-adjoint $\mathbf{L}$: IBP^n to get $\mathbf{L}^*, BC^*$ then $\mathbf{L}^*\psi = -\lambda\psi$ for $\langle \phi_j, \psi_k \rangle_2 = 0$ if $j \neq k$, λ_k can be complex

2. n^{th} order self-adjoint $\mathbf{L}$: $\mathbf{L}^* = \mathbf{L}, BC^* = BC$ so $\psi_k = \phi_k$, and $\langle \phi_j, \phi_k \rangle_2 = 0$ if $j \neq k$, and all λ_k real.

3. 2nd order general $\mathbf{L}$ with Dir/Neu/Rob BC's: $\mathbf{L}\phi = -\lambda\phi$, $\mathbf{L}^*\psi = -\lambda\psi$ if not L^2 self-adjoint can always convert to SL form (via σ), so all λ_k are real!

4. SL operators: $\boxed{\widetilde{\mathbf{L}}u \equiv \dfrac{d}{dx}\left(p(x)\dfrac{du}{dx}\right) + q(x)u}$ on finite $a \leq x \leq b$ with Dir/Neu/Rob BC's at $x = a$ and $x = b$: L^2 self-adjoint, all λ_k are real, weighted eig-val prob: $\widetilde{\mathbf{L}}\phi = -\lambda\sigma\phi$, self-orth in σ-weighted inner prod: $\langle \phi_j, \phi_k \rangle_\sigma = 0$ if $j \neq k$

And extra properties!

Important SL properties: $\widetilde{\mathrm{L}}u \equiv (p(x)u')' + q(x)u$ and $\widetilde{\mathrm{L}}\phi = -\lambda\sigma\phi$

1. Regular SL problems: If $p(x) > 0$ on $a \leq x \leq b$ then
   - $-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k \to \infty$
   - $\phi_k(x)$'s are a complete basis for L^2 expansions
   - k index gives a count of zeroes (nodes) in ϕ_k

2. Regular$^+$ SL problems: If $p(x) > 0$ and $q(x) \leq 0$ and **BC's** are: Dirichlet ($u(a) = 0$) or Neumann ($u'(a) = 0$) or "Robin$^+$": $u'(a) - \alpha^2 u(a) = 0$ and $u'(b) + \beta^2 u(b) = 0$ then regular properties and $\boxed{\text{all } \lambda_k \geq 0}$

3. **Singular SL problems**: If any of the assumptions/conditions for regular SL problems are *violated* then the problem is called "singular" and some of the results (on λ's or $\phi(x)$'s) may work differently. 3 types of "violations":
   (a) **Singular endpoints**: If $p(x) = 0$ at a bdry ($x = a$ or $x = b$) then one of the two linearly inde. hom. solns in the general soln may be "singular" ($\phi_1 \to \infty$ or $\phi_1' \to \infty$ or ...) there and is un-usable for building the eigenfcn. Then no BC at the singular endpoint: just make $\phi_k(x)$ **bounded** and use the other BC!
   (b) **Mixed BC's**: with $x = a, x = b$ terms in same eqn ($u'(a) + u(b) = 0$)
   (c) **Singular domain**: $a \leq x \leq b$ with $a = -\infty$ or $b = \infty$

Examples of SL problems (I): $\{a, b, \widetilde{\mathbf{L}}[p, q, \sigma], \widetilde{BC}[\alpha_1, \alpha_2, \beta_1, \beta_2]\}$

1. $0 \le x \le 1$ with $p(x) \equiv 1$ and $q(x) \equiv 0$ and $\sigma(x) \equiv 1$

$$\widetilde{\mathbf{L}}u = \frac{d^2u}{dx^2} \quad : \quad \widetilde{\mathbf{L}}\phi = -\lambda\sigma\phi \quad \Longrightarrow \quad \frac{d^2\phi}{dx^2} = -\lambda\phi$$

Pick Regular$^+$ BC's: as Dir, Neu, or Rob$^+$ then $\lambda \ge 0$, and general soln:

$$\phi_{\mathrm{gen}}(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

Subcases:

1. Dirichlet BC's ($\alpha_2 = \beta_2 = 0$): $\phi(0) = 0, \phi(1) = 0$
   This is the **Fourier Sine Series** – Regular SL problem.
   SL theory: $\{\phi_k(x), k = 1, 2, \cdots\}$ is **complete** for L^2 expansions
2. Neumann BC's ($\alpha_1 = \beta_1 = 0$): $\phi'(0) = 0, \phi'(1) = 0$
   This is the **Fourier Cosine Series** – Regular SL problem.
   SL theory: $\{\phi_k(x), k = 0, 1, 2, \cdots\}$ is **complete** for L^2 expansions
3. Others with Dir BC on one end, Neu BC on other...

Singular SL examples (II): $\{a, b, \widetilde{\mathbf{L}}[p, q, \sigma], \widetilde{BC}[\alpha_1, \alpha_2, \beta_1, \beta_2]\}$

3b. $0 \leq x \leq 1$ with $p(x) \equiv 1$ and $q(x) \equiv 0$ and $\sigma(x) \equiv 1$

$$\widetilde{\mathbf{L}}u = \frac{d^2u}{dx^2} \qquad : \qquad \widetilde{\mathbf{L}}\phi = -\lambda\sigma\phi \quad \Longrightarrow \quad \frac{d^2\phi}{dx^2} = -\lambda\phi$$

with periodic boundary conditions

$$\phi(0) = \phi(1) \qquad \phi'(0) = \phi'(1)$$

- Mixed BC's (info from $x = 0, x = 1$ together in same eqns
- Violates Regular SL un-mixed BC's condition, so it is a Singular problem!
- This is the **Full Fourier Series** – Singular SL problem.
- But most SL results still hold:
  - $\widetilde{\mathbf{L}}$ self-adjoint
  - λ real and $\lambda \geq 0$ too!
  - $\{\phi_k(x)\}$ **complete** for L^2 expansions
- The only thing the is violated is now there are **TWO** eigenfunctions for each $k = 1, 2, \cdots$ (and $\phi_0(x) \equiv 1$ for $\lambda_0 = 0$)

$$\lambda_k = (2\pi k)^2 \qquad \phi_k(x) = \cos(2\pi kx) \qquad \phi_k(x) = \sin(2\pi kx)$$

<u>**Singular SL examples**</u> (III): $\{a, b, \widetilde{\mathrm{L}}[p, q, \sigma], \widetilde{BC}[\alpha_1, \alpha_2, \beta_1, \beta_2]\}$

3a. A singular endpoint problem for formal SL operator:

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) - \frac{1}{4x}\phi = -\lambda x\phi \qquad 0 \le x \le 1$$

Can read off SL coeff fcns: $\quad p(x) = x \qquad q(x) = -\dfrac{1}{4x} \qquad \sigma(x) = x$

- It is a singular problem because $p(0) = 0$ violates $p > 0$ condition at $x = 0$
- This problem has general solution

$$\phi_{\mathbf{gen}}(x) = c_1\frac{\sin(\sqrt{\lambda}\,x)}{\sqrt{x}} + c_2\frac{\cos(\sqrt{\lambda}\,x)}{\sqrt{x}}$$

- $\displaystyle\lim_{x\to 0}\frac{\sin(\sqrt{\lambda}\,x)}{\sqrt{x}} = 0$ (OK) $\qquad \displaystyle\lim_{x\to 0}\frac{\cos(\sqrt{\lambda}\,x)}{\sqrt{x}} = \infty$ (Blows-up, **BAD!**)
- So $\phi(x) = \sin(\sqrt{\lambda}\,x)/\sqrt{x}$ is the only acceptable bounded soln.
  No need for a BC at $x = 0$, just kill the “bad” soln ($c_2 = 0$).
- In this problem only need one BC to pick λ's, like $\phi(1) = 0$.

$\boxed{\textbf{Part 2}}$: The "Big Questions" for all linear problems $\mathbf{L}u = f$

1. Uniqueness: *Is there only one solution?*
2. Existence: *Is there really a solution, or is there no answer?*

Uniqueness: Check for contradictions – assume there are two solns, u_1 and u_2,

$$\mathbf{L}u_1 = f \qquad BC_a u_1 = c \qquad BC_b u_1 = d$$

$$\mathbf{L}u_2 = f \qquad BC_a u_2 = c \qquad BC_b u_2 = d$$

Call the difference $w = u_1 - u_2$.
Subtract the eqns to show that w solves the homogeneous problem:

$$\mathbf{L}w = 0 \qquad BC_a w = 0 \qquad BC_b w = 0$$

One solution of this is the trivial soln, $w \equiv 0$, but that would mean that $u_2 = u_1$ are the same. Are there other solutions?
Yes, if $\lambda = 0$ is an eigenvalue of $\mathbf{L}\phi = -\lambda\sigma\phi$.
Compare the w-problem to the eigenvalue problem:

$$\mathbf{L}\phi = -\lambda\sigma\phi \qquad BC_a\phi = 0 \qquad BC_b\phi = 0$$

It matches for $\lambda_0 = 0$, so the ϕ_0 eigenfunction gives $w(x) = c\phi_0(x)$!

Uniqueness (concluded)

- So, if $u_1(x)$ is one soln, then
  $u_2(x) = u_1(x) - c\phi_0(x)$ is another soln for any c!
- $\boxed{\text{Multiple solutions exist if } \mathbf{L} \text{ has a zero eigenvalue!}}$
- If zero is NOT an eigenvalue then soln u of the BVP is unique.[a]

Existence of solutions of $\mathbf{L}u = f$: via a "constructive proof"

If you can show how to construct the complete solution, then it exists.

Use the usual soln process, ortho-project the full problem on the adjoint eigfcns ψ_k:

$$\langle \mathbf{L}u, \psi_k \rangle_2 = \langle f, \psi_k \rangle_2$$

then using IBP yields $\qquad B_k - \lambda_k c_k \langle \phi_k, \psi_k \rangle_2 = \langle f, \psi_k \rangle_2$

for each c_k in

$$u(x) = \sum_k c_k \phi_k(x) \qquad c_k = \frac{\langle f, \psi_k \rangle_2 - B_k}{-\lambda_k \langle \phi_k, \psi_k \rangle_2}$$

Also described in the **Fredholm Alternative Theorem (FAT)**: $\lambda_0 = 0$?

[a] Usually quicker/easier to check if $\lambda = 0$? rather than finding all of the other λ_k's

$\boxed{\textbf{Part 3}}$: **Integral equations** (IE): another class of linear operator problems

1. ODE problems = main equation and side conditions (IC/BC)
2. All ODE $\implies$ IE: all linear ODE problems can be re-written as integral eqns. IE have all information in a single equation. (No separate BC's)
3. But not all IE's come from ODE's (other problems too) IE theory is more general, shares main ideas with ODE/SL/Linear-Operator theory

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Examples of ODE $\implies$ IE: initial value problems (IVP)

1. First order ODE problem for u: $$\frac{du}{dx} = f(x, u(x)) \qquad u(a) = c$$

Integrate : $$\int_{t=a}^{t=x} \left(\frac{du}{dt} = f(t, u(t)) \right) dt \qquad (t \text{ 'dummy' integration variable})$$
$$u(x) - u(a) = \int_a^x f(t, u(t))\, dt$$

$\implies$ IE for u: $$u(x) = c + \int_a^x f(t, u(t))\, dt$$

Integral equations: Examples of ODE $\Longrightarrow$ IE (continued)

2. Second order ODE IVP problem for u on $x \geq a$:

$$\frac{d^2u}{dx^2} = -x^2 u \qquad u(a) = c \qquad u'(a) = d$$

Integrate twice and interchange order of integration...

$\Longrightarrow$ IE for u: $$u(x) = c + d(x-a) - \int_a^x t^2(x-t)u(t)\,dt$$

$\boxed{\textbf{Volterra IE}}$ (VIE): IE's equivalent to ODE IVP's for $x \geq a$

How to recognize VIE: variable x is a limit of integ. $u(x) = c + \int_a^{\boxed{x}} f(t, u(t))\,dt$

VIE's wont be covered in Math 551.

$\boxed{\textbf{Fredholm IE}}$ (FIE): IE's produced by ODE BVP's on $a \leq x \leq b$

How to recognize FIE: the integral is over the entire domain $\int_a^b \cdots\, dt$

Integral equation problems: solve $Lu = f(x)$ for $u(x)$

FIE's will lead to understanding Green's functions methods for ODE BVP's!

Fredholm Integral Equation (FIE) problems for unknown $u(x)$: $Lu = f(x)$

Types of linear operators in Fredholm integral equations:

- **1st kind FIE** (FIE_1): $Lu \equiv$ integral only

$$\int_a^b K(x,t)u(t)\,dt = f(x) \qquad \text{“kernel” function: } K(x,t) \text{ given}$$

- **2nd kind FIE** (FIE_2): $Lu \equiv$ integral plus multiple of soln

$$g(x)u(x) + \int_a^b K(x,t)u(t)\,dt = f(x) \qquad K(x,t), g(x) \text{ given}$$

Simplest problems: separable (“degenerate”) kernel functions

$$K(x,t) = \text{finite sum of separation of variables products} = \boxed{\sum_{j=1}^{n} \alpha_j(x)\beta_j(t)}$$

Harder problems: non-degenerate/non-separable kernels ($n = \infty$): $K(x,t) = \frac{1}{x-t}$ Hilbert, $K = e^{-xt}$ Laplace, $K = J_0(xt)$ Hankel, and $K = e^{-ixt}$ Fourier integral transforms (later)