From **Lecture 4**

Claim: If L is self-adjoint ($L^* = L$) then all of its eigenvalues λ are real.

Proof

Assume L-op is a real operator, meaning that it doesn't have any complex coeffs in it

like $Lu = a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu$ with a, b, c real #s

so if $u(x)$ is a real fcn then $v = Lu$, $v(x)$ is real too.

The eigenvalue eqn is $\boxed{L\phi = -\lambda\phi}$ (1)

Allow the possibility that $\lambda, \phi(x)$ might be complex.

Take the conjugate of (1) $\overline{L\phi} = -\overline{\lambda\phi}$

Since no imaginary parts in L, $\bar{L} = L$

$$L\bar{\phi} = -\bar{\lambda}\bar{\phi} \quad (2)$$

so $\bar{\lambda}$ is also an eigenvalue, with eig fcn $\overline{\phi(x)}$

Use the real inner product $\langle f, g\rangle = \int_0^1 fg\,dx$

(3) inner prod of (2) w/ ϕ $\quad \langle\phi, L\bar{\phi}\rangle = \langle\phi, -\bar{\lambda}\bar{\phi}\rangle = -\bar{\lambda}\langle\phi, \bar{\phi}\rangle$

(4) inner prod of (1) w/ $\bar{\phi}$ $\quad \langle\bar{\phi}, L\phi\rangle = \langle\bar{\phi}, -\lambda\phi\rangle = -\lambda\langle\bar{\phi}, \phi\rangle$

also need the adjoint relation:

$$\langle\phi, L\bar{\phi}\rangle = \langle L^*\phi, \bar{\phi}\rangle = \langle L\phi, \bar{\phi}\rangle$$

since $L^* = L$ self-adjoint

and that the real inner product commutes:

$$\langle f, g\rangle = \int_0^1 fg\,dx = \int_0^1 gf\,dx = \langle g, f\rangle$$

so (3) − (4) $= \langle\phi, L\bar{\phi}\rangle - \langle\bar{\phi}, L\phi\rangle$

$= \langle L\phi, \bar{\phi}\rangle - \langle\bar{\phi}, L\phi\rangle$ ← commutes

$= \langle L\phi, \bar{\phi}\rangle - \langle L\phi, \bar{\phi}\rangle$

$= 0$

$\lambda = a + ib$

$\bar{\lambda} = a - ib$

(3) − (4) also $= -\bar{\lambda}\langle\phi, \bar{\phi}\rangle + \lambda\langle\bar{\phi}, \phi\rangle = (\lambda - \bar{\lambda})\int_0^1 |\phi|^2 dx$

so $\lambda - \bar{\lambda} = 0 \rightarrow$ λ is real $\quad = 0$