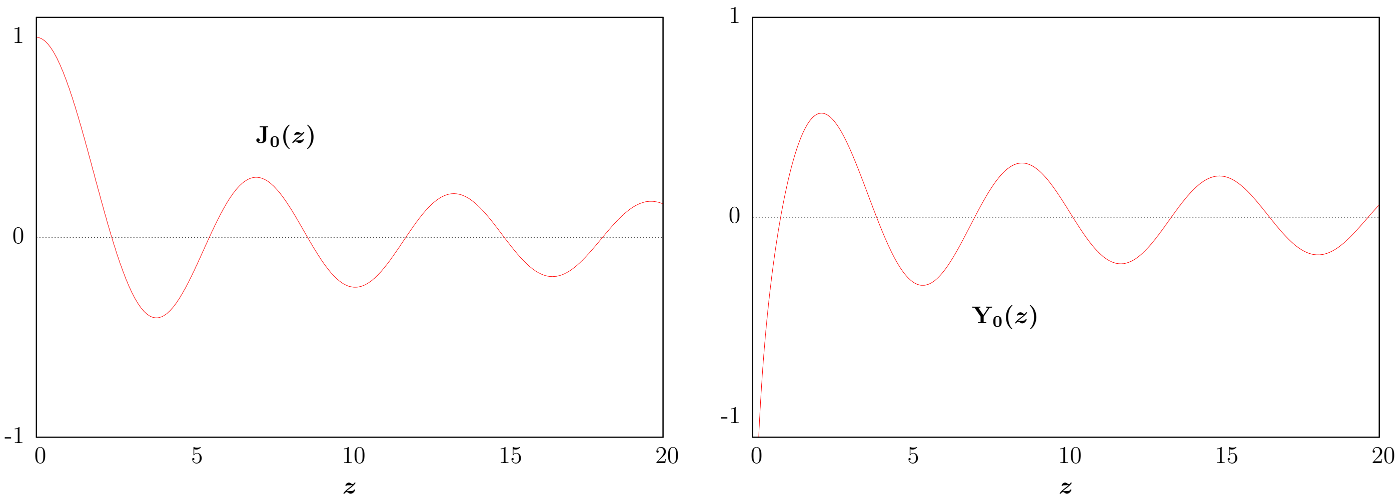
The Bessel functions are solutions of various forms of Bessel's ODE. Bessel equations in terms of the radial variable are produced by separation of variables in polar coordinates for the Helmholtz equation, $\nabla^2\phi = -\lambda\phi$. See Haberman, Chapter 7.7 (sections 7.7.3–7.7.7).

1 Bessel's equation of order zero

The Bessel equation of order zero comes from separation of variables for problems that have no dependence on θ, yielding an r-ODE for $f(r)$. General form ODE and corresponding general solution:

$$\boxed{\frac{d}{dz}\left(z\frac{dy}{dz}\right) + zy = 0} \qquad \to \qquad \boxed{y(z) = c_1 \mathrm{J}_0(z) + c_2 \mathrm{Y}_0(z)} \tag{1}$$

where $\mathrm{J}_0, \mathrm{Y}_0$ are called the first- and second-kind Bessel functions of order zero.



In many ways this is very analogous to the harmonic oscillator equation:

$$\frac{d^2y}{dz^2} + y = 0 \qquad \to \qquad y(z) = c_1 \cos z + c_2 \sin z \tag{2}$$

$\mathrm{J}_0, \mathrm{Y}_0$ are oscillatory functions like the trigonometric functions. And starting with the general solution and applying the boundary conditions given in problems allows us to reduce the general solution to the eigenfunction for each specific problem.

BUT there are some important differences too:

1. Matching Bessel's equation to Sturm-Liouville form shows that it has $p(z) = z$, so it is a singular SL problem and we expect one solution to be singular at $z = 0$. The $\mathrm{Y}_0(z)$ solution is singular there with $\mathrm{Y}_0 \to -\infty$ as $z \to 0$.
2. Because the oscillator equation (2) is autonomous (all of the coefficients of y-terms are constants), its solutions can be written in "translated" or "phase shifted" forms, like $c_1 \cos z + c_2 \sin z = A\cos(z - z_*)$ (using trig. identities). This is **NOT true** for Bessel's equation (because of the z-dependent coefficients in the eqn), its solutions **CAN NOT** be shifted: $\mathrm{J}_0(z - z_*)$ is NOT a solution of (1) and there is no identity formula to combine J's and Y's together.
3. $\mathrm{J}_0, \mathrm{Y}_0$ are oscillatory functions but their zeros and critical points (zeros of the first derivative) are not equally-spaced.
   For solving BVP's using eigenfunction expansions, we are mainly only interested in the positions of the zeros and critical points of these functions. These values do not have nice formulas (unlike $\sin(z_k) = 0$ at $z_k = k\pi$), but they have been computerized and tabulated:

For Dirichlet BC's

zeros	$J_0(z_k) = 0$
z_k	value
z_1	2.048255
z_2	5.520078
z_3	8.865373
z_4	11.791534
z_5	14.930917
	$\cdots$

zeros	$Y_0(z_k) = 0$
$\hat{z}_k$	value
$\hat{z}_1$	0.893577
$\hat{z}_2$	3.957678
$\hat{z}_3$	7.086051
$\hat{z}_4$	10.222345
$\hat{z}_5$	13.361097
	$\cdots$

For Neumann BC's

crit pts	$J_0'(\bar{z}_k) = 0$
$\bar{z}_k$	value
$\bar{z}_0$	0.000000
$\bar{z}_1$	3.831706
$\bar{z}_2$	7.015586
$\bar{z}_3$	10.173468
$\bar{z}_4$	13.323692
	$\cdots$

2 Bessel's equation of order m

The general version of Bessel's equation, that comes from the separation of variables of the Helmholtz equation with coupling to $\{\cos(m\theta), \sin(m\theta)\}$, is

$$\boxed{\frac{d}{dz}\left(z\frac{dy}{dz}\right) + \left(z - \frac{m^2}{z}\right) y = 0} \qquad \to \qquad \boxed{y(z) = c_1 \mathrm{J}_m(z) + c_2 \mathrm{Y}_m(z)} \tag{3}$$

where for $m \geq 0$, J_m is called the Bessel function of order m of first kind, and Y_m is the Bessel function of order m of second kind.

$\mathrm{J}_m, \mathrm{Y}_m$ are similar in spirit to $\mathrm{J}_0, \mathrm{Y}_0$ for $m = 0$:

- Like J_0, all of the J_m are bounded at $z = 0$.
- Like Y_0, all of the Y_m are singular at $z = 0$.
- All $\mathrm{J}_m, \mathrm{Y}_m$ are oscillatory functions.

The zeros and critical points for $k = 1, 2, 3, \cdots$ of these functions depend on the value of the m-order parameter: there are tables of values for $z_{m,k}, \hat{z}_{m,k}, \bar{z}_{m,k}, \cdots$ for each value of $m \geq 0$, for $m =$ integers or any real number.

3 Modified Bessel equations/functions

The relation of the harmonic oscillator equation (2) and its oscillatory solutions to the modified equation (with the sign of the y term flipped) and its non-oscillatory solutions:

$$\frac{d^2y}{dz^2} - y = 0 \qquad \to \qquad y(z) = c_1 \cosh z + c_2 \sinh z \tag{4}$$

also carries over to Bessel equation with the sign of the y term flipped.
The **modified** Bessel equation of order zero is

$$\boxed{\frac{d}{dz}\left(z\frac{dy}{dz}\right) - zy = 0} \qquad \to \qquad \boxed{y(z) = c_1 \mathrm{I}_0(z) + c_2 \mathrm{K}_0(z)} \tag{5}$$

with the general solution in terms of the modified Bessel functions of first and second kinds:

- I_0 is like J_0 because it doesn't blow up at $z = 0$
- K_0 is like Y_0 because it DOES blow up at $z = 0$
- I_0, K_0 are like $\cosh, \sinh$ because they don't oscillate

The modified version of the Bessel equation of order m is

$$\boxed{\frac{d}{dz}\left(z\frac{dy}{dz}\right) + \left(-z - \frac{m^2}{z}\right) y = 0} \qquad \to \qquad \boxed{y(z) = c_1 \mathrm{I}_m(z) + c_2 \mathrm{K}_m(z)} \tag{6}$$

Notice the flipped sign of the zy term in (5) and (6) compared to the regular Bessel eqns (1) and (3).

For more information on modified Bessel functions, see Haberman, section 7.9.5.