Part 1: L19 Recap: SV and 1-d eigen-expansion for Laplace BVP's

- Separation of variables trial soln: $u(r,\theta)=f(r)g(\theta)$
- Laplace's eqn $\nabla^2 u = 0$ in polar coords:

$$\frac{1}{r}\partial_r(r\partial_r u)+\frac{1}{r^2}\partial_{\theta\theta}u=0 \qquad \Longrightarrow \qquad \frac{r(rf')'}{f}=-\frac{g''}{g}=s_k$$

- Superposition, break-up into single-edge inhom BC problems
- Dirichlet problems on separable domain $a\le r\le b$ and $0\le\theta\le\omega$
- (a) BC's: $u=A(\theta)$ on $r=a$ and $u=0$ on other edges
  - $g(0)=0$, $g(\omega)=0$ Hom BC's $\Longrightarrow$ solve $g''+\lambda g=0$ first: $g_k(\theta)=\sin(k\pi\theta/\omega)$ with $\lambda_k=k^2\pi^2/\omega^2$
  - then CE problem for $f(r)$, BC $f(b)=0$, shifted soln form:[a]

$$f_k(r)=(r/b)^{\sqrt{\lambda_k}}-(r/b)^{-\sqrt{\lambda_k}}$$

  - $u=\sum_k c_k f_k(r)g_k(\theta)$ apply BC $u(r=a)=A(\theta)$

$$c_k=\frac{\langle A,g_k\rangle}{f_k(a)\langle g_k,g_k\rangle}=\frac{2}{\omega f_k(a)}\int_0^\omega A(\theta)g_k(\theta)\,d\theta$$

[a] Or as less-simple-looking $f_k(r)=\sinh(\sqrt{\lambda_k}\,\ln(r/b))$ $\qquad (p^q=(e^{\ln p})^q=e^{q\ln p})$

Laplace-Dirichlet BVP (concluded): $\frac{r(rf')'}{f} = -\frac{g''}{g} = s_k$

- (b) BC's: $u = C(r)$ on $\theta = \omega$ and $u = 0$ on other edges
  - $f(a) = 0$, $f(b) = 0$ Hom BC's $\Longrightarrow$ solve $f(r)$ ODE first:

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) = -\frac{\lambda}{r}f \qquad \text{SL: } p(r) = r, \sigma(r) = 1/r$$

    and use its CE form: $r^2 f'' + rf' + \lambda f = 0$ to get shifted soln form:

$$f(r) = c_1 \cos(\sqrt{\lambda}\ln(r/a)) + c_2 \sin(\sqrt{\lambda}\ln(r/a))$$

    BC $f(a) = 0$ gives $c_1 = 0$ then BC $f(b) = 0$:

$$f_k(r) = \sin(\sqrt{\lambda_k}\ln(r/a)) \qquad \lambda_k = \left(\frac{k\pi}{\ln(b/a)}\right)^2$$

  - then LCC problem for $g(\theta)$: $g_k'' - \lambda_k g_k = 0$, BC $g(0) = 0$:

$$g_k(\theta) = \sinh(\sqrt{\lambda_k}\,\theta)$$

  - $u = \sum_k c_k f_k(r) g_k(\theta)$ apply BC $u(\theta = \omega) = C(r)$

$$c_k = \frac{\langle C, f_k\rangle_\sigma}{g_k(\omega)\langle f_k, f_k\rangle_\sigma} = \frac{2}{g_k(\omega)\ln(b/a)}\int_a^b C(r) f_k(r)\,\frac{dr}{r}$$

Part 2: Special properties of polar coordinates

- What if sector angle is $\omega = 2\pi$?
  - No boundaries in the θ-direction
  - ~~No BC's in θ needed on the whole donut?~~
  - NO, solution must be **periodic**: $g(\theta + 2\pi) = g(\theta)$

  Eigenfcns become the full Fourier series on $0 \le \theta \le 2\pi$:

$$g_k(\theta) = a_k \sin(k\theta) + b_k \cos(k\theta) \qquad \lambda_k = k^2$$

- What if inner radius is $a = 0$ (the origin)?
  - Need to remove the $f_k(r)$ soln terms that would be singular at $r = 0$:

  $f = c_1 r^{\sqrt{\lambda}} + c_2 r^{-\sqrt{\lambda}} \quad \Longrightarrow \quad c_2 = 0$

  Double root case too: $f_0(r) = c_1 + c_2 \ln(r)$!

  Bounded solution reduces to:

$$f_k(r) = a_k r^{\sqrt{\lambda_k}} \qquad 0 \le r \le b$$

Reminder: Green's second identity

$$\underbrace{\iint_D v \nabla^2 u \, dA}_{\langle v, \mathbf{L}u \rangle} = \underbrace{\oint_C \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds}_{\text{boundary terms}} + \underbrace{\iint_D u \nabla^2 v \, dA}_{\langle u, \mathbf{L}^* v \rangle}$$