1 The Laplacian in different coordinate systems

- Rectangular coordinates (1D) $u = u(x)$ (2D) $u = u(x, y)$ or (3D) $u = u(x, y, z)$

$$\boxed{\nabla^2 u = \frac{d^2u}{dx^2}} \qquad \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}} \qquad \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}}$$

- Cylindrical polar coordinates (2D) $u = u(r, \theta)$ or (3D) $u = u(r, \theta, z)$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad z = z$$

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} \qquad \boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}}$$

- Spherical polar coordinates $u = u(\rho, \theta, \phi)$

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \arctan(y/x) \qquad \phi = \arctan(r/z)$$

$$\boxed{\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right)}$$

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2 ODE eigenvalue problems from sep. of vars. for $\nabla^2\phi = -\lambda\phi$

1) General self-adjoint 2nd order eigenvalue problems
Sturm-Liouville equation on $a \le x \le b$

$$\boxed{\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = -\lambda\sigma(x)y} \qquad Ly = -\lambda\sigma y$$

Singular problem:
If $p(a) = 0$, then no BC at $x = a$ (similarly at $x = b$)
One solution may be singular (remove it), requiring solns with good behavior yields an effective BC there.
General solution (hom soln of $Ly + \lambda\sigma y = 0$)

$$\boxed{y(x) = c_1 w_1(x) + c_2 w_2(x)}$$

Homogeneous BC's applied to the general soln selects the eigensolns $\big(\lambda_k,\, y_k(x)\big)$ for $k = 0, 1, 2, \cdots$.
Inner product with weight function $\sigma(x)$

$$\langle y_k, y_\ell\rangle_\sigma \equiv \int_a^b y_k(x)y_\ell(x)\sigma(x)\,dx$$

Orthogonality $\langle y_k, y_\ell\rangle_\sigma = 0$ if $k \ne \ell$
Norm $||y_k||^2 \equiv \langle y_k, y_k\rangle_\sigma$
Series expansion, coefficients

$$f(x) = \sum_{k=0}^{\infty} c_k y_k(x), \qquad c_k = \frac{\langle f(x), y_k(x)\rangle_\sigma}{||y_k||^2}$$

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2) Rectangular coordinates: $f(x), g(y)$ or $g(\theta)$
Harmonic oscillator equation on $0 \le x \le b$

$$\boxed{y'' = -\lambda y}$$

SL coefficient fcns: $p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$
General solution $\lambda > 0$ ($m = \pm i\sqrt{\lambda}$ in $y = e^{mx}$)

$$\boxed{y(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)}$$

Inner product with weight function $\sigma(x) = 1$

$$\langle y_k, y_\ell\rangle \equiv \int_0^b y_k(x)y_\ell(x)\,dx$$

Useful formulas: (for Dir. or Neu. BC's)

$$\sin(k\pi) = 0, \qquad \cos(k\pi) = (-1)^k$$

$$\sin\left(\tfrac{(2k+1)\pi}{2}\right) = (-1)^k, \qquad \cos\left(\tfrac{(2k+1)\pi}{2}\right) = 0$$

$$||\sin(\sqrt{\lambda}x)||^2 = ||\cos(\sqrt{\lambda}x)||^2 = b/2, \qquad \lambda > 0$$

$$||1||^2 = b, \qquad (\cos(\sqrt{\lambda}x) \equiv 1 \text{ for } \lambda = 0)$$

3) Polar coordinates (Laplace): $f(r)$
Cauchy-Euler equation on $a \le x \le b$

$$\boxed{(xy')' = -\lambda x^{-1} y} \qquad x^2 y'' + xy' = -\lambda y$$

SL coefficient fcns: $p(x) = x$, $q(x) = 0$, $\sigma(x) = 1/x$
Singular at $x = 0$: $p(0) = 0$
General solution $\lambda > 0$ ($m = \pm i\sqrt{\lambda}$ in $y = x^m$)

$$\boxed{y(x) = c_1 \cos\left(\sqrt{\lambda}\ln(x)\right) + c_2 \sin\left(\sqrt{\lambda}\ln(x)\right)}$$

Inner product with weight function $\sigma(x) = 1/x$

$$\langle y_k, y_\ell \rangle_\sigma \equiv \int_a^b y_k(x) y_\ell(x) \frac{1}{x}\, dx$$

Useful formulas: (for Dir. or Neu. BC's)

$$||\sin\left(\sqrt{\lambda}\ln(x)\right)||^2 = ||\cos\left(\sqrt{\lambda}\ln(x)\right)||^2 = \frac{1}{2}\ln(b/a)$$

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4) Polar coordinates (Helmholtz): $f(r)$
Bessel's equation of order m on $a \le x \le b$

$$\boxed{(xy')' - \frac{m^2}{x} y = -\lambda x y} \qquad x^2 y'' + xy' - m^2 y = -\lambda x^2 y$$

SL coefficient fcns: $p(x) = x$, $q(x) = -m^2/x$, $\sigma(x) = x$
Singular at $x = 0$: $p(0) = 0$
General solution $\lambda > 0$

$$\boxed{y(x) = c_1 \mathrm{J}_m(\sqrt{\lambda}x) + c_2 \mathrm{Y}_m(\sqrt{\lambda}x)}$$

Inner product with weight function $\sigma(x) = x$

$$\langle y_k, y_\ell \rangle_\sigma \equiv \int_a^b y_k(x) y_\ell(x) x\, dx$$

Useful formulas:

$$\mathrm{J}_0(0) = 1, \qquad \mathrm{J}_0'(0) = 0, \qquad \mathrm{Y}_0(0) = -\infty$$

$$\mathrm{J}_m(0) = 0, \qquad \mathrm{Y}_m(0) = -\infty, \qquad m > 0$$

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5) Spherical coordinates (Helmholtz): $f(\rho)$
spherical Bessel's equation

$$\boxed{(x^2 y')' - \mu y = -\lambda x^2 y}$$

SL coefficient fcns: $p(x) = x^2$, $q(x) = -\mu$, $\sigma(x) = x^2$
Singular at $x = 0$: $p(0) = 0$
General solution $\lambda > 0$, Bessel order $m = \sqrt{\mu + \frac{1}{4}}$

$$\boxed{y(x) = c_1 \frac{\mathrm{J}_m(\sqrt{\lambda}x)}{\sqrt{x}} + c_2 \frac{\mathrm{Y}_m(\sqrt{\lambda}x)}{\sqrt{x}}}$$

Inner product $\langle y_k, y_\ell \rangle_\sigma \equiv \int_a^b y_k(x) y_\ell(x) x^2\, dx$
Useful formulas:
If $\mu = n(n+1)$ then $m = \sqrt{n^2 + n + 1/4} = n + 1/2$

$$\frac{\mathrm{J}_{n+1/2}(x)}{\sqrt{x}} = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left[\frac{\sin x}{x}\right] \sqrt{\frac{2}{\pi}}$$

$$\frac{\mathrm{J}_{1/2}(x)}{\sqrt{x}} = \frac{\sin x}{x}\sqrt{\frac{2}{\pi}} \qquad \frac{\mathrm{J}_{3/2}(x)}{\sqrt{x}} = \frac{\sin x - x\cos x}{x^2}\sqrt{\frac{2}{\pi}}$$

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6) Spherical coordinates: $g(\phi)$
Legendre's eqn of order m on $-1 \le x \le 1$

$$\boxed{\left((1 - x^2) y'\right)' - \frac{m^2}{1 - x^2} y = -\lambda y}$$

SL: $p(x) = 1 - x^2$, $q(x) = -m^2/(1 - x^2)$, $\sigma(x) = 1$
Singular at $x = \pm 1$: $p(\pm 1) = 0$
General solution $\lambda > 0$

$$\boxed{y(x) = c_1 \mathrm{P}_\lambda^m(x) + c_2 \mathrm{Q}_\lambda^m(x)}$$

Inner product with weight function $\sigma(x) = 1$

$$\langle y_k, y_\ell \rangle_\sigma \equiv \int_a^b y_k(x) y_\ell(x)\, dx$$

Useful formulas:

$$\boxed{g(\phi) = y(\cos\phi), \qquad x = \cos\phi}$$

$$\boxed{(\sin\phi\, g')' + \left(\lambda\sin\phi - \frac{m^2}{\sin\phi}\right) g = 0}$$

$\mathrm{P}_\lambda^m(x) =$ "1st kind Legendre fcn", $\qquad \mathrm{Q}_\lambda^m(\pm 1) = \infty$

For $-1 \le x \le 1$

$$\lambda_n = n(n+1), \qquad y_n(x) = P_n^m(x), \qquad n = 0, 1, 2, \cdots$$

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

For $m = 0$: $P_n^m(x) = P_n(x)$ Legendre polynomials
Even/odd polynomials with order n

For $0 \le x \le 1$ (need BC at $x = 0$)

$$y'(0) = 0, \qquad \lambda_n = 2n(2n+1), \qquad y_n(x) = P_{2n}^m(x)$$

$$y(0) = 0, \qquad \lambda_n = 2(n+1)(2n+1), \qquad y_n(x) = P_{2n+1}^m(x)$$

Eqns, Solns	$\lambda > 0$: Oscillatory solutions	$\lambda = -\alpha^2 < 0$: Non-oscillatory solns
Harmonic, $f(x)$, $g(\theta)$	$\sin(\sqrt{\lambda}x), \cos(\sqrt{\lambda}x)$	$\sinh(\alpha x), \cosh(\alpha x)$ or $e^{\pm\alpha x}$
Cauchy-Euler, $f(r)$	$\sin(\sqrt{\lambda}\ln r), \cos(\sqrt{\lambda}\ln r)$	$r^\alpha, r^{-\alpha}$
Bessel order m, $f(r)$	$\mathrm{J}_m(\sqrt{\lambda}r), \mathrm{Y}_m(\sqrt{\lambda}r)$	$\mathrm{I}_m(\alpha r), \mathrm{K}_m(\alpha r)$
Legendre, $g(\phi)$	$\mathrm{P}_n(\cos\phi), \mathrm{Q}_n(\cos\phi)$	—