L1 recap Solving **O**rdinary **D**ifferential **E**quation **I**nitial **V**alue **P**roblems

A nxn matrix

ODE: $\frac{d\vec{u}}{dt} = A\vec{u}$ Initial Condition (IC) $\vec{u}(t=0) = \vec{b}$

Separation of Variables trial soln:

$$\vec{u}(t) = \sum_{k=1}^{n} T_k(t)\vec{v}_k \qquad \begin{pmatrix} T_k(t)\ ? \\ \vec{v}_k\ ? \end{pmatrix}$$

ODE $\frac{d\vec{u}}{dt} = A\vec{u} \Rightarrow \sum_{k=1}^{n} \frac{dT_k}{dt}\vec{v}_k = \sum_{k=1}^{n} T_k A\vec{v}_k$

- assume equal term-by-term: $\sum_{k=1}^{n}\left(\frac{dT_k}{dt}\vec{v}_k = TA\vec{v}_k\right)$
- divide across to separate t-fcns vs vecs $\underbrace{\left(\frac{1}{T_k}\frac{dT_k}{dt}\right)}_{\text{const}}\vec{v}_k = \underbrace{A\vec{v}_k}_{\text{no } t\text{'s}}$ ⇐

separation constant $s_k = \frac{1}{T_k}\frac{dT_k}{dt}$

Problem is separated into 2 easier subproblems @ each k

① $\frac{dT_k}{dt} = s_k T_k \rightarrow T_k(t) = c_k e^{s_k t}$ ← const of integration

② $s_k\vec{v}_k = A\vec{v}_k$ → matches $A\vec{\psi}_k = \lambda_k\vec{\psi}_k$
$k = 1, 2, \ldots n$
$s_k = \lambda_k$ eigenvalues of A
$\vec{v}_k = \vec{\psi}_k$ eigenvectors of A

Re-assemble soln: $\boxed{\vec{u}(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t}\vec{\psi}_k}$

use IC $\vec{u}(t=0) = \vec{b} = \underbrace{\sum_{k=1}^{n} c_k \cdot 1 \cdot \vec{\psi}_k}$ $(e^0 = 1)$

use orth. expansion of $\vec{b}$ onto $\vec{\psi}_k$'s → $\boxed{c_k = \frac{\langle \psi_k, b\rangle}{\langle \psi_k, \psi_k\rangle}}$

SoV is a "divide + conquer" method. It splits up the prob into easier subproblems for $(s_k, v_k, T_k(t), c_k)$ to get whole soln. $k = 1, 2, \ldots n$