Math 551, Duke University — Lecture 4

Week 1 recap: Eigen-expansion solution of matrix/vector problem $\mathbf{L}\vec{\mathbf{u}} = \vec{\mathbf{f}}$ is

$$\vec{\mathbf{u}} = \sum_{k=1}^{n} c_k \phi_k \qquad \text{with coeffs} \qquad c_k = c(\vec{\mathbf{f}}, \lambda_k, \phi_k, \psi_k)$$

Steps:

1. Find λ_k, ϕ_k from $\mathbf{L}\phi = \lambda\phi$
2. Determine the adjoint $\mathbf{L}^*$, this depends on the inner product
   If $\mathbf{L}^* = \mathbf{L}$ then $\psi = \phi$ else find ψ's from $(\mathbf{L}^* - \lambda\mathbf{I})\psi_k = 0$
3. Determine c_k's from $\langle \psi_k, \mathbf{L}\vec{\mathbf{u}} \rangle = \langle \psi_k, \vec{\mathbf{f}} \rangle$

Extending Linear Algebra Theory to Linear Differential Eqns

- Solutions/domains: $\mathbf{u} \in \mathbb{R}^n \quad \to \quad L^2$ fcn $u(x)$ on $a \le x \le b$
- Inner products: $\mathbf{u} \cdot \mathbf{v} \quad \to \quad \int_a^b u(x)v(x)\,dx$
- Orthogonality: $\mathbf{u} \cdot \mathbf{v} = 0 \quad \to \quad \int_a^b uv\,dx = 0$
- Linear operators: matrix $\mathbf{L}_{n\times n} \quad \to \quad \mathbf{L} =$ LHS of ODE & BC's
- Problems: $\mathbf{L}\mathbf{u} = \mathbf{f} \quad \to \quad Lu(x) = f(x)$

ODE Boundary Value Problems (BVP) (part I)

$$\text{Ex :} \qquad \begin{cases} \dfrac{d^2u}{dx^2} + 4x\dfrac{du}{dx} + 5u = e^{3x} & 0 \le x \le 2 \\ u(0) = 6 & u'(2) - 3u(2) = 0 \end{cases}$$

Every ODE BVP has 4 parts:

1. The domain ($a \le x \le b$)
2. The differential equation which holds for all points in the domain (ODE)
3. The boundary equations that hold at each endpoint (BC's)
4. Righthandside forcing ($f(x)$ fcn in ODE, consts in BC's)

- Each equation in the problem has:

  LHS $u =$ operator acting on soln $\qquad$ RHS $= \begin{cases} 0 & \text{homogeneous case} \\ \neq 0 & \text{inhomogeneous case} \end{cases}$

- If all eqns' RHS's are homogeneous, then the problem is homogeneous
- If any eqn has an inhomogeneous RHS, then the problem is inhomogeneous.
- If both the ODE and BC's are homogeneous, then $u(x) \equiv 0$ is a soln, but it is not helpful (called "the trivial soln"). Only NON-trivial solns are useful.

ODE BVP (part II): definitions

1. **Linear differential operator** = linear sum of soln and derivatives times coeffs
   Ex: $\mathbf{L}u = 4u'' + 3xu' + 5e^{2x}u$

2. **Formal linear operator** = LHS of ODE only
   Complete linear operator = LHS's of (ODE and all BC's)
   Without BC's $\mathbf{L}u = f$ has many solns. With BC's, final soln is *"specific"*.

3. **Classes of BC's** (see Haberman, page 156) (inhom. / hom. versions)
   (a) **Dirichlet** (1st kind) Sets value of solution: $u(a) = g \quad / \quad u(b) = 0$
   (b) **Neumann** (2nd kind) Sets value of derivative: $u'(a) = h \quad / \quad u'(b) = 0$
   (c) **Robin** (3rd kind) Sets value of a linear combination:
   $$u'(a) + ku(a) = d \qquad / \qquad u'(b) - pu(b) = 0$$
   (d) BVP can mix different kinds of BC's at left ($x = a$) & right ($x = b$) boundaries
   (e) Separated BC's: (Isolated) not involving the other boundary (ex: 1st, 2nd, 3rd kind)
   (f) Mixed BC's: coupling properties from both boundaries in a single equation

   Ex: **Periodic BC's** $\quad = \quad \{\, u(b) - u(a) = 0, \quad u'(b) - u'(a) = 0 \,\}$

ODE Adjoint problems: needed to calculate c_k's (need ψ's for $\langle f, \psi\rangle$ and $\langle \phi, \psi\rangle$)

- ODE BVP: How do you determine the complete adjoint operator?

$$\mathbf{L}^* \text{ from } \mathbf{L} \qquad \text{and} \qquad BC^* \text{ from } BC$$

The complete adjoint $\mathbf{L}^*$ is always defined by the **Inner product adjoint relation**:

$$\boxed{\langle v(x), \mathbf{L}u(x)\rangle = \langle \mathbf{L}^* v(x), u(x)\rangle} \qquad \text{for all } u, v\text{'s with **hom.** BC's}$$

For diff eqns, called the Lagrange identity or Green's formula in some books.

- If $\mathbf{L}^* = \mathbf{L}$ in $\langle v, \mathbf{L}u\rangle = \langle \mathbf{L}^* v, u\rangle$ (i.e. $\mathbf{L}^* v$ on v is the same as $\mathbf{L}u$ on u) then $\mathbf{L}$ is called **formally self-adjoint**
- If $\mathbf{L}^* = \mathbf{L}$ and $BC^* = BC$ then the **complete operator is self-adjoint (a.k.a. fully self-adjoint)**

Important results for (fully) self-adjoint operators:[a]

1. Self-orthogonal set of eigenfunctions: $\langle \phi_k, \phi_j\rangle = 0$ if $k \neq j$.
2. All eigenvalues λ_k are real (see `Sakai/Resources/L04b.pdf`)

These results can make some problems MUCH easier to solve.
(also see `Sakai/Resources/L04a.pdf`)

[a] Usual general proofs from the inner product adjoint relation for any type of inner product.

ODE BVP Eigenvalue problems on domain $a \le x \le b$:

$$L\phi = -\lambda\phi \qquad BC_1\phi(a) = 0 \qquad BC_2\phi(b) = 0 \qquad \text{(version 1.0)}$$

Problem structure:

- LHS=Linear operator applied to eigenfcn ϕ and RHS= $-$(eigenvalue)ϕ
- Homogeneous version of BC's applied to ϕ
- $L\phi + \lambda\phi = 0$ is a homogeneous problem: it has nontrivial solutions (eigenfunctions $||\phi||^2 \neq 0$) only for special choices for λ values (eigenvalues)

Eigenvalue problems: Matrix vs. ODE BVP comparisons

Matrix: $\mathbf{L}\vec{\phi} = \lambda\vec{\phi}$ vs. ODE BVP: $\{\ L\phi(x) = -\lambda\phi(x),\ BC\phi = 0\ \}$

- Historical tradition: ODE eigenvalue equation has an extra minus sign
- Matrix eigenvalues for $\mathbf{L}_{n\times n}$: determinant $|\mathbf{L} - \lambda\mathbf{I}| = 0$ has n eig-vals
  ODE BVP eigenvalues: has an infinite number of eig-vals. (How? IOU)

Solving ODE BVP eigenvalue problems – the general approach:

1. Get the general solution of the homogeneous ODE.
   The general homogeneous soln of an n^{th} order ODE is a linear combo of n independent solns:

$$\phi_{\text{gen}}(x) = b_1\phi_1(x) + b_2\phi_2(x) + \cdots + b_n\phi_n(x)$$

   $b_1, b_2, \cdots, b_n$ constants not determined yet (BC's not applied yet)

2. Apply the homogeneous BC's to the general soln to determine the condition for the eigenvalues, and the form of the eigenfunctions.

   A n^{th} order problem should have n BC's:

   - Will give n eqns for n parameters: λ and $(n-1)$ of the b_k's
   - Can set the last remaining coefficient to be $b_{\text{last}} = 1$
     (eigenfcns can always be scaled, like in HW#1 Q1)

Two classes of ODE's with simple solutions: plugging-in the right form of **trial solution** reduces the homogeneous ODE to an algebraic equation (roots of the **characteristic polynomial**, $P(m) = 0$) for all independent solns:

1. Linear Constant Coefficient (LCC): $Lu =$ sum of constants times derivatives

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = 0 \qquad \text{trial soln: } u = e^{mx}$$

2. Cauchy-Euler (CE): $Lu =$ sum of products of constants times x^n times n^{th} derivatives

$$ax^2\frac{d^2u}{dx^2} + bx\frac{du}{dx} + cu = 0 \qquad \text{trial soln: } u = x^m$$

- The general soln of the homogeneous ODE will be a linear combination of all the trial solutions solving the char-poly (with const coeffs to be determined from BC's)
- If the char-poly has repeated roots, those solutions get modified
- For complex roots of the char-poly, solutions should always be combined appropriately to be in terms of real-valued fcns. (like $e^{\pm ix} \to \cos(x), \sin(x)$)

(See Summary Sheet for LCC and CE eqns: `Sheets/LCC_CE_ODE.pdf`)