CEE628 Project Report

Uncertainty quantification on a one-dimensional pipe flow system

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1 Introduction

The incompressible pipe flow is a fundamental problem in fluid mechanics that are encountered in many industrial and engineering applications. The flow is governed by the Navier-Stokes equations, which are a set of partial differential equations. These equations are often solved numerically using deterministic solvers. However, there are uncertainties associated with the properties of the fluid, the geometry of the pipe, and the boundary conditions. These uncertainties can have a significant impact on the behavior of the flow and can lead to large errors in the predicted results. Therefore, it is important to develop stochastic models that can capture these uncertainties and provide more accurate predictions of flow behavior. The main equations are listed below:

$$\nabla \cdot \boldsymbol{u} = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}$$

In this study, we focus on the implementation of two propagation techniques for propagating the uncertainty, the Monte Carlo method and the Stochastic Galerkin method. To simplify the problem, we have adopted one-dimensional pipe flow.

Figure 1: Simple Model for 1D Pipe Flow

2 Methods

2.1 Methods for Fluid Simulation

we will introduce the methodology used to simulate one-dimensional pipe flow in this part. In this study, we use FDM to discretize the Navier-Stokes equations for incompressible flow in one-dimensional pipe flow. The spatial domain is discretized into a set of evenly spaced points, and the solution is advanced in time using a time-stepping scheme.The second-order windward formula is used to discretize the space, and the time scheme uses the Crank-Nicolson method. The pressure field is then updated using the pressure correction equation[1][2][3].

Second-order upwind scheme(This method enhances the stability and accuracy of the numerical solution by considering the influence of the convective terms' direction, thus reducing numerical diffusion in the process):

$$u\frac{\partial u}{\partial x} \approx \begin{cases} \frac{-u_{i+1}^2+4u_i^2-3u_{i-1}^2}{4\Delta x}, & \text{if } u_i \geq 0 \\ \frac{u_{i-1}^2-4u_i^2+3u_{i+1}^2}{4\Delta x}, & \text{if } u_i < 0 \end{cases}$$

Crank-Nicolson method(This semi-implicit method combines the advantages of both implicit and explicit schemes, providing a good balance between stability and computational efficiency for the time integration of fluid flow equations):

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + \frac{1}{2}(u_i^{n+1}+u_i^n)\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}$$

Pressure correction equation(This equation helps to decouple the velocity and pressure fields in incompressible fluid flow problems, allowing for a more efficient solution of the Navier-Stokes equations by iteratively updating the pressure field to maintain mass conservation):

$$p_{i+1}^{n+1} - p_{i-1}^{n+1} = \rho \Delta x (u_{i+1}^{n+1} - u_{i-1}^{n+1})$$

In order to improve the stability of the calculation, the simulation code uses an adaptive step size.
Adaptive time-stepping scheme:

$$\Delta t = C \min \left(\frac{\Delta x}{1.2 u_{\max}}, \frac{\Delta x^2}{\nu} \right)$$

The friction factor is an important parameter in pipe flow simulation as it accounts for the resistance due to the pipe wall roughness. The provided code calculates the friction factor using the Hagen-Poiseuille equation for laminar flow (Re $<$ 2000) and the Swamee-Jain equation for turbulent flow (Re $>$ 2000).
Friction factor calculation:

$$f = \begin{cases} \frac{64}{Re}, & \text{if } Re < 2000 \\ \frac{0.25}{(\log_{10}(\frac{\epsilon}{3.7D} + \frac{5.74}{Re^{0.9}}))^2}, & \text{if } Re \geq 2000 \end{cases}$$

2.2 Stochastic representation of the Inputs

In this study, we consider two input parameters that exhibit uncertainty: fluid viscosity and pipe diameter. We model the uncertainty of these parameters using probability distributions as follows.

Input variable and U.M.	pdf	mean $E(\cdot)$	σ	range
D, pipe diameter, (m)	uniform	0.022, 0.0228	–	$E(D) \pm 0.001$
L, pipe length, (m)	uniform	9	–	$E(L) \pm 0.01$
β_1 , inclination angle 1, (°)	uniform	0°, 1°	–	$E(\beta_1) \pm 0.1$
β_2, inclination angle 2, (°)	uniform	5°	–	$E(\beta_2) \pm 0.05$
Q_g, gas (air) flow rate, (m^3/s)	normal	3.8×10^{-4}, 7.6×10^{-4} $3.8 \times 10^{-4} \div 2.7 \times 10^{-3}$ (Fig. 11)	1.25×10^{-5}	$E(Q_g) \pm 3\sigma$
Q_l, liquid flow rate, (m^3/s)	normal	3.8×10^{-5}, 8.94×10^{-6} 7.6×10^{-4}, 7.6×10^{-5} 1.9×10^{-4} $7.6 \times 10^{-6} \div 7.6 \times 10^{-5}$ (Fig. 11)	0.7×10^{-5}	$E(Q_l) \pm 3\sigma$
ρ_g, gas (air) density, (kg/m^3)	normal	1.2	0.1	$E(\rho_g) \pm 3\sigma$
ρ_l, liquid density, (kg/m^3)	normal	1000, 886, 1002	0.5	$E(\rho_l) \pm 3\sigma$
μ_g, gas (air) viscosity, (Pa· s)	normal	1.8×10^{-5}	2×10^{-6}	$E(\mu_g) \pm 3\sigma$
μ_l, water viscosity, (Pa· s)	normal	0.001	6.25×10^{-5}	$E(\mu_l) \pm 3\sigma$
m_1, consistency index F1, (Pa· s^{n_1})	normal	0.05	6.25×10^{-5}	$E(n_1) \pm 3\sigma$
n_1, behaviour index F1, (–)	normal	0.6	0.005	$E(n_1) \pm 3\sigma$
m_2, consistency index F2, (Pa· s^{n_2})	normal	0.006	0.00025	$E(m_2) \pm 3\sigma$
n_2, behaviour index F2, (–)	normal	0.6	0.025	$E(n_2) \pm 3\sigma$
m_3, consistency index CMC6, (Pa· s^{n_3})	normal	0.264	0.01	$E(m_3) \pm 3\sigma$
n_3, behaviour index CMC6, (–)	normal	0.76	0.0125	$E(n_3) \pm 3\sigma$
σ_{ow}, oil-water surface tension, (N/m)	normal	0.05	0.001	$E(\sigma_{ow}) \pm 3\sigma$
σ_{a2}, air-F2 surface tension, (N/m)	normal	0.072	0.0025	$E(\sigma_{a2}) \pm 3\sigma$

Figure 2: Prabability Distribution of Inputs From Reference[4]

Fluid viscosity: We assume that the fluid viscosity follows a lognormal distribution, which is commonly used to model positive variables that have a skewed distribution.

$$f(x;\mu,\sigma) = \frac{1}{x \cdot 0.096809 \cdot \sqrt{2\pi}} e^{-\frac{(\ln x + 6.907755)^2}{2 \cdot 0.096809^2}}$$

Pipe diameter: We assume that the pipe diameter follows a uniform distribution, which is commonly used to model variables that are equally likely to take any value within a given range.

$$f(x;a,b) = \frac{1}{0.031 - 0.029}, \quad 0.029 \leq x \leq 0.031$$

These probability distributions are used to generate a set of input samples for the numerical simulations, which are used to propagate the

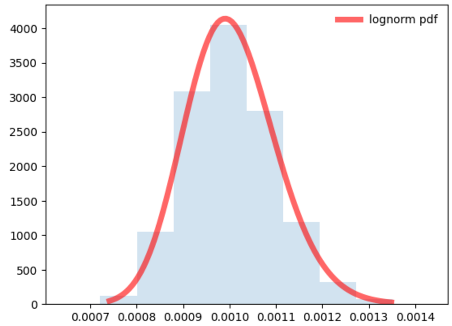

Figure 3: PDF of Fluid Viscosity

uncertainties through the model and perform global sensitivity analysis.

In this case, we are given the probability distributions of two input parameters (fluid viscosity and pipe diameter), and they are independent of each other. Therefore, we can directly use the joint probability density function of these distributions to generate samples. This allows us to propagate the uncertainties through the model and perform global sensitivity analysis. Here, we do not need to use the maximum entropy principle, as the known input distributions are sufficient for quantifying the model's uncertainty.

For fluid viscosity and pipe diameter, the joint probability density function (PDF) can be obtained by multiplying the two independent probability density functions. Specifically, let $f_\nu(\nu)$ represent the probability density function of fluid viscosity, and $f_D(D)$ represent the probability density function of pipe diameter, then the joint probability density function $f_{\nu,D}(\nu, D)$ is:

$$f_{\nu,D}(\nu, D) = f_\nu(\nu) f_D(D)$$

Here, $f_\nu(\nu)$ is the log-normal distribution of fluid viscosity, and $f_D(D)$ is the uniform distribution of pipe diameter. This two-dimensional joint probability density function can be used to generate input samples for the model, allowing for uncertainty propagation and global sensitivity analysis.

2.3 Stochastic Solver

In this study, we employ two methods to propagate the uncertainty through the model: the non-intrusive Monte Carlo method and the intrusive Stochastic Galerkin method[5].

1.The Monte Carlo method is a non-intrusive simple approach for uncertainty propagation,

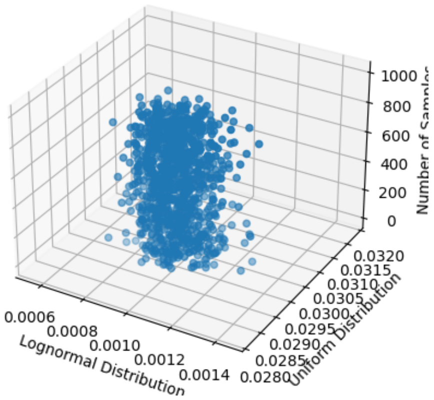

Figure 4: Possible Samples of The Random Filed

where random samples are directly generated according to the input probability distributions. Although the Stochastic Collocation Methods can reduce the number of samples required, they are not suitable for non-smooth distributions, such as the lognormal distribution used in this study. In the Monte Carlo method, multiple simulations are performed using randomly generated input samples, and the results are used to estimate the output statistics.

2.The Stochastic Galerkin method(SGMs or GMs) is an intrusive method that approximates the model response as a series of orthogonal polynomials. The PCE expansion coefficients are determined by solving a set of equations obtained by projecting the model equations onto the polynomial space. The benefit of using PCE is that it provides an approximate expression for the output, allowing for the evaluation of the response at any point in the input space without the need for further simulations. For the lognormal distribution of fluid viscosity, the appropriate orthogonal polynomials are logarithmic orthogonal polynomials, while for the uniform distribution of pipe diameter, Legendre polynomials are employed. The PCE expansion is given by:

$$Y(\boldsymbol{\xi}) \approx \sum_{j=1}^{N_{pce}-1} y_j \Psi_j(\boldsymbol{\xi})$$

where $Y(\boldsymbol{\xi})$ is the model response, $\boldsymbol{\xi}$ denotes the input parameters, P is the order of the PCE, y_j are the expansion coefficients, and $\Psi_j(\boldsymbol{\xi})$ are the orthogonal polynomials.

The PCE coefficients y_j are determined by solving a set of equations derived from the model equations. These coefficients can be computed using Monte Carlo integration.

$$y_j =< Y(\boldsymbol{\xi}), \Psi_i(\boldsymbol{\xi}) >_H \approx \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} Y(\boldsymbol{\xi}(\boldsymbol{\theta_i}))\Psi_i(\boldsymbol{\xi}(\boldsymbol{\theta_i}))$$

Once the PCE expansion is obtained, it can be used to compute the statistics of the model output.

The SGMs with PCE provides an efficient way to propagate uncertainty through the model while maintaining the accuracy of the approximation.

3 Results

The average flow velocity in the pipeline is choosen as the output variable for analysis.

Firstly, the pure monte carlo simulation is executed. The convergence of Monte Carlo simulation is performed, and the convergence result of the first-moment and second-moment can be seen in Figure 5.

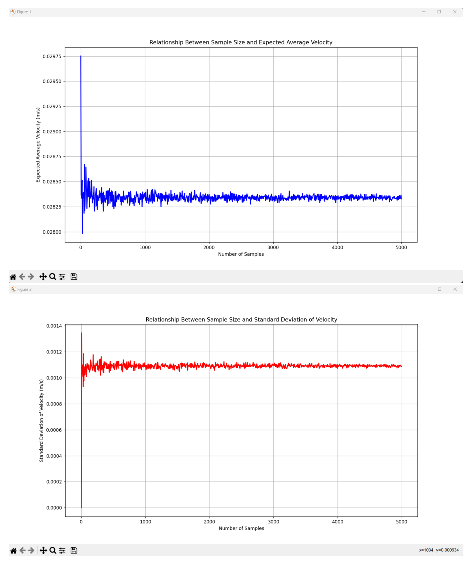

Figure 5: Monte Carlo Simulation Results

It shows that the convergence is reached at approximately 4000 samples.

Secondly, We use SGMs to propagate the uncertainties. In this part, we used 100 samples to calculated the coefficients for our PCE The coefficients are printed on the output which can be seen in UQ Code. The convergence results of different polynomial degrees of SGM are shown in Figure 6.

However, the probability distribution of second order PCE (or higher) is consistent with the

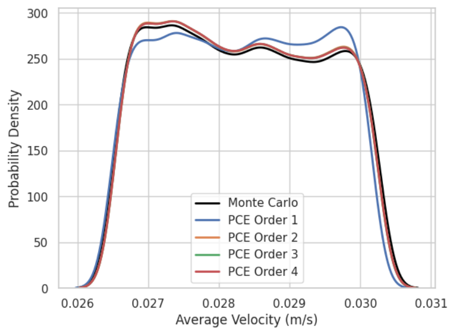

Figure 6: Monte Carlo Simulation Results

Monte Carlo simulations quite well, which indicates that only with polynomial degrees higher than 2 one can obtain the convergent results.

After that, I tested the performance of the second-order PCE under different sample input conditions, and found that it was very close to the results obtained by directly sampling into the model operation, and both the first moment and the second moment converged faster. The result is shown in Figure 7.

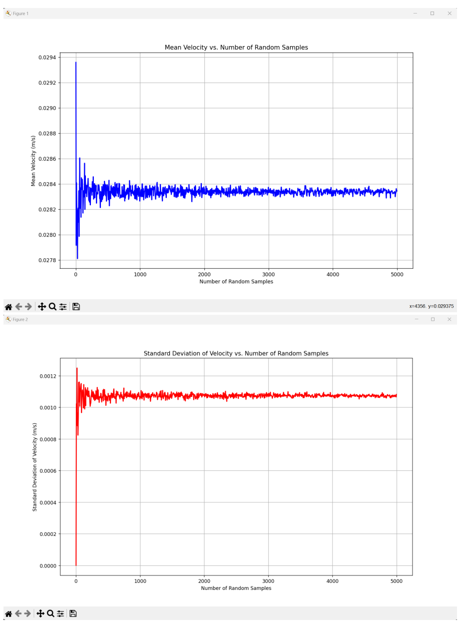

Figure 7: Monte Carlo Simulation Results

4 Conclusions

In this study, we have investigated uncertainty quantification in a one-dimensional pipe flow system by implementing two propagation techniques: the Monte Carlo method and the Stochastic Galerkin method. The problem was simplified by adopting one-dimensional pipe flow and focusing on two uncertain input parameters: fluid viscosity and pipe diameter.

The non-intrusive Monte Carlo method was employed to propagate uncertainty through the model by directly generating random samples according to the input probability distributions. The convergence of the Monte Carlo simulation was demonstrated with approximately 4000 samples.

The intrusive Stochastic Galerkin method, which uses Polynomial Chaos Expansion (PCE), provided an efficient way to propagate uncertainty through the model while maintaining the accuracy of the approximation. Convergence tests for the polynomial degree of the Stochastic Galerkin method showed that the second-order PCE is enough to provide accurate results.

This study demonstrates the effectiveness of these two methods in uncertainty quantification for fluid flow problems and provides a basis for further research in more complex fluid systems.

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