1 The full trigonometric Fourier series

For a given L^2 function $f(x)$ on $-\ell \le x \le \ell$

$$f(x) \text{ “=” } \sum_{k=0}^{\infty} a_k \cos\left(\tfrac{k\pi}{\ell}x\right) + \sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi}{\ell}x\right), \qquad (\text{“=” means } \overset{\text{a.e.}}{=\!=}) \tag{1a}$$

If $f(x)$ is continuous everywhere (usually the case for solutions of differential equations) then pointwise equality holds, so the “=” will be a real regular equal sign (else, at discontinuities of f, the series converges to the average of the left/right limits). The Fourier coefficients are given by the usual expansion formula for self-adjoint orthogonal eigenfunctions applied to the cosines as the basis functions:

$$a_k = \frac{\langle f(x), \cos(k\pi x/\ell)\rangle}{||\cos(k\pi x/\ell)||^2} = \begin{cases} \dfrac{1}{2\ell}\displaystyle\int_{-\ell}^{\ell} f(x)\,dx & k=0 \\[2ex] \dfrac{1}{\ell}\displaystyle\int_{-\ell}^{\ell} f(x)\cos(k\pi x/\ell)\,dx & k=1,2,3,\cdots \end{cases} \tag{1b}$$

and similarly for the coefficients of the sine terms:

$$b_k = \frac{\langle f(x), \sin(k\pi x/\ell)\rangle}{||\sin(k\pi x/\ell)||^2} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(k\pi x/\ell)\,dx. \tag{1c}$$

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2 Connections to Even/Odd functions

The relations between the values for functions for negative vs. positive values of x:

1. If $f(x)$ is a given function on a symmetric interval, $-\ell \le x \le \ell$ then
   (a) $f(x)$ is an even function if $f(-x) = f(x)$
   (b) $f(x)$ is an odd function if $f(-x) = -f(x)$
   (c) Every function can be broken up into a sum of even and odd parts

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x), \qquad f_{\text{even}}(x) = \tfrac{1}{2}\left[f(x) + f(-x)\right] \qquad f_{\text{odd}}(x) = \tfrac{1}{2}\left[f(x) - f(-x)\right] \tag{2}$$

2. If $f(x)$ is given on the positive half-interval, $0 \le x \le \ell$, even/odd extensions on $-\ell \le x \le \ell$ can be defined by

$$f_{\text{even}}(x) = \begin{cases} f(x) & x \ge 0 \\ f(-x) & x < 0 \end{cases} \qquad\qquad f_{\text{odd}}(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

3. Integration on symmetric domains:

$$\int_{-\ell}^{\ell} \text{odd}(x)\,dx = 0 \qquad\qquad \int_{-\ell}^{\ell} \text{even}(x)\,dx = 2\int_0^{\ell} \text{even}(x)\,dx. \tag{3}$$

4. Products of functions:

$$\text{odd}(x)\cdot\text{odd}(x) = \text{even}(x) \qquad \text{even}(x)\cdot\text{odd}(x) = \text{odd}(x) \qquad \text{even}(x)\cdot\text{even}(x) = \text{even}(x) \tag{4}$$

5. $\cos(kx)$ is even, $\sin(kx)$ is odd.
6. If $f(x)$ is odd, then by $(4)_2$ and $(3)_1$ equation (1b), all $a_k = 0$, and b_k in (1c) is doubled the half-interval value by $(3)_2$, so the full trig. Fourier series reduces down to the Fourier sine series:

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi}{\ell}x\right) \qquad b_k = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(k\pi x/\ell)\,dx. \tag{5}$$

   If $f(x)$ is even, then equation (1c) reduces to $b_k = 0$ and a_k is doubled from the half-interval value so the full trig. Fourier series reduces down to the Fourier cosine series:

$$f(x) = \sum_{k=0}^{\infty} a_k \cos\left(\tfrac{k\pi}{\ell}x\right) \qquad a_k = \frac{2}{\ell}\int_0^{\ell} f(x)\cos(k\pi x/\ell)\,dx \qquad a_0 = \frac{1}{\ell}\int_0^{\ell} f(x)\,dx. \tag{6}$$

3 Useful integration properties for calculating Fourier coefficients

- Commonly occurring special values, $\cos(0) = 1$, $\sin(0) = 0$,

$$\cos(k\pi) = (-1)^k \qquad \sin(k\pi) = 0 \qquad \cos\left(\frac{(2k+1)\pi}{2}\right) = 0 \qquad \sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k \qquad k = 0, 1, 2, 3, \cdots$$

- Piecewise-defined functions should be integrated piece by piece (subinterval by subinterval):

$$\text{if} \quad f(x) = \begin{cases} f_1(x) & a \le x < x_1 \\ f_2(x) & x_1 \le x < x_2 \\ f_3(x) & x_2 \le x < b \end{cases} \quad \text{then} \quad \int_a^b f(x)g(x)\,dx = \int_a^{x_1} f_1(x)g(x)\,dx + \int_{x_1}^{x_2} f_2(x)g(x)\,dx + \int_{x_2}^{b} f_3(x)g(x)\,dx$$

- Integration by parts

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx.$$

- See the review sheets for methods of integration and trig identities.

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4 Convergence of infinite series: theory and convergence tests

- A partial sum is the sum of the first N terms from the infinite series,

$$S_N(x) = \sum_{k=1}^{N} c_k \phi_k(x) \tag{7}$$

- **Point-wise convergence**: examine the series at a fixed point x_0, then the terms become constants, $a_k = c_k\phi_k(x_0)$

$$S_N(x_0) = \sum_{k=1}^{N} c_k\phi_k(x_0) = \sum_{k=1}^{N} a_k$$

then convergence at x_0 means that

$$\lim_{N\to\infty} S_N(x_0) = f(x_0)$$

and questions of convergence one x-value-at-a-time are reduced to calculus questions about series of numbers:

  - k^{th} term test: a necessary condition for convergence is that

$$\lim_{k\to\infty} a_k = 0$$

    (i.e. if $a_\infty \neq 0$ then the series definitely does not converge). For alternating series (terms alternate in sign), if $|a_k|$ is decreasing as k increases and if this test is satisfied, then the series converges.

  - Important series

$$\boxed{\textbf{Geometric series}}: \quad \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad \text{converges if } |r| < 1 \qquad \boxed{p\textbf{-series}}: \quad \sum_{k=1}^{\infty} \frac{1}{k^p} \quad \text{converges if } p > 1$$

    and the $\boxed{\textbf{Harmonic series}}$: $\sum_{k=1}^{\infty}(1/k)$, diverges – it is a p-series with $p = 1$.

  - Limit comparison test: if $\sum_k b_k$ is known to converge or diverge, then if

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \text{finite nonzero value}$$

    then the $\sum_k a_k$ series will converge/diverge just like $\sum_k b_k$.

  - Absolute ratio test: examine the limit of the ratio of successive terms,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1: \quad \text{converges}, \qquad \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1: \quad \text{diverges}, \qquad \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1: \quad \text{use a different test.}$$

  - If $\sum_k |a_k|$ converges, then the series $\sum_k a_k$ is called absolutely convergent.

- **Uniform convergence**: "$S_N(x)$ converges uniformly to $f(x)$ on $a \le x \le b$" means that we define the sequence $M_k = \max_x |S_k(x) - f(x)|$ and it is true that $M_k \to 0$ as $k \to \infty$. Uniform convergence is an important property: if your Fourier Series for $f(x)$ is uniformly convergent, then it is possible to guarantee when operations (like derivatives) on the series will give the correct (convergent) results.