Introduction to functions of complex variables: motivation

- An extension of $\mathbb{R}$eal numbers to form a "complete set": all possible solutions of algebraic problems (quadratic eqn and up...) are $\mathbb{C}$omplex-numbers
- Extensions of: algebra, functions, derivatives, **integrals**.
- Complex-valued extensions of usual real functions make some calculations of integrals and differential equations easier.

Complex algebra

- The unit imaginary number $\boxed{\mathrm{i} = \sqrt{-1}}$ with: $\boxed{\mathrm{i}^2 = -1}$
- If x, y are two real numbers ($\mathbb{R}$) then $\boxed{z = x + \mathrm{i}y}$ is a complex number ($\mathbb{C}$)
- $z \in \mathbb{C}$ have some "2D vector-ish" properties
  - Separable components: Real/Imaginary parts $\mathbf{Re}(z) = x$ and $\mathbf{Im}(z) = y$
  - Equality component-wise: $z_1 = z_2 \qquad \leftrightarrow \qquad x_1 = x_2$ and $y_1 = y_2$
  - Component-wise addition: $z_1 + z_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$
- But products are done as usual algebra with i (not "vector-ish")

$$(a + \mathrm{i}b)(c + \mathrm{i}d) = ac + \mathrm{i}ad + \mathrm{i}bc + \mathrm{i}^2 bd = (ac - bd) + \mathrm{i}(ad + bc)$$

Complex algebra (continued)

- Conjugation: flip sign of imaginary part: $\boxed{\overline{\mathbf{i}} = -\mathbf{i}}$ Example: $\overline{3 + \mathbf{i}4} = 3 - \mathbf{i}4$
- Complex conjugate of $z = x + \mathbf{i}y$ is $\overline{z} = x - \mathbf{i}y$, $\qquad \overline{z_1 z_2} = (\overline{z}_1)(\overline{z}_2)$
- Formulas for components of $z = x + \mathbf{i}y$:

$$\mathrm{Re}(z) \equiv \frac{z + \overline{z}}{2} = \frac{\begin{pmatrix} x + \mathrm{i}y \\ +x - \mathrm{i}y \end{pmatrix}}{2} = x \quad \mathrm{Im}(z) \equiv \frac{z - \overline{z}}{2\mathrm{i}} = \frac{\begin{pmatrix} x + \mathrm{i}y \\ -x + \mathrm{i}y \end{pmatrix}}{2\mathrm{i}} = y$$

- Modulus (magnitude or length) of z:

$$|z|^2 \equiv z\overline{z} = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2 \qquad |z| = \sqrt{x^2 + y^2}$$

- Division (multiply top/bottom by $\overline{\mathbf{denom}}$)

$$\frac{1}{z} = \frac{1}{z}\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{x - \mathrm{i}y}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \mathrm{i}\frac{y}{x^2 + y^2}$$

- $z = x + \mathbf{i}y$ is the rectangular coordinate form for a complex number (unique)
- Rectangular to polar coord conversions ("modulus" (r) and "argument" (θ))

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad z = r\cos\theta + \mathrm{i}r\sin\theta \qquad \left\{\begin{aligned} r &= |z| = \sqrt{x^2 + y^2} \geq 0 \\ \theta &= \arg(z) = \tan^{-1}(y/x) \end{aligned}\right.$$

Complex algebra (continued)

Euler's formula: $\boxed{e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi}$ Use Taylor series for $\phi \to 0$:

$$
\begin{array}{ccccccccccccccccc}
e^{\phi} & = & 1 & + & \phi & + & \frac{\phi^2}{2!} & + & \frac{\phi^3}{3!} & + & \frac{\phi^4}{4!} & + & \frac{\phi^5}{5!} & + & \frac{\phi^n}{n!} \\
\\
e^{\mathrm{i}\phi} & = & 1 & + & \mathrm{i}\phi & + & \frac{\mathrm{i}^2\phi^2}{2!} & + & \frac{\mathrm{i}^3\phi^3}{3!} & + & \frac{\mathrm{i}^4\phi^4}{4!} & + & \frac{\mathrm{i}^5\phi^5}{5!} & + & \frac{\mathrm{i}^n\phi^n}{n!} \\
e^{\mathrm{i}\phi} & = & 1 & + & \mathrm{i}\phi & - & \frac{\phi^2}{2!} & - & \mathrm{i}\frac{\phi^3}{3!} & + & \frac{\phi^4}{4!} & + & \mathrm{i}\frac{\phi^5}{5!} & + & \frac{\mathrm{i}^n\phi^n}{n!} \\
\\
\cos\phi & = & 1 & & & - & \frac{\phi^2}{2!} & & & + & \frac{\phi^4}{4!} & & & + & \frac{(-1)^n\phi^{2n}}{(2n)!} \\
\\
\mathrm{i}\sin\phi & = & & & \mathrm{i}\phi & & & - & \mathrm{i}\frac{\phi^3}{3!} & & & + & \mathrm{i}\frac{\phi^5}{5!} & + & \mathrm{i}\frac{(-1)^n\phi^{2n+1}}{(2n+1)!}
\end{array}
$$

$\boxed{e^{iz} = \cos z + \mathrm{i}\sin z}$ is true for all $z \in \mathbb{C}$ then $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ ...

$\boxed{z = re^{\mathrm{i}\theta}}$ $= r(\cos\theta + \mathrm{i}\sin\theta)$ is the polar coordinate form: $r \geq 0$ and $\theta \in \mathbb{R}$

(non-unique!? – replace $\theta \to \theta + 2\pi k$ yields same $z = re^{\mathrm{i}\theta}$)

- Multiplication and division is easier in polar form:

$$z_1 z_2 = (r_1 e^{\mathrm{i}\theta_1})(r_2 e^{\mathrm{i}\theta_2}) = (r_1 r_2) e^{\mathrm{i}(\theta_1+\theta_2)} \qquad \frac{1}{z} = \frac{1}{r} e^{-\mathrm{i}\theta}$$

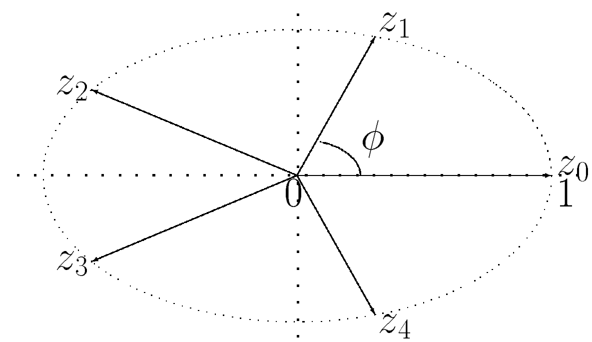
Complex algebra (continued)

- $\boxed{e^{i2\pi k} = 1 \text{ for any } k = \text{integer}}$ $(1 = 1 + i0 = \cos(2\pi k) + i\sin(2\pi k))$
- Solving (easy) algebra equations: "Solve $z^n = 1$" ("roots of unity") really means:

$$\text{solve} \qquad z^n = e^{i2\pi k} \qquad \text{for all integers } k$$

To get solns: take $(1/n)$-th power of both sides: $z^{n/n} = e^{i2\pi k/n}$.
Let $\phi = 2\pi/n$ then $\boxed{z_k = e^{ik\phi}}$ gives n different roots for $k = 0, 1, 2, \cdots, n-1$ (then repeats)

Example: $n = 5$, $\phi = 2\pi/5 = 72°$



Similarly for $z^n = a + ib$ (convert RHS to polar form $a + ib = \rho e^{i\alpha}$ first)
Similarly for $(z-c)^n = a + ib$ (shift by c after n-th root)

$$\begin{aligned}
(z-c)^n &= a + ib = \rho e^{i\alpha} \\
(z-c)^n &= \rho e^{i\alpha} e^{i2\pi k} \\
(z-c) &= \rho^{1/n} e^{i\alpha/n} e^{i2\pi k/n} \\
z_k &= c + \rho^{1/n} e^{i\alpha/n} e^{ik\phi} \qquad k = 0, 1, 2, \cdots, n-1
\end{aligned}$$