Math 551, Applied PDE's — Lecture 22

Recap: PDE's in polar coords, heat and wave eqn problems (H 7.7)

An axisymmetric problem for the heat eqn: $\partial_t u = \nabla^2 u$ (H 7.7.9)

- Axisymmetric domain: $0 \le r \le b$: (disk of radius b)
- Axisymmetric BC: $u(r=b)=0$ ("cold" Dirichlet BC, value same for all θ)
- Axisymmetric IC: $u(t=0)=A(r)$
- No part of the problem makes the soln depend on θ
  so seek a θ-independent soln $u=u(r,t)$ of the PDE

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

- Separation of variables trial soln: $u(r,t)=f(r)h(t)$

$$\frac{h'(t)}{h(t)} = \frac{(rf'(r))'}{rf(r)} = -\lambda$$

- $h'/h = -\lambda$ solution: $h(t)=ce^{-\lambda t}$ with λ unknown

Axisymmetric heat problem (continued)

The $f(r)$ ODE problem on $0 \le r \le b$:

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) = -\lambda r f$$

- In Sturm-Liouville form:
  $p(r) = r$ (singular at $r = 0$), $q = 0$, $\sigma = r$ (weight fcn)
- BC's: $u(r = b) = 0 \quad \Longrightarrow \quad f(b) = 0$ and
  singular SL at $r = 0 \Longrightarrow$ seek $f(0) =$ bounded $\neq \infty$
- The $\lambda r f$ term does not match with Cauchy-Euler form; this is NOT CE.
  This eqn is called **Bessel's eqn of order zero** (BE-0).
  This is the simplest of a whole family of Bessel ODE's.... (to be continued).

Digression: **Matching ODE's to "standard forms"** - scaling in/out $\boldsymbol{\lambda}$ parameters

- Example 1: Original equation $\dfrac{d^2g}{dx^2}+\lambda g=0$

  Let $z=\sqrt{\lambda}\,x$ and $y(z)=g(x)$.
  Use chain rule ($x=z/\sqrt{\lambda}$ and $g(x)=y(\sqrt{\lambda}\,x)$) then the ODE becomes:

  $$(\sqrt{\lambda})^2\frac{d^2y}{dz^2}+\lambda y=0 \qquad \Longrightarrow \qquad \frac{d^2y}{dz^2}+y=0 \quad \text{Standard Osc eqn}$$

  General soln of the scaled problem: $y(z)=c_1\cos z+c_2\sin z$
  $\cos z, \sin z$ are called the trigonometric fcns of 1st, 2nd kind
  Un-do scaling to get general soln of original problem

  $$g(x)=c_1\cos(\sqrt{\lambda}\,x)+c_2\sin(\sqrt{\lambda}\,x)$$

- Example 2: Original equation $\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right)+\lambda r f=0$

  Let $z=\sqrt{\lambda}\,r$ and $y(z)=f(r)$.
  Use chain rule ($r=z/\sqrt{\lambda}$ and $f(r)=y(\sqrt{\lambda}\,r)$) then ODE becomes:

  $$\sqrt{\lambda}\frac{d}{dz}\left(\frac{z}{\sqrt{\lambda}}\sqrt{\lambda}\frac{dy}{dz}\right)+\lambda\frac{z}{\sqrt{\lambda}}y=0 \qquad \Longrightarrow \qquad \frac{d}{dz}\left(z\frac{dy}{dz}\right)+zy=0$$

Standard form Bessel eqn of order zero: $\displaystyle \frac{d}{dz}\left(z\frac{dy}{dz}\right) + zy = 0$

- General solution:

$$y(z) = c_1 J_0(z) + c_2 Y_0(z)$$

  $J_0(z)$: the 1st kind Bessel fcn of order zero
  $Y_0(z)$: the 2nd kind Bessel fcn of order zero
  They are specific "special fcns" with tabulated properties, just like $\cos, \sin$ (and are oscillatory fcns like $\cos, \sin$ too).
  Other key properties: $J_0(0) = 1$ and $Y_0(z \to 0) \to -\infty$ and $J_0, Y_0 \to 0$ for $z \to \infty$ (decaying oscillations)

- Un-do scaling to get general soln of original problem

$$f(r) = c_1 J_0(\sqrt{\lambda}\, r) + c_2 Y_0(\sqrt{\lambda}\, r)$$
