Math 551 Green's fcn Example

Solve $u'' + u = f(x) \quad 0 \le x \le 1$

$u'(0) = P \quad u(1) = Q$

Green's fcn $G(x)$: $\dfrac{d^2 G}{dx^2} + G = \delta(x - s)$ $\quad G_x(0) = 0$, $\quad G(1) = 0$

$$G(x) = \begin{cases} G_-(x) & 0 \le x < s \\ G_+(x) & s < x \le 1 \end{cases}$$

Left part $x < s$: $\dfrac{d^2 G_-}{dx^2} + G_- = 0$ osc eqn

$G_-(x) = A \sin x + B \cos x$

$G_-'(x) = A \cos x - B \sin x$

Left BC $G_-'(0) = A = 0$ $\quad \boxed{G_-(x,s) = B\cos x}$

Right part $s < x \le 1$: $\dfrac{d^2 G_+}{dx^2} + G_+ = 0$ LCC/osc eqn

$G_+ = C \sin x + D \cos x$

$G_+ = \tilde{C} \sin(x-1) + \tilde{D} \cos(x-1)$ $\quad x-1=0$ @ BC

LCC translational shift $x \to x-1$ to make some algebra cleaner :)

Right BC $G_+(x=1) = 0 + \tilde{D}\cos(0) = 0 \quad \tilde{D} = 0$

$$\boxed{G_+(x,s) = C \sin(x-1)}$$

Jump Conditions @ $x = s$ $\quad$ ($n=2$ 2nd order ODE)

(n-1) $\dfrac{dG_+}{dx} - \dfrac{dG_-}{dx} = 1 \qquad C\cos(s-1) + B\sin(s) = 1$

(n-2) $G_+(s) - G_-(s) = 0 \qquad C\sin(s-1) - B\cos(s) = 0$

$$\begin{pmatrix} \sin(s) & \cos(s-1) \\ -\cos(s) & \sin(s-1) \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Alg + Trig}$$

Trig: $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

$$\longrightarrow B = \frac{\sin(s-1)}{\cos(1)} \qquad C = \frac{\cos(s)}{\cos(1)}$$

$$G(x,s) = \frac{1}{\cos(1)} \begin{cases} \sin(s-1)\cos(x) & 0 \le x < s \quad G_- \\ \cos(s)\sin(x-1) & s < x \le 1 \quad G_+ \end{cases}$$

<u>Return to full problem</u> $\langle Lu, G \rangle = \langle f, G \rangle$

$$\int_0^1 (\underbrace{u_{ss}}_{IBP^2} + u) G(x,s)\,ds = \int_0^1 f(s) G(x,s)\,ds$$

$$G(x,s)\frac{du(s)}{ds} - \frac{\partial G}{\partial s} u(s) \Big|_{s=0}^{s=1} + \langle u, \underbrace{L^* G}_{\delta(s-x)} \rangle = \langle f, G \rangle$$

BC's

<u>$s=0$</u> $\quad u'(0) = P \qquad u(0)$ unknown

$$G_+(x,0) = \frac{\cos(0)\sin(x-1)}{\cos(1)} = \frac{\sin(x-1)}{\cos(1)}$$

$$\frac{\partial G_+}{\partial s}(x,0) = \frac{-\sin(0)\sin(x-1)}{\cos(1)} = 0$$

<u>$s=1$</u> $\quad u(1) = Q \qquad u'(1)$ unknown

$$G_-(x,1) = \frac{\sin(1-1)\cos(x)}{\cos(1)} = 0$$

$$\frac{\partial G_-}{\partial s}(x,1) = \frac{\cos(1-1)\cos(x)}{\cos(1)} = \frac{\cos(x)}{\cos(1)}$$

$$\left(0 - \frac{P\sin(x-1)}{\cos(1)}\right) - \left(\frac{Q\cos(x)}{\cos(1)} - 0\right) + u(x) = \langle f, G \rangle$$

$$u(x) = \underbrace{\frac{P\sin(x-1)}{\cos(1)} + \frac{Q\cos(x)}{\cos(1)}}_{\text{homogeneous soln with BC's}} + \underbrace{\int_0^1 G(x,s) f(s)\,ds}_{\text{~~f(x)~~ forcing (particular soln)}}$$