$\boxed{\text{Part 1}}$: Inner products of fcns, orthogonality of fcns

Math 551, Lecture 3

- Pick a form for the inner product: $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$ **L2 recap**
- If $\{\phi_k(x)\}$ is a self-orthogonal set then $\langle \phi_j, \phi_k\rangle = 0$ for $j \neq k$
- If $\{\phi_k(x)\}$ is a complete set then can write orthogonal expansion of L^2 fcns:

$$f(x) \stackrel{ae}{=\!=} \sum_{k=1}^{\infty} c_k\phi_k(x) \qquad c_k = \frac{\langle f, \phi_k\rangle}{||\phi_k||^2}$$

- Each $\{\phi_k(x)\}$ is a set of eigenfunctions[a] from a (self-adjoint) linear operator L

Three examples: The classic Trig Fourier series

1. For $0 \le x \le \pi$, let $\boxed{\phi_k(x) = \sin(kx)}$ with $k = 1, 2, 3, \dots$ Fourier sine series

$$\langle \phi_j, \phi_k\rangle = \int_0^{\pi} \sin(kx)\sin(jx)\,dx = 0 \qquad ||\phi_k||^2 = \int_0^{\pi} \sin^2(kx)\,dx = \frac{\pi}{2}$$

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(kx) \qquad c_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx$$

Completeness? (IOU) What L operator do the $\phi_k(x)$'s come from? (IOU)

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[a] just like eigenvectors $\{\vec{\phi}_k\}$ from a self-adjoint matrix $\mathbf{L}$

<u>Three examples</u>: The classic Trig Fourier series (Haberman Chap 3)

2. For $0 \le x \le \pi$, let $\boxed{\phi_k(x) = \cos(kx)}$ with $k = \boxed{0}, 1, 2, 3, \ldots$

   <u>Fourier cosine series</u> $\{\phi_k\} = \{1, \cos(x), \cos(2x), \cos(3x), \cdots\}$

$$\int_0^\pi \cos(kx)\cos(jx)\,dx = 0 \qquad \int_0^\pi \cos^2(kx)\,dx = \begin{cases} \pi & k = 0 \\ \frac{\pi}{2} & k = 1, 2, \cdots \end{cases}$$

$$f(x) = \sum_{k=0}^{\infty} c_k \cos(kx) \qquad c_k = \frac{1}{||\phi_k||^2}\int_0^\pi f(x)\cos(kx)\,dx$$

(a.k.a.) $f(x) = \frac{1}{2}\widetilde{c}_0 + \sum_{k=1}^{\infty} \widetilde{c}_k \cos(kx) \qquad \widetilde{c}_k = \frac{2}{\pi}\int_0^\pi f(x)\cos(kx)\,dx$

   Completeness? (IOU) These ϕ_k come from a different L, what is it? (IOU)

3. For $\boxed{-\pi} \le x \le \pi$, let $\boxed{\phi_k = \{\sin(kx), \cos(kx)\}}$, $k = 0, 1, 2, \cdots$

   <u>The Full Fourier series</u> $f(x) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k \cos(kx) + d_k \sin(kx)$

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \qquad d_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

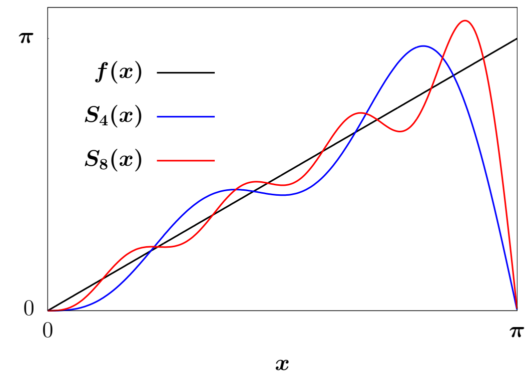
   Completeness? (IOU again) and a third different L operator (IOU)

$\boxed{\text{Part 2}}$: Properties of Fourier series **(Haberman Chap 3)**

For L^2 functions: $f(x) \stackrel{ae}{=\!=} \sum_{k=1}^{\infty} c_k \phi_k(x)$ $\qquad c_k = \dfrac{\langle f, \phi_k \rangle}{||\phi_k||^2}$ $\quad$ ($\phi_k(x)$ self-orth. set)

Call the N-term partial sum approximation: $S_N(x) = \sum_{k=1}^{N} c_k \phi_k(x)$

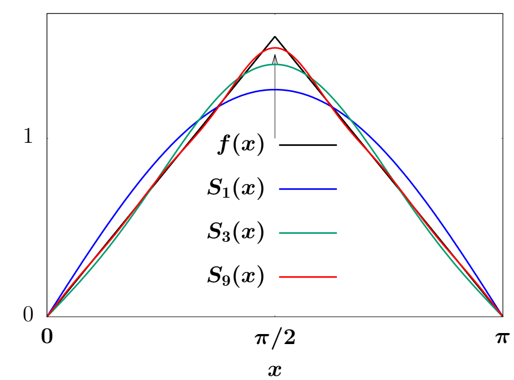
Examples: Use Sine series, $\phi_k(x) = \sin(kx)$ on $0 < x < \pi$



(a) $f_a(x) = x$: c_k via $\boxed{\textbf{Integration By Parts}}$ (IBP)

$$c_k = \frac{2}{\pi}\int_0^{\pi} \underbrace{x}_{u} \underbrace{\sin(kx)\,dx}_{dv} = (-1)^{k+1}\frac{2}{k}$$

(b) $f_b(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \le \pi \end{cases}$ Sub-intervals then IBP



$$c_k = \frac{2}{\pi}\left[\int_0^{\pi/2} x\phi_k + \int_{\pi/2}^{\pi} (\pi - x)\phi_k\right] = \frac{4}{\pi k^2}\sin(\tfrac{k\pi}{2})$$

(c) $f_c(x) = \begin{cases} \frac{1}{2} & 0 \le x \le \pi/2 \\ 1 & \pi/2 < x \le \pi \end{cases}$ Sub-intervals $\quad c_k = \dfrac{1 + \cos(\frac{k\pi}{2}) - 2(-1)^k}{k\pi}$

Special properties of Trigonometric Fourier Series

1. $\sin(kx), \cos(kx)$ are 2π-periodic functions, $\phi_k(x + 2\pi) = \phi_k(x) \quad \forall x$.
   Most other $\phi_k(x)$ are only defined on an interval $a \le x \le b$.
2. Sine/Cosine/Full Fourier series describe periodic extensions of the original $f(x)$ piece given on $x \in [-\pi, \pi)$, with

$$\boxed{f(x + 2n\pi) = f(x)} \qquad n\text{: all integers}$$

3. If $f(x)$ is given on $x \in [0, \pi)$ then the Sine series gives the expansion of the 2π-periodic $\boxed{\text{odd}}$ extension of $f(x)$:

$$f_{\text{odd}}(x) = \begin{cases} f(x) & x \in [0, \pi) \\ -f(-x) & x \in [-\pi, 0) \end{cases} \qquad f_{\text{odd}}(x + 2n\pi) = f_{\text{odd}}(x)$$

4. If $f(x)$ is given on $x \in [0, \pi)$ then the Cosine series gives the expansion of the 2π-periodic $\boxed{\text{even}}$ extension of $f(x)$:

$$f_{\text{even}}(x) = \begin{cases} f(x) & x \in [0, \pi) \\ f(-x) & x \in [-\pi, 0) \end{cases} \qquad f_{\text{even}}(x + 2n\pi) = f_{\text{even}}(x)$$

5. These extensions (periodic and odd/even) can produce discontinuities or changes in the smoothness of the function at the edges of the original base interval.

1. **Parseval's theorem**: $||f||^2 = \sum_{k=1}^{\infty} c_k^2 ||\phi_k||^2$

General Convergence Theory

$$\int_a^b f^2\,dx = \left\langle \sum_k c_k \phi_k, \sum_j c_j \phi_j \right\rangle = \sum_k c_k \left\langle \phi_k, \sum_j c_j \phi_j \right\rangle = \sum_k c_k^2 \langle \phi_k, \phi_k \rangle$$

2. **L^2 convergence** of N-term approx of $f(x)$ as $N \to \infty$: show $S_N \xrightarrow{ae} f$

$$f(x) = \boxed{\sum_{k=1}^{N} c_k \phi_k(x)} + \sum_{k=N+1}^{\infty} c_k \phi_k(x) \qquad S_N(x) = \sum_{k=1}^{N} c_k \phi_k(x)$$

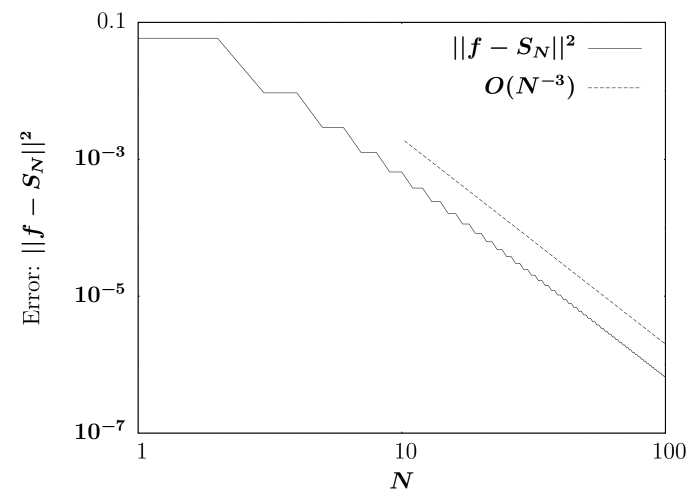
Show that the L^2 norm of the remainder $\to 0$ as $N \to \infty$

$$\lim_{N\to\infty} ||f - S_N||^2 = \lim_{N\to\infty} \left|\left| \sum_{k=N+1}^{\infty} c_k \phi_k(x) \right|\right|^2 = \lim_{N\to\infty} \sum_{k=N+1}^{\infty} c_k^2 ||\phi_k||^2 = 0$$

If $\{\phi_k(x)\}$ not complete set, then error $\not\to 0$

Example: $f_b(x)$: $\qquad c_k = \frac{4}{\pi k^2} \sin(k\pi/2)$

$$||f - S_N||^2 = \sum_{k=N+1}^{\infty} \frac{8}{\pi^2 k^4}$$



3. **Pointwise convergence**: If the original fcn $f(x)$ is continuous at a ("good") point x_0 then the Fourier series converges to the fcn's value there

$$\lim_{N\to\infty} S_N(x_0) = f(x_0)$$

4. If $f(x)$ is discontinuous at a ("bad") pt x_0 (has a "jump"), meaning that

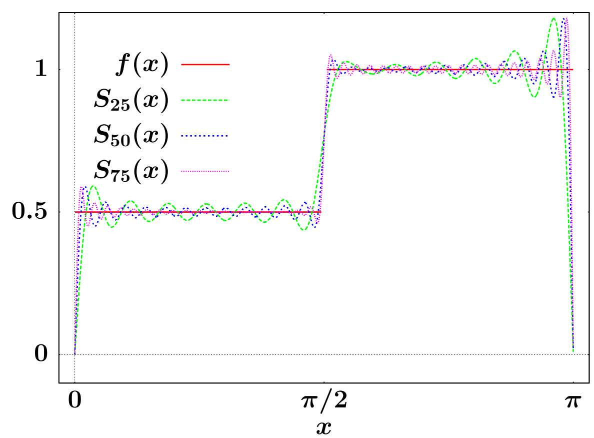
$$\lim_{x\to x_0^-} f(x) \neq f(x_0) \neq \lim_{x\to x_0^+} f(x)$$

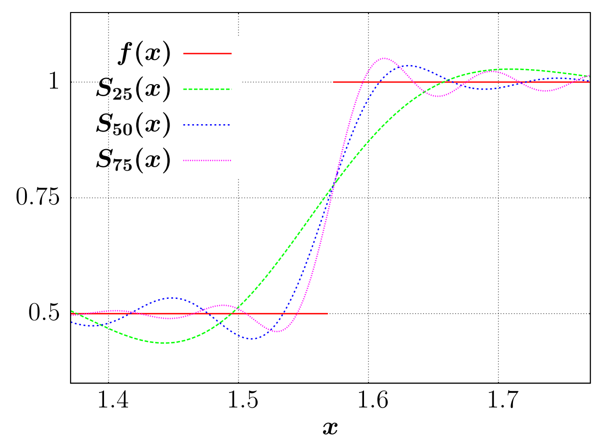
then Fourier series converges to $\boxed{\text{the avg of the left/right-limit values at the jump}}$

$$\lim_{N\to\infty} S_N(x_0) = \tfrac{1}{2}[f(x_0^-) + f(x_0^+)]$$

5. $\boxed{\textbf{Gibbs phenomenon}}$: the 8.9% overshoot and "ringing" of the Fourier series approx. at a bad pt x_0 where the extended function $f(x)$ has a discont/jump.

Example: Sine series of $f_c(x)$





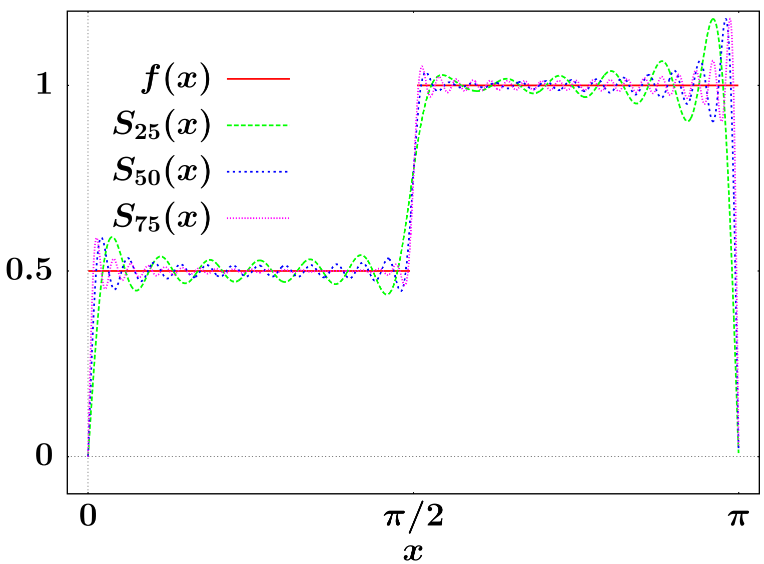
6. **Properties of the c_k coefficients** (and the smoothness of $f(x)$)

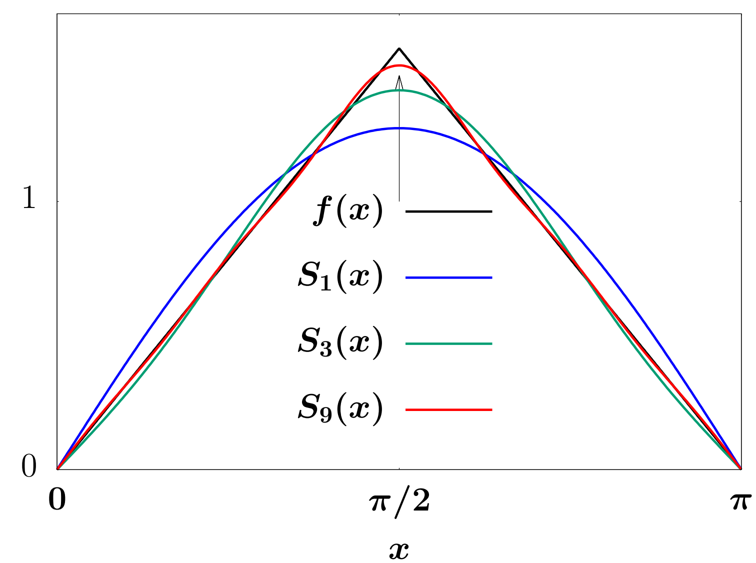
(0) If $f(x)$ has jump(s) then its Fourier series will have "*coefficients that decay like* $\boxed{1/k}$ *as* $k \to \infty$", meaning that the limit of the ratio satisfies:

$$\lim_{k\to\infty} \frac{c_k}{(1/k)} = \text{finite value}$$

(1) If $f(x)$ is continuous everywhere but it has a corner (so $f'(x)$ has a jump somewhere) then $f(x)$'s Fourier series c_k's decay like $\boxed{1/k^2}$ as $k \to \infty$.

(n) If the $f^{(n)}(x)$ derivative is the first one to have a jump, then the Fourier series for $f(x)$ has coefficients c_k that decay like $\boxed{1/k^{n+1}}$ as $k \to \infty$.





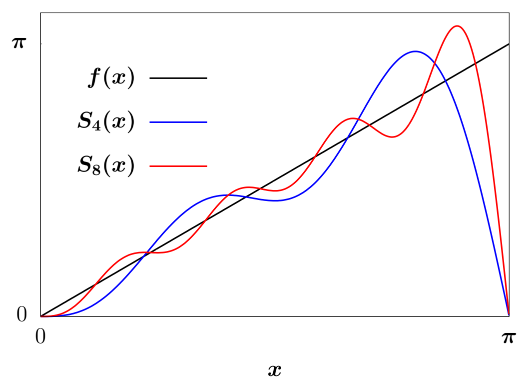
Valid operations on converging Fourier series that can be done term-by-term:

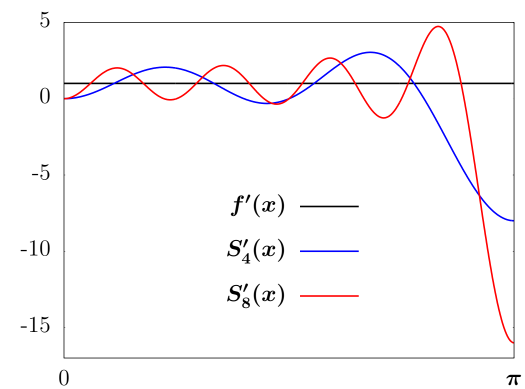
- Scalar multiplication: $af(x) = \sum_k (ac_k)\phi_k(x)$
- Addition: $f(x) + g(x) = \sum_k (c_k + d_k)\phi_k(x)$
- Integration: $\int f(x)\, dx = \sum_k c_k \left(\int \phi_k(x)\, dx\right)$

BUT $\boxed{\textbf{Differentiation}}$ might lead to series that do not converge (i.e. are **wrong!**)

Example: $f_a(x) = x$ then call $g(x) = f'(x) = 1$

$$\frac{d}{dx}\left[x = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{2}{k}\sin(kx)\right] \qquad \to \qquad 1 \neq \sum_{k=1}^{\infty} 2(-1)^{k+1}\cos(kx)$$





- The cosine series of $g(x) = 1$ is $g(x) = 1 + \sum_k 0\cos(kx)$
- $c_k = \pm 2$ does not satisfy $|c_k| \to 0$ as $k \to \infty$ (n^{th} term convergence test)
- $\frac{d}{dx}$ (sine series of $f(x)$) DOES NOT converge to the cosine series for $f'(x)$(!?!)

A big issue for Math 551 is understanding how to work around this problem to obtain guaranteed-correct Fourier series for solutions of differential equations.

$\boxed{\text{Part 3}}$: Extending Linear Algebra Theory to Linear Differential Eqns

- Solutions/domains: $\mathbf{u} \in \mathbb{R}^n \quad \to \quad u(x)$ on $a \le x \le b$
- Inner products: $\mathbf{u} \cdot \mathbf{v} \quad \to \quad \int_a^b u(x)v(x)\,dx$
- Orthogonality: $\mathbf{u} \cdot \mathbf{v} = 0 \quad \to \quad \int_a^b uv\,dx = 0$
- Linear operators: matrix $\mathbf{L} \quad \to \quad L =$ LHS of ODE

  Example: $Lu(x) \equiv p\frac{d^2u}{dx^2} + q\frac{du}{dx} + ru$
- Problems: $\mathbf{Lu} = \mathbf{b} \quad \to \quad Lu(x) = f(x)$

Boundary value problems (BVP) for ODE's have four basic parts:

1. Domain $a \le x \le b$
2. Linear operator L: LHS of eqn $Lu = f$
3. Inhomogeneous forcing function $f(x)$: RHS of eqn $Lu = f$
4. Hom./Inhomogeneous Boundary Conditions (BC) at $x = a$ and $x = b$

Without BC's, $Lu = f$ has many solns. Adding BC's picks one $\boxed{\text{final solution}}$

- L without specific BC's = "**Formal Linear Operator**" (LHS only)
- L with specific Homogeneous BC's = "**Complete Linear Operator**"

Eigenvalue problems: overview

$$\text{Matrix: } \mathbf{L}\vec{\phi} = \lambda\vec{\phi} \qquad \rightarrow \qquad \text{ODE BVP: } L\phi(x) = -\lambda\phi(x)$$

- Historical tradition: ODE eigenvalue equation has an extra minus sign
- Matrix eigenvalues for $\mathbf{L}_{n\times n}$: determinant $|\mathbf{L} - \lambda\mathbf{I}| = \mathbf{0}$ has n eig-vals
- ODE BVP eigenvalues: has an infinite number of eig-vals. (How? IOU)

Adjoint problems: needed to calculate c_k's (need ψ's for $\langle f, \psi\rangle$ and $\langle \phi, \psi\rangle$)

- Matrix case: Make the adjoint $\mathbf{L}^*$ from original $\mathbf{L}$:

  if $\mathbf{L}$ real, then $\mathbf{L}^* = \mathbf{L}^T$ if $\mathbf{L}$ complex, then $\mathbf{L}^* = \mathbf{L}^H$
- ODE BVP: How do you make the complete adjoint operator?

$$L^* \text{ from } L \qquad \text{and} \qquad BC^* \text{ from } BC$$

- The complete adjoint L^* is always defined by the **Inner product adjoint relation**:

$$\boxed{\langle v(x), Lu(x)\rangle = \langle L^*v(x), u(x)\rangle} \qquad \text{for all } u, v\text{'s}$$

For differential equations, this relation is also called the Lagrange identity or Green's formula in some books.