Vector Derivative Identities and Properties

$$f = f(x,y,z) = f(\vec{\mathbf{x}}) \qquad \vec{\mathbf{x}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\mathbf{F}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$$

$$\nabla \equiv \hat{\mathbf{i}}\partial_x + \hat{\mathbf{j}}\partial_y + \hat{\mathbf{k}}\partial_z$$

$$\text{grad } f \equiv \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad \text{(vector)}$$

$$\text{div } \mathbf{F} \equiv \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad \text{(scalar)}$$

$$\text{curl } \mathbf{F} \equiv \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} \qquad \text{(vector)}$$

$\mathbf{F}, \mathbf{G}$: vector fields, f: scalar function, ψ: either

1. Orthogonality of the gradient to level curves or surfaces: $\nabla f(\vec{\mathbf{x}}_0) \perp \{f(\vec{\mathbf{x}}) = f_0\}$ at $\vec{\mathbf{x}}_0$

2. Vector forms of the product rule (derivative of first times second plus first times derivative of second):

$$\nabla \cdot (f\mathbf{F}) = \mathbf{F} \cdot \nabla f + f\,\nabla \cdot \mathbf{F} \tag{1a}$$

$$\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f\,(\nabla \times \mathbf{F}) \tag{1b}$$

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = \begin{array}{c} (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F}) \\ + (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F} \times (\nabla \times \mathbf{G}) \end{array} \tag{1c}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \tag{1d}$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \begin{array}{c} (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} \\ - (\mathbf{F} \cdot \nabla)\mathbf{G} + (\nabla \cdot \mathbf{G})\mathbf{F} \end{array} \tag{1e}$$

$$\nabla \cdot (\mathbf{F}\mathbf{G}) = (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{F} \cdot \nabla)\mathbf{G} \tag{1f}$$

3. Curl-free property of gradient fields ($\forall f$): $\boxed{\nabla \times \nabla f = \mathbf{0}}$

4. Divergence-free property of curl fields ($\forall \mathbf{F}$): $\boxed{\nabla \cdot (\nabla \times \mathbf{F}) = 0}$

5. div(grad)=Laplacian: $\boxed{\nabla \cdot \nabla\psi = \nabla^2\psi} \equiv \Delta\psi$ (alternative math notation)

6. curl-curl-grad(div)-Laplacian identity: $\boxed{\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}}$

Domains and boundaries

1D) domain=interval $a \le x \le b$, bdry=endpts $x = a$, $x = b$
2D) domain=region in xy, bdry=oriented closed curve
Righthand rule defines the "inside" of the curve
3D) domain=region in xyz, bdry=closed surface

Integral Theorems of Vector Calculus

Green's theorem in the xy plane

For region D inside closed-curve C with no singularities of $P(x,y), Q(x,y)$ inside C:

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

If singularities are present, then the values of the line and double integral are not promised to be equal.

2D Vector versions of Green's theorem

In the xy plane, with $\mathbf{F} = (P(x,y), Q(x,y), 0)$:

- **2D Stokes' theorem** (**Green's Work Theorem**)

$$\text{Work} = \boxed{\oint_C \mathbf{F} \cdot \vec{\mathbf{T}}\,ds = \iint_D (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}}\,dA}$$

unit tangent vector, $\vec{\mathbf{T}} = (x'(t), y'(t))/|\vec{\mathbf{x}}'(t)|$

- **2D Divergence theorem** (**Green's Flux Theorem**)

$$\text{Flux} = \boxed{\oint_C \mathbf{F} \cdot \vec{\mathbf{n}}\,ds = \iint_D \nabla \cdot \mathbf{F}\,dA}$$

unit outward normal vector, $\vec{\mathbf{n}} = (y'(t), -x'(t))/|\vec{\mathbf{x}}'(t)|$

$$\oint_C -Q\,dx + P\,dy = \iint_D (P_x + Q_y)\,dA$$

3D Stokes' theorem: Surface S with edge curve C

$$\text{Work} = \boxed{\oint_C \mathbf{F} \cdot \vec{\mathbf{T}}\,ds = \iint_S (\nabla \times \mathbf{F}) \cdot \vec{\mathbf{n}}\,dS}$$

Surface integral: $\vec{\mathbf{n}}$ is the "right-thumb" unit normal to S with edge curve C with fingers gripped in the direction of C

3D Divergence theorem: Volume D enclosed by closed surface S

$$\text{Flux} = \boxed{\iint_S \mathbf{F} \cdot \vec{\mathbf{n}}\,dS = \iiint_D \nabla \cdot \mathbf{F}\,dV}$$

Other basic results

Line integrals on parametric curve $C = \{t : a \to b\}$

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

unit tangent $\vec{\mathbf{T}} = \vec{\mathbf{x}}'(t)/|\vec{\mathbf{x}}'(t)|$, arclength $ds = |\vec{\mathbf{x}}'(t)|\,dt$

$$\text{Work} = \int_C \mathbf{F} \cdot \vec{\mathbf{T}}\,ds = \int_C \mathbf{F} \cdot d\vec{\mathbf{x}}$$
$$= \int_a^b \mathbf{F}(\vec{\mathbf{x}}(t)) \cdot \vec{\mathbf{x}}'(t)\,dt = \int_C P\,dx + Q\,dy + R\,dz$$

Surface integrals

$$\text{Flux} = \iint_S \mathbf{F} \cdot \vec{\mathbf{n}}\,dS = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

Parametric curves

Position

$$\vec{\mathbf{x}}(t) = (x(t), y(t), z(t))$$

Velocity

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{x}}}{dt} = (x'(t), y'(t), z'(t))$$

unit **Tangent** vector

$$\vec{\mathbf{T}} = \hat{\mathbf{v}} = \frac{1}{|\vec{\mathbf{v}}(t)|}\vec{\mathbf{v}}(t)$$

In 2D: unit tangent and normal $\vec{\mathbf{T}} \cdot \vec{\mathbf{n}} = 0$

$$\vec{\mathbf{T}} = \frac{(x'(t), y'(t))}{\sqrt{x'(t)^2 + y'(t)^2}} \qquad \vec{\mathbf{n}} = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}$$

Derivative product rules applied to parametric curves $\vec{\mathbf{x}} = \mathbf{p}(t)$, $\vec{\mathbf{x}} = \mathbf{q}(t)$ and scalar function $k(t)$:

$$\frac{d}{dt}(k\mathbf{p}) = \frac{dk}{dt}\mathbf{p} + k\frac{d\mathbf{p}}{dt} \tag{2a}$$

$$\frac{d}{dt}(\mathbf{p} \cdot \mathbf{q}) = \frac{d\mathbf{p}}{dt} \cdot \mathbf{q} + \mathbf{p} \cdot \frac{d\mathbf{q}}{dt} \tag{2b}$$

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{q}) = \frac{d\mathbf{p}}{dt} \times \mathbf{q} + \mathbf{p} \times \frac{d\mathbf{q}}{dt} \tag{2c}$$

Surface integrals: list of unit normal vectors $\vec{\mathbf{n}}$

Parametric surface: $x = x(u,v), y = y(u,v), z = z(u,v)$

$$\vec{\mathbf{N}} = \frac{\partial\vec{\mathbf{x}}}{\partial u} \times \frac{\partial\vec{\mathbf{x}}}{\partial v} \qquad \longrightarrow \qquad \vec{\mathbf{n}} = \frac{\vec{\mathbf{N}}}{|\vec{\mathbf{N}}|}$$

Graph of fcn: $z = g(x,y)$ $\qquad \vec{\mathbf{n}} = \dfrac{(-g_x, -g_y, 1)}{\sqrt{1 + g_x^2 + g_y^2}}$

Plane: $ax + by + cz = d$ $\qquad \vec{\mathbf{n}} = \dfrac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$

Sphere: $\rho = a$ $\qquad \vec{\mathbf{n}} = \dfrac{1}{a}(x,y,z)$

Circular cylinder: $r = a$ $\qquad \vec{\mathbf{n}} = \dfrac{1}{a}(x,y,0)$

Area (2D) $\vec{\mathbf{x}}(u,v) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \qquad x = x(u,v),\ y = y(u,v)$

$$\boxed{dA = J\,du\,dv \qquad J = \left|\frac{\partial\vec{\mathbf{x}}}{\partial u} \times \frac{\partial\vec{\mathbf{x}}}{\partial v}\right| = \left\|\begin{matrix} x_u & y_u \\ x_v & y_v \end{matrix}\right\|}$$

Rectangular Area $\boxed{dA = dy\,dx = dx\,dy}$

Polar Area $\boxed{dA = r\,dr\,d\theta}$

Volume (3D) $\vec{\mathbf{x}}(u,v,w) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

$$x = x(u,v,w),\ y = y(u,v,w),\ z = z(u,v,w)$$

$$\boxed{dV = J\,du\,dv\,dw}$$

$$\boxed{J = \left|\frac{\partial\vec{\mathbf{x}}}{\partial w} \cdot \left(\frac{\partial\vec{\mathbf{x}}}{\partial u} \times \frac{\partial\vec{\mathbf{x}}}{\partial v}\right)\right| = \left\|\begin{matrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{matrix}\right\|}$$

Rectangular Volume $\boxed{dV = dz\,dy\,dx}$

Cylindrical Volume $\boxed{dV = r\,dz\,dr\,d\theta}$

Spherical Volume $\boxed{dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta}$

Arclength (1D) Param curves $\vec{\mathbf{x}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$

$$\boxed{ds = |\vec{\mathbf{x}}'(t)|\,dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt}$$

Graph of fcn: $x = t,\ y = f(x),\ z = 0$ $\boxed{ds = \sqrt{1 + f'(x)^2}\,dx}$

Surface Area (2D) Param surfaces $\vec{\mathbf{x}}(u,v) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ $x = x(u,v),\ y = y(u,v),\ z = z(u,v)$ [1]

$$\boxed{dS = \left|\frac{\partial\vec{\mathbf{x}}}{\partial u} \times \frac{\partial\vec{\mathbf{x}}}{\partial v}\right| du\,dv = \left\|\begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{matrix}\right\| du\,dv}$$

Graph of fcn: $x = u, y = v, z = f(x,y)$ $\vec{\mathbf{x}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + f(x,y)\hat{\mathbf{k}}$

$$\boxed{dS = \sqrt{1 + f_x^2 + f_y^2}\,dy\,dx}$$

Area in xy plane: $x = u,\ y = v,\ z = 0$ $\boxed{dS = dA}$

Circular Cylinder surface, radius $r = a$:

- Using Cyl coords: $z = u, \theta = v$ $\boxed{dS = a\,dz\,d\theta}$

Sphere surface, radius $\rho = a$:

- Using Sph coords: $\theta = u, \phi = v$ $\boxed{dS = a^2 \sin\phi\,d\phi\,d\theta}$

Cone surface, angle $\phi = \alpha$:

- Using Sph coords: $\theta = u, \rho = v$ $\boxed{dS = \rho\sin\alpha\,d\rho\,d\theta}$
- Using Cyl coords: $\theta = u, r = v, z = r/\tan\alpha$ $\boxed{dS = r\csc\alpha\,dr\,d\theta}$

Rectangular	Cylindrical	Spherical	General
$x = x$	$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$x = x(u,v,w)$
$y = y$	$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$y = y(u,v,w)$
$z = z$	$z = z$	$z = \rho\cos\phi$	$z = z(u,v,w)$

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[1] Note: The "||stuff||" in the general formula for dS means "the length of the cross product vector given by the determinant." For the Jacobian in dA and dV the double bars mean the absolute value of the determinant.