Math 551, Applied PDE's — Lecture 19

Laplace's eqn $\nabla^2 u = 0$ in polar coords (continued): $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$

Separation of variables trial soln: $u(r,\theta) = f(r)g(\theta)$

(a) Positive separation const, **oscillations in θ**: $\dfrac{r(rf')'}{f} = -\dfrac{g''}{g} = \lambda \geq 0$

$g'' + \lambda g = 0$ osc-eqn, eigenvalue problem picks λ_n's, from general soln and BC's:

$$g(\theta) = A\cos(\sqrt{\lambda}\,\theta) + B\sin(\sqrt{\lambda}\,\theta)$$

[Also LCC and SL: $p = 1, q = 0, \sigma = 1$]

then r-ODE: $r^2 f'' + rf' - \lambda f = 0$, CE, trial soln $f = r^m, m = \pm\sqrt{\lambda}$

$$f(r) = Cr^{\sqrt{\lambda}} + Dr^{-\sqrt{\lambda}} \qquad \text{(growing/decaying)}$$

(b) Negative separation const, **oscillations in r**: $\dfrac{r(rf')'}{f} = -\dfrac{g''}{g} = -\lambda \leq 0$

$r^2 f'' + rf' + \lambda f = 0$, CE, $f = r^m$, with $m = \pm i\sqrt{\lambda}$, general soln:

$$f(r) = A\cos(\sqrt{\lambda}\ln(r)) + B\sin(\sqrt{\lambda}\ln(r))$$

[Also SL: $p = r, q = 0, \sigma = 1/r$]

use BC's to pick λ_n's then θ-ODE: $g'' - \lambda g = 0$ LCC, general soln:

$$g(\theta) = C\cosh(\sqrt{\lambda}\,\theta) + D\sinh(\sqrt{\lambda}\,\theta) \qquad (= \tilde{C}e^{\sqrt{\lambda}\,\theta} + \tilde{D}e^{-\sqrt{\lambda}\,\theta})$$

A brief LCC/CE review: basics vs best-practices (I) $\quad (x_{\mathrm{LCC}} = \ln(x_{\mathrm{CE}}))$

LCC : $y_{\text{distinct}}(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ $\qquad y_{\text{double}}(x) = C_1 e^{m_1 x} + C_2 e^{m_1 x} x$

CE : $y_{\text{distinct}}(x) = C_1 x^{m_1} + C_2 x^{m_2}$ $\qquad y_{\text{double}}(x) = C_1 x^{m_1} + C_2 x^{m_1} \ln(x)$

Solving ODE BVP: For real probs, solns should be real-valued fcns even if $m_{1,2}$ complex roots. Roots will always be complex conjugate pairs $m_\pm = \alpha \pm i\beta$. Justifying general soln **final forms** :

$$
\begin{aligned}
y_{\text{gen,LCC}}(x) &= C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x} \\
&= e^{\alpha x}(C_1[\cos\beta x + i\sin\beta x] + C_2[\cos\beta x - i\sin\beta x]) \\
&= e^{\alpha x}([C_1 + C_2]\cos\beta x + i[C_1 - C_2]\sin\beta x) \\
&= \boxed{Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)}
\end{aligned}
$$

$$
\begin{aligned}
y_{\text{gen,CE}}(x) &= C_1 x^{\alpha} x^{i\beta} + C_2 x^{\alpha} x^{-i\beta} \\
&= x^{\alpha}(C_1 e^{i\beta\ln x} + C_2 e^{-i\beta\ln x}) = \cdots \\
&= \boxed{Ax^{\alpha}\cos(\beta\ln x) + Bx^{\alpha}\sin(\beta\ln x)}
\end{aligned}
$$

For real $m = \pm\gamma$:

$$y_{\mathrm{LCC}} = A\cosh(\gamma x) + B\sinh(\gamma x) \qquad\qquad y_{\mathrm{CE}} = Ax^{\gamma} + Bx^{-\gamma}$$

A brief LCC/CE review: basics vs best-practices (II)

$$y_{\mathrm{LCC}} = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x) \quad y_{\mathrm{CE}} = Ax^{\alpha}\cos(\beta\ln x) + Bx^{\alpha}\sin(\beta\ln x)$$

Applying BC's: basic forms of gen solns set-up for easy eval of BC's at $x = 0$.
If **one BC** at $x = a$ or $x = b$ is homogeneous, may want to re-write general soln in **c-shifted final form** (back-justified) :

- **For LCC**: replace all x's with $(x - c)$'s

$$\begin{aligned} y_{\mathrm{gen,LCC}}(x) &= \boxed{Ae^{\alpha(x-c)}\cos(\beta(x-c)) + Be^{\alpha(x-c)}\sin(\beta(x-c))} \\ &= e^{\alpha x}[Ae^{-\alpha c}\cos(\beta x - \beta c) + Be^{-\alpha c}\sin(\beta x - \beta c)] = \cdots \\ &= C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x) \end{aligned}$$

- **For CE**: replace all x's with (x/c)'s

$$\begin{aligned} y_{\mathrm{gen,CE}}(x) &= \boxed{A(x/c)^{\alpha}\cos(\beta\ln(x/c)) + B(x/c)^{\alpha}\sin(\beta\ln(x/c))} \\ &= x^{\alpha}[A/c^{\alpha}\cos(\beta\ln(x) - \beta\ln(c)) + B/c^{\alpha}(\cdots)] = \cdots \\ &= C_1 x^{\alpha}\cos(\beta\ln x) + C_2 x^{\alpha}\sin(\beta\ln x) \end{aligned}$$

- Same way for real-case growing/decaying solns:

$$y_{\mathrm{LCC}} = A\cosh(\gamma(x-c)) + B\sinh(\gamma(x-c)) \qquad y_{\mathrm{CE}} = A\left(\frac{x}{c}\right)^{\gamma} + B\left(\frac{c}{x}\right)^{\gamma}$$

- Once you pick one shifted form ($c = a$ or $c = b$), you <u>cannot</u> change it later!